IRREDUCIBLE EXTENSIONS OF CHARACTERS

by

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ABSTRACT

Suppose that H is a finite group and ξ is a not necessarily irreducible character of H. In this note, we study the question of whether or not there exist a finite group G containing H and an irreducible character χ of G such that the restriction of χ to H is ξ. We also investigate some related questions.

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1. Introduction. Given a (not necessarily irreducible) character ξ of a finite group H, we investigate the question of whether or not there exist a finite group G containing H and an *irreducible* character χ of G such that $\chi_H = \xi$. If G and χ exist, we refer to χ as an **irreducible extension** of ξ. Given the huge variety of different groups G that contain a given group H, it may be surprising that sometimes, the character ξ may fail to have an irreducible extension, but this can happen.

THEOREM A. *Let $H = \mathrm{SL}(2,3)$, and suppose that $\xi = 1_H + \varphi$, where φ is the unique real-valued degree 2 irreducible character of H. Then ξ has no irreducible extension.*

In fact, $\mathrm{SL}(2,3)$ is the smallest group that has a character with no irreducible extension. Also, $\mathrm{SL}(2,3)$ is the smallest group that fails to be an M-group, and the following theorem shows that this is not entirely a coincidence. (Recall that by definition, a not necessarily irreducible character φ of a group H is **monomial** if $\varphi = \lambda^H$ for some linear character λ of a subgroup of H. Also, H is an **M-group** if every irreducible character of H is monomial.

THEOREM B. *If H is an M-group, then every character of H has an irreducible extension.*

It is not clear whether or not, conversely, it is true that if every character of some group H has an irreducible extension, then H must be an M-group, but the following is a step in that direction.

THEOREM C. *Suppose that H is a solvable group that is not an M-group, and let C be a cyclic group of order at least 7. Then some character of $H \times C$ fails to have an irreducible extension.*

The following theorem suggests that perhaps Theorem C would remain true if we drop the hypothesis that H is solvable.

THEOREM D. *Suppose that H has a non-monomial irreducible character φ such that $\varphi + 1_H$ is not a permutation character, and let C be a cyclic group of order at least 7. Then some character of $H \times C$ fails to have an irreducible extension.*

For example, suppose we take H to be the alternating group A_5. If φ is one of the degree 3 irreducible characters of H, then φ is not monomial and $\varphi + 1_H$ is not a permutation character. (Both of these assertions follow from the fact that A_5 has no proper subgroup with index 3 or 4.) We can thus apply Theorem D to conclude that some character of $A_5 \times C$ fails to have an irreducible extension, where C is an arbitrary cyclic group of order at least 7.

Finally, we consider the question of whether or not a given character ξ of a finite group H has an irreducible extension to a *solvable* group G. Of course, if ξ has an extension to a solvable group, then H would have to be solvable, but as the next theorem suggests, even in the case where H is solvable, the requirement that G should be solvable appears to be quite restrictive.

THEOREM E. *Let $\alpha, \lambda \in \mathrm{Irr}(H)$, where λ is linear, and write $\xi = \alpha + \lambda$. Then ξ has no irreducible extension to a solvable group unless $\xi(1)$ is a prime power.*

We do have one affirmative result concerning solvable extensions, however.

THEOREM F. *If H is a abelian, then every character of H has an irreducible extension to a solvable group.*

Since an abelian group is certainly an M-group, it follows by Theorem B that every character of an abelian group has an irreducible extension. The supergroup constructed in the proof of Theorem B, however, is generally very far from being solvable, so we will use a completely different argument to establish Theorem F.

We mention that we do not know if there exists a nonabelian group with the property that every character has an irreducible extension to a solvable group.

2. Examples where no irreducible extension exists. Given a character ξ of a group H and a group G containing H, we shall say that a character $\chi \in \mathrm{Irr}(G)$ such that $\chi_H = \xi$ is a **monomial extension** of ξ if χ is a monomial character. Also, we say that $\chi \in \mathrm{Irr}(G)$ is a **primitive extension** of ξ if χ is a **primitive character**, which means that that there is no character α of a proper subgroup of G such that $\alpha^G = \chi$. Note that if $J \subseteq G$ and $\alpha \in \mathrm{Irr}(J)$ with $\alpha^G = \chi$, then $\chi(1) = |G : J|\alpha(1)$, so if $\chi(1)$ is prime, then either $\alpha(1) = 1$ and χ is monomial, or else $|G : J| = 1$ and $J = G$. It follows that if $\xi(1)$ is prime, then an irreducible extension of ξ must be either a monomial extension or a primitive extension.

It is easy to establish a condition that is sufficient to guarantee that a character ξ of H has no monomial extension.

(2.1) LEMMA. *Let ξ be a faithful character of a group H, and write $\xi(1) = n$. If ξ has an irreducible monomial extension, then there exists an abelian subgroup $A \triangleleft H$ such that H/A is isomorphic to a subgroup of the symmetric group S_n.*

Proof. Let $H \subseteq G$, and suppose that $\chi \in \mathrm{Irr}(G)$ is a monomial character of G such that $\xi = \chi_H$. Since χ is monomial, there exists a subgroup $J \subseteq G$ and a linear character λ of J such that $\lambda^G = \chi$. Now $n = \xi(1) = \chi(1) = |G : J|\lambda(1) = |G : J|$, so writing $K = \mathrm{core}_G(J)$, we see that G/K is isomorphic to a subgroup of S_n. Then $H/(K \cap H) \cong KH/K$ is also isomorphic to a subgroup of S_n, and so it suffices to observe that $K \cap H$ is abelian.

Now λ_K is a linear constituent of χ_K, and since χ is irreducible and $K \triangleleft G$, it follows that every irreducible constituent of χ_K is linear, and thus $K' \subseteq \ker(\chi)$. Then $(K \cap H)' \subseteq K' \cap H \subseteq \ker(\chi) \cap H = \ker(\xi) = 1$, and thus $K \cap H$ is abelian, as required. ∎

We can now prove Theorem A, which we restate here.

(2.2) THEOREM. *Let $H = \mathrm{SL}(2, 3)$, and suppose that $\xi = 1_H + \varphi$, where φ is the unique real-valued degree 2 irreducible character of H. Then ξ has no irreducible extension.*

Proof. Assuming that the theorem is false, let $G \supseteq H$ and $\chi \in \mathrm{Irr}(G)$ with $\chi_H = \xi$, and choose G to have the smallest possible order. Now $\xi(1) = 3$ and ξ is faithful, but since no normal abelian subgroup of H has index dividing $3! = 6$, Lemma 2.1 guarantees that ξ has no monomial irreducible extension. Since $\xi(1)$ is prime, it follows that χ must be primitive.

We have $H\cap\ker(\chi) = \ker(\xi) = 1$, and thus H is isomorphic to a subgroup of $G/\ker(\chi)$. Viewing χ as a character of $G/\ker(\chi)$, we see that χ is an irreducible extension of the character corresponding to ξ of the isomorphic copy of H in $G/\ker(\chi)$. By the minimality of G, therefore, we have $\ker(\chi) = 1$, so χ is faithful, and thus G is a degree 3 complex linear group.

It follows by Theorem A of **[1]** that $|G : Z| \leq 360$, where $Z = \mathbf{Z}(G)$. Also, we see that if $N \triangleleft G$ and $N \not\subseteq Z$, then N must be nonabelian, and since $\chi(1)$ is prime, we conclude that χ_N is irreducible, and thus $\mathbf{C}_G(N) = Z$.

Now $\det(\varphi)$ is trivial, so $\det(\xi)$ is also trivial, and writing $D = \ker(\det(\chi))$, we see that $H \subseteq D \triangleleft G$, and thus $D \not\subseteq Z$. Then χ_D is irreducible, and it follows by the minimality of G that $D = G$, and thus $\det(\chi)$ is trivial.

We see that $\chi_Z = 3\mu$ for some faithful linear character μ of Z, and since $\det(\chi)$ is trivial, we have $\mu^3 = 1_Z$, and thus $|Z|$ divides 3. Also, $Z \cap H \subseteq \mathbf{Z}(H)$, and since $|\mathbf{Z}(H)| = 2$, we deduce that $Z \cap H = 1$.

Let N/Z be a minimal normal subgroup of G/Z, so $Z < N \triangleleft G$. As we have seen, it follows that N is nonabelian and χ_N is irreducible, and thus χ_{NH} is irreducible. We conclude by the minimality of G that $NH = G$.

We argue next that N/Z does not contain a subgroup isomorphic to the quaternion group Q_8. This is obvious if N/Z is abelian, and otherwise, since $|N/Z| \leq |G/Z| \leq 360$, we see that N/Z must be isomorphic to one of the simple groups A_5, $\mathrm{PSL}(3,2)$ or A_6. Since none of these groups contains a copy of Q_8, our assertion holds in this case too.

We have

$$N/Z \supseteq (N\cap H)Z/Z \cong (N\cap H)/(N\cap H\cap Z) = N\cap H\,,$$

and thus $N\cap H$ does not contain the copy of Q_8 contained in H. Since $N\cap H$ is a normal subgroup of H, it follows that $|N\cap H| \leq 2$, and thus $|H : N\cap H| \geq 12$. Then

$$360 \geq |G:Z| \geq |NH:Z| = |NH:N||N:Z| = |H:N\cap H||N:Z| \geq 12||N:Z|\,,$$

and hence $|N/Z| \leq 30$. Then N/Z is not one of the groups A_5, $\mathrm{PSL}(3,2)$ or A_6, and so N/Z must be abelian. Since N is not abelian, Z must be nontrivial, and thus $|Z| = 3$.

Now χ_N is irreducible and N/Z is an abelian chief factor of G, and since χ_Z is a multiple of a linear character, it follows that χ_N is fully ramified with respect to N/Z. We deduce that $|N/Z| = \chi(1)^2 = 9$, and thus $|N| = 27$.

Since H has no nonidentity normal subgroup whose order is a power of 3, we have $N\cap H = 1$, and thus $|G| = |NH| = |N||H| = 27{\cdot}24 = 648$. In order to identify the group G using the GAP or Magma software packages, we observe that N is the full Fitting subgroup of G. (This follows since $\mathbf{Z}(H)$ is the unique minimal normal subgroup of H and $\mathbf{Z}(H)$ does not centralize N because $\mathbf{C}_G(N) = Z$, and $H\cap Z = 1$.)

Using GAP or Magma, we can check that (up to isomorphism) there are exactly three groups X of order 648 such that $\mathbf{F}(X)$ is nonabelian and has order 27, and in only one of these, namely `SmallGroup(648,533)`, is the Fitting subgroup complemented. We deduce that our group G must be `SmallGroup(648,533)`.

Querying the software further, we find that G has exactly six faithful irreducible characters of degree 3, and that there are exactly three conjugacy classes of complements to $\mathbf{F}(G)$ in G. For each such complement K, exactly two of the six restrictions χ_K have a principal constituent. In each of these cases, χ_K has a unique nonprincipal irreducible constituent of degree 2, but this character is never real. This contradiction completes the proof. ∎

It is interesting to note that according to the computer, the complements for the Fitting subgroup of `SmallGroup(648,533)` actually are isomorphic to $\mathrm{SL}(2,3)$. Also, the computations show that if we were to replace φ with either of the two non-real irreducible degree 2 characters of $H = \mathrm{SL}(2,3)$, the resulting character ξ definitely would have an irreducible extension.

We mention that it is also possible to complete the proof of Theorem 2.2 without the computer, using some fairly deep theory instead. The following is a very brief sketch of the argument. If $T \subseteq H$ is a subgroup of order 3, then in the language of Chapter 8 of [**3**], we see that ZT is a distinguished complement for N relative to Z in NT. It follows that χ_{TZ} has a unique irreducible constituent with odd multiplicity, and thus χ_T also has this property. Now $\chi_T = \xi_T = \varphi_T + 1_T$, and φ_T is the sum of two nonprincipal linear characters. Then χ_T is a sum of three distinct linear characters, and this is the desired contradiction.

(2.3) COROLLARY. *Let $K = \mathrm{GL}(2,3)$, and let $\theta, \lambda \in \mathrm{Irr}(K)$, where λ is linear and θ is faithful and has degree 2. Then $\theta + \lambda$ has no irreducible extension.*

Proof. Let $H = \mathrm{SL}(2,3)$ so $H \subseteq K$. It is easy to check that the restriction of $\theta + \lambda$ to H is the character ξ of Theorem 2.2. An irreducible extension of $\theta + \lambda$, therefore, would be an irreducible extension of ξ, and since by Theorem 2.2, no such irreducible extension exists, it follows that $\theta + \lambda$ has no irreducible extension. ∎

Similarly, we have the following, which provides another example of a nonsolvable group having a character with no irreducible extension.

(2.4) COROLLARY. *Let $K = \mathrm{SL}(2,5)$, and let $\theta \in \mathrm{Irr}(K)$ have degree 2. Then $\theta + 1_K$ has no irreducible extension.*

Proof. Observe that K has a subgroup H isomorphic to $\mathrm{SL}(2,3)$ and that the restriction of $\theta + 1_K$ to H is the character ξ of Theorem 2.2. Since ξ has no irreducible extension, the result follows. ∎

3. Sums of monomial characters. The main result of this section is the following.

(3.1) THEOREM. *Let $\theta_1, \theta_2, \ldots, \theta_m$ be monomial (but not necessarily irreducible) characters of some group H. Then $\theta_1 + \theta_2 + \cdots + \theta_m$ has a monomial irreducible extension.*

If H is an M-group, then since every irreducible character of H is monomial, it follows that every character of H is a sum of monomial characters. Assuming Theorem 3.1, therefore, we see that every character of H has an irreducible extension, and this is Theorem B.

We begin working toward a proof of Theorem 3.1 with some definitions. Given a finite group H, we say that a $\mathbb{C}[H]$-module V having a basis B is a **monomial module** with **monomial basis** B if for each basis vector $b \in B$ and each group element $h \in H$, there exists a vector $c \in B$ and a root of unity ε such that $bh = \varepsilon c$. Note that if V is a monomial module for H with monomial basis B, then there is a natural associated permutation action of H on the set B, and if this action is transitive, we say that the monomial module V is **transitive**.

Next, we say that a square matrix M over $\mathbb{C}$ is a **monomial matrix** if each row and each column of M contains exactly one nonzero entry, and all of the nonzero entries of M are roots of unity. Finally, a matrix representation $\mathcal{X}$ of H is a **monomial representation** if $\mathcal{X}(H)$ consists of monomial matrices.

Observe that a $\mathbb{C}[H]$-module V is a monomial module with respect to some basis B if and only if the corresponding matrix representation of H (with respect to the basis B) is a monomial representation. It is easy to see that if θ is a (not necessarily irreducible) monomial character of H, then θ is afforded by a monomial representation, and the corresponding monomial module is transitive on the relevant basis B. (In fact, if $\theta = \lambda^H$, where λ is a linear character of a subgroup J of H, then the permutation action of H on B is permutation isomorphic to the natural action of H on the right cosets of J in H.) Conversely, the character afforded by a monomial representation corresponding to a transitive monomial module is guaranteed to be a monomial character.

Given positive integers n and r, we write $\mathrm{Mon}(n,r)$ to denote the group of $n \times n$ monomial matrices whose nonzero entries are rth roots of unity. Since each monomial $n \times n$ matrix has exactly n nonzero entries, we see that $|\mathrm{Mon}(n,r)| = n!{\cdot}r^n$, and in particular $\mathrm{Mon}(n,r)$ is a finite subgroup of $\mathrm{GL}(n,\mathbb{C})$. (Although we shall not need this fact, we mention that $\mathrm{Mon}(n,r)$ is isomorphic to the wreath product $C_r \wr S_n$, where C_r is the cyclic group of order r and S_n is the symmetric group of degree n.)

(3.2) LEMMA. *If $r > 1$, then the natural degree n monomial representation of* $\mathrm{Mon}(n,r)$ *is irreducible.*

One way to prove this would be to observe that $\mathrm{Mon}(n,r)$ contains the full group of $n \times n$ permutation matrices, and it is well known that if $n > 1$, then the corresponding n-dimensional permutation module has exactly two nonzero proper submodules: one of dimension 1 and the other of dimension $n-1$. To prove Lemma 3.2, therefore, it suffices to show that if $r > 1$, then neither of these submodules is invariant under $\mathrm{Mon}(n,r)$. We prefer, however, to give the following direct argument.

Proof of Lemma 3.2. Let V be the natural monomial module for $\mathrm{Mon}(n,r)$, where $\dim_{\mathbb{C}}(V) = n$, and let B be the corresponding monomial basis. To show that V is irreducible (in fact, that it is absolutely irreducible) it suffices to show that every linear operator $f : V \to V$ that commutes with the action of $\mathrm{Mon}(n,r)$ is multiplication by some scalar.

Let $b \in B$, and let $g \in \mathrm{Mon}(n,r)$ be the linear operator such that $bg = b$ and $cg = \varepsilon c$, where ε is an rth root of unity different from 1. Then $\mathbb{C}b$ is the fixed-point subspace of g in V, and since f commutes with g we see that f must map $\mathbb{C}b$ into itself, and thus $bf = \beta b$ for some scalar β. This shows that the matrix of f with respect to the basis B is diagonal.

If b and c are arbitrary members of B, we can write $bf = \beta b$ and $cf = \gamma c$ for scalars β and γ, and it suffices to show that $\beta = \gamma$. To see this, let $h \in \mathrm{Mon}(n,r)$, where $bh = c$ and $ch = b$. Then $\beta c = \beta bh = bfh = bhf = cf = \gamma c$, and it follows that $\beta = \gamma$, as required. ∎

Proof of Theorem 3.1. Write $\xi = \sum \theta_i$, so we must show that there exists an injective homomorphism π from H into some finite group G, where G has a monomial irreducible character χ and $\chi(\pi(h)) = \xi(h)$ for all $h \in H$.

Suppose first that ξ is faithful. Since θ_i is monomial for $1 \le i \le m$, each of the characters θ_i is afforded by a monomial representation $\mathcal{X}_i$, where $\mathcal{X}_i$ corresponds to a (transitive) monomial module V_i having a monomial basis B_i. Viewing the bases B_i as disjoint sets, let $B = \bigcup B_i$, so $|B| = n$, where $n = \xi(1)$. Let V be the $\mathbb{C}$-linear span of B, so V is the direct sum of its subspaces V_i.

Now if $b \in B$ then b lies in one of the sets B_i, and thus bh is defined. In fact, if we fix a sufficiently large integer $r > 1$, we can write $bh = \varepsilon c$ where $c \in B_i \subseteq B$ and ε is an rth root of unity. Then V is a monomial $\mathbb{C}[H]$-module, with monomial basis B, and this construction defines a homomorphism $\pi : H \to \mathrm{Mon}(n, r)$ such that $\ker(\pi) = \bigcap \ker(\mathcal{X}_i) = \ker(\xi) = 1$.

Now let $\mathcal{M}$ be the natural monomial representation of $\mathrm{Mon}(n, r)$, and let χ be the character of $\mathrm{Mon}(n, r)$ afforded by $\mathcal{M}$. Then χ is irreducible by Lemma 3.2, and χ is monomial because V is transitive as a monomial module for $\mathrm{Mon}(n, r)$. Also,

$$\chi(\pi(h)) = \mathrm{tr}(\mathcal{M}(\pi(h))) = \sum_i \mathrm{tr}(\mathcal{X}_i(h)) = \sum_i \theta_i(h) = \xi(h)\,,$$

and this completes the proof in the case where ξ is faithful.

In the general case, we can view ξ is a faithful character of $\overline{H} = H/\ker(\xi)$, so by the first part of the proof, there is a group K containing $\overline{H}$ and a monomial irreducible character ψ of K that extends ξ. Now let $G = H \times K$ and let $\chi = 1_H \times \psi$, so $\chi \in \mathrm{Irr}(G)$ and χ is monomial.

If $h \in H$, let $\overline{h}$ denote the image of h in $\overline{H}$, and observe that $(h, \overline{h})$ lies in $H \times \overline{H} \subseteq H \times K = G$. The map $\pi : h \mapsto (h, \overline{h})$ is an injective homomorphism from H into G, and for $h \in H$, we have $\chi(\pi(h)) = \chi(h, \overline{h}) = \psi(h) = \xi(h)$, as wanted. ∎

(3.3) COROLLARY. *Let ξ be a character of a group H. Then there exists a finite group $G \supseteq H$ and an irreducible character χ of G such that $\chi_H = \xi + \psi$ for some character ψ of H*

Proof. There clearly exists a character ψ of H such that $\xi + \psi$ is a multiple $m\rho$ of the regular character ρ of H. Now $\rho = (1_1)^H$ so ρ is monomial, and thus $\xi + \psi = m\rho$ has an irreducible extension by Theorem 3.1. ∎

4. Cyclic groups. Our proof of Theorem 2.2 indicates that it can be quite difficult to establish the nonexistence of a primitive extension of a character of some given group H. The following lemma, however, shows that if H is cyclic of sufficiently large order, it is easy to find characters that have no primitive extension.

(4.1) LEMMA. *Let C be a cyclic group with $|C| \ge 7$, and let ξ be a character of C that has exactly two distinct irreducible constituents: the principal character and some faithful linear character λ. Then ξ has no primitive extension.*

Proof. Since λ is faithful, there exists an element $c \in C$ such that $\lambda(c) = e^{2\pi i/n}$, where $n = |C| \geq 7$. Assuming that there exist a finite group $G \supseteq C$ and a primitive character $\chi \in \mathrm{Irr}(G)$ such that $\chi_C = \xi$, we work to obtain a contradiction.

Let $\mathcal{X}$ be a representation affording χ, so $\mathcal{X}(G)$ is a finite complex linear group, and the matrix $\mathcal{X}(c)$ has exactly two distinct eigenvalues: 1 and $e^{2\pi i/n}$. All of the eigenvalues of $\mathcal{X}(c)$, therefore, lie on the unit circle in an arc of length $2\pi/n < \pi/3$.

By a result of Frobenius (Theorem 14.15(b) of **[2]**), we deduce that the conjugates of $\mathcal{X}(c)$ in $\mathcal{X}(G)$ centralize each other, and hence these conjugates generate an abelian normal subgroup A of $\mathcal{X}(G)$. Since the representation $\mathcal{X}$ is primitive, the subgroup A must be central in $\mathcal{X}(G)$, and hence the irreducibility of $\mathcal{X}$ guarantees that A consists of scalar matrices. This is a contradiction, however, because $\mathcal{X}(c)$ lies in A and $\mathcal{X}(c)$ has two distinct eigenvalues. ∎

Next, we establish a result sufficient to prove that certain characters have no irreducible monomial extension.

(4.2) LEMMA. *Let $\varphi \in \mathrm{Irr}(H)$, where φ is not monomial and $\varphi + 1_H$ is not a permutation character, and let $\xi = \varphi + m1_H$, where m is a nonnegative integer. Then ξ has no irreducible monomial extension.*

Proof. Working to obtain a contradiction, suppose there exist a group $G \supseteq H$ and a character $\chi \in \mathrm{Irr}(G)$ such that $\chi_H = \xi$ and χ is monomial. We have $\chi = \lambda^G$, where λ is a linear character of some subgroup $J \subseteq G$, and so by Mackey's lemma, we have

$$\varphi + m1_H = \xi = \chi_H = \sum_{x \in R} ((\lambda^x)_{H \cap J^x})^H ,$$

where R is a set of representatives for the double cosets of the form JxH in G, and where λ^x is the linear character of J^x defined by the formula $\lambda^x(j^x) = \lambda(j)$ for $j \in J$. It follows that there is some element $x \in R$ such that φ is a constituent of δ^H, where $\delta = (\lambda^x)_D$ and $D = H \cap J^x$.

Now $\delta^H \neq \varphi$ because by assumption, φ is not monomial, and thus $\delta^H = \varphi + t1_H$ for some integer t with $0 < t \leq m$. Then

$$0 < t = [\delta^H, 1_H] = [\delta, 1_D] \leq 1 ,$$

where the final inequality holds since δ is irreducible because it is the restriction to D of the linear character λ^x. It follows that $t = 1$ and $\delta = 1_D$, and thus $\varphi + 1_H = \delta^H = (1_D)^H$. This is a contradiction since by assumption, $\varphi + 1_H$ is not a permutation character. ∎

We can now prove Theorem D, which we restate here.

(4.3) THEOREM. *Suppose that H has a non-monomial irreducible character φ such that $\varphi + 1_H$ is not a permutation character, and let C be a cyclic group of order at least 7. Then some character of $H \times C$ fails to have an irreducible extension.*

Proof. Define $\alpha, \beta \in \mathrm{Irr}(H \times C)$ by setting $\alpha = \varphi \times 1_C$ and $\beta = 1_H \times \lambda$, where λ is some faithful linear character of C. Next, choose an integer $m > 0$ so that $\varphi(1) + m$ is prime, and let $\xi = \alpha + m\beta$. We argue by contradiction that ξ has no irreducible extension.

Suppose that $G \supseteq (H \times C)$ and that $\chi \in \mathrm{Irr}(G)$ extends ξ. Then $\chi(1) = \xi(1) = \alpha(1) + m\beta(1) = \varphi(1) + m$, and since this is a prime number, we see that either χ must be either monomial or is primitive. Lemma 4.1 applies because χ is an extension of the character $\varphi(1)1_C + m\lambda$ of C, and we deduce that χ cannot be primitive. Also, Lemma 4.2 applies because χ is an extension of the character $\varphi + m1_H$ of H, and it follows that χ cannot be monomial. This is the desired contradiction. ∎

Next, we restate Theorem C.

(4.4) THEOREM. *Suppose that H is a solvable group that is not an M-group, and let C be a cyclic group of order at least 7. Then some character of $H \times C$ fails to have an irreducible extension.*

Since H is not an M-group in Theorem 4.4, there exists a non-monomial character $\varphi \in \mathrm{Irr}(H)$. To prove the theorem, therefore, it suffices by Theorem 4.3 to show that $\varphi + 1_H$ is not a permutation character. Since H is solvable and φ is not monomial, we see that Theorem 4.4 is a consequence of the following.

(4.5) LEMMA. *Let $\varphi \in \mathrm{Irr}(H)$, where H is solvable, and suppose that $\varphi + 1_H$ is a permutation character. Then φ is monomial.*

Proof. Let $K = \ker(\varphi)$. Since $\varphi + 1_H$ is a permutation character, we can write $\varphi + 1_H = (1_J)^H$ for some subgroup J of G, and thus $K = \ker(\varphi) \subseteq \ker((1_J)^H)$, and hence $K \subseteq J$. Now viewing φ as a character of H/K, we have $\varphi + 1_{H/K} = (1_{J/K})^{H/K}$, so $\varphi + 1_{H/K}$ is a permutation character of H/K. Replacing H by H/K, we can assume that φ is faithful, and we see that the permutation character $\varphi + 1_H$ is also faithful.

Now J is the stabilizer of a point in the corresponding faithful permutation representation, and so $\mathrm{core}_H(J) = 1$. Since φ is irreducible, the permutation representation is doubly transitive, and hence it is primitive, and thus J is a maximal subgroup of H.

Let N be a minimal normal subgroup of H, and observe that $N \not\subseteq J$, so $NJ = H$. Also, N is abelian because H is solvable, so an irreducible constituent λ of φ_N is linear.

Let T be the stabilizer of λ in H, and let $\eta \in \mathrm{Irr}(T)$ be the Clifford correspondent of φ with respect to λ. Then $\varphi = \eta^H$ and $\eta_N = e\lambda$, where $e = [\varphi_N, \lambda]$. Now $\eta(1) = e\lambda(1) = e$, and $\eta^H = \varphi$, so to prove that φ is monomial, it suffices to show that $e = 1$.

Since $H = NJ$, we have

$$e = [\varphi_N, \lambda] \le [(\varphi + 1_H)_N, \lambda] = [((1_J)^H)_N, \lambda] = [(1_{N\cap J})^N, \lambda] = [1_{N\cap J}, \lambda_{N\cap J}] \le 1\,,$$

where the final inequality holds since λ is linear, and thus $\lambda_{N\cap J}$ is irreducible. It follows that $e = 1$, as wanted. ∎

5. Solvable extensions. In this section we consider the question of when a character ξ of a group H can have an irreducible extension to a solvable group containing H. The following is Theorem E.

(5.1) THEOREM. *Let $\alpha, \lambda \in \mathrm{Irr}(H)$, where λ is linear, and write $\xi = \alpha + \lambda$. Then ξ has no irreducible extension to a solvable group unless $\xi(1)$ is a prime power.*

Proof. Suppose that $\chi \in \mathrm{Irr}(G)$, where G is solvable. It suffices to show that if there exists a subgroup $H \subseteq G$ such that χ_H has the form $\alpha + \lambda$, where $\alpha \in \mathrm{Irr}(H)$ and λ is linear, then, $\chi(1)$ must be a power of a prime. We can assume that χ is faithful, and we proceed by induction on $|G : H|$.

Suppose $H < X < G$ and let $\psi \in \mathrm{Irr}(X)$ lie under χ and over α. Then either $\psi_H = \alpha$ or $\psi_H = \alpha + \lambda$. If $\psi_H = \alpha$, then $\chi_X = \psi + \nu$ for some linear character ν of X, and thus since $|G : X| < |G : H|$, the inductive hypothesis applied with X in place of H yields that $\chi(1)$ is a prime power, as required. Otherwise, $\psi_H = \alpha + \lambda$, and since $|X : H| < |G : H|$, the inductive hypothesis with X in place of G implies that $\psi(1)$ is a prime power, and thus $\chi(1)$ is a prime power, and we are done in this case too. We may assume, therefore, that H is a maximal subgroup of G.

Now let $L = \mathrm{core}_G(H)$, and let K/L be a chief factor of G. Then $KH = G$ and K/L is abelian, and it follows that $K \cap H = L$. Write $\varphi = \lambda_L$ and note that because λ is linear, φ must be linear and invariant in H. If H is the full stabilizer of φ in G, then $\lambda^G = \chi$ by the Clifford correspondence, and thus $\chi(1) = |G : H| = |K : L|$, which is a prime power. We can thus assume that φ is invariant in G, and thus either φ extends to K, or else φ is fully ramified in K. (See, for example, Corollary 7.4 of **[3]**.)

We can assume that $K < G$ since otherwise, $H = L \triangleleft G$ and thus the irreducible constituents α and λ of χ_H have equal degree, and thus $\chi(1) = 2$, and there is nothing further to prove. We can thus choose a chief factor N/K of G, and we argue that χ_N is irreducible.

Write $M = N \cap H$. Then $M > L$, and writing $\mu = \lambda_M$, we see that μ is invariant in H. Let $\beta \in \mathrm{Irr}(N)$ lie under χ and over μ. Since β lies over μ and μ is invariant in H, we see that every H-conjugate of β also lies over μ. Also, $NH = G$, and thus every G-conjugate of β lies over μ. Equivalently, every irreducible constituent of χ_N lies over μ.

If χ_N is not irreducible, it follows that the multiplicity of μ as a constituent of χ_M is at least 2, and thus μ lies under α as well as under λ. Since μ is invariant in H, it follows that α_M is a multiple of μ, and thus χ_M is a multiple of μ. Then M is central in G because χ is faithful, and in particular, $M \triangleleft G$. This is a contradiction, however, because $H \supseteq M > L = \mathrm{core}_G(H)$. We deduce that χ_N is irreducible, as claimed, and we have $\chi_N = \beta$

If φ extends to K, then all characters of K lying over φ are linear, and thus all irreducible constituents of χ_K are linear, and hence K is abelian. It follows that β has degree dividing $|N : K|$, which is a prime power, and since $\chi(1) = \beta(1)$, we are done in this case.

We can now assume that φ is fully ramified with respect to K/L, so we can write $|K : L| = e^2$ for some integer e, and we see that e is a prime power. Also, we can assume that $|K : L|$ and $|N : K|$ are relatively prime because otherwise, N/L is a p-group for some prime p, and thus $\chi(1) = \beta(1)$ is a power of p.

Now M/L is the unique (up to conjugacy) complement for K/L relative to L in N/L, and hence it follows by Theorem 8.4 of **[3]** that there is a bijection from $\mathrm{Irr}(N|\varphi)$ onto $\mathrm{Irr}(M|\varphi)$, where if $\sigma \mapsto \tau$, then $\sigma(1) = e\tau(1)$. Now λ_M is a linear extension of φ to M, and since M/L is abelian, it follows that all members of $\mathrm{Irr}(M|\varphi)$ are linear, and thus the degree of each member of $\mathrm{Irr}(N|\varphi)$ is e. In particular, $\chi(1) = \beta(1) = e$, and this is a prime power. ∎

Finally, we establish Theorem F, which we restate here.

(5.2) THEOREM. *If H is a abelian, then every character of H has an irreducible extension to a solvable group.*

In fact, we prove the following somewhat more general result.

(5.3) THEOREM. *Let ξ be a character of a solvable group H, and suppose that every irreducible constituent of ξ is linear. Then ξ has an irreducible extension to a solvable group.*

Proof. We can certainly assume that $\xi(1) > 1$, and we proceed by induction on $\xi(1)$. First, if $\xi = m\eta$, for some integer $m > 1$ and character η of H, then by the inductive hypothesis, there exist a group $K \supseteq H$ and a character $\psi \in \mathrm{Irr}(K)$ such that $\psi_H = \eta$. Now let U be a solvable group having some irreducible character φ with $\varphi(1) = m$. (For example, we can take U to be the wreath product $A \wr C$, where A is a nontrivial abelian group and C is cyclic of order m.) Now let $G = K \times U$ and $\chi = \psi \times \eta$, so $\chi \in \mathrm{Irr}(G)$. Then $\chi_H = \varphi(1)\eta = m\eta = \xi$, and we are done in this case.

We can now assume that ξ is not a proper multiple of any character, and we write $\xi = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, where the λ_i are linear characters of H. Now let $G = H \wr C$, where C is cyclic of order n, and observe that G is solvable. Let B be the base group of the wreath product G, so we can write $B = H_1 \times H_2 \times \cdots \times H_n$, where the H_i are isomorphic to H, and we view (by abuse of notation) λ_i as a character of H_i. Now let $\lambda = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$, so λ is a linear character of B.

We argue next that the stabilizer of λ in C is trivial. To see this, let c be a generator of C, and assume that the notation has been chosen so that $(H_i)^c = H_{i+1}$ and $(\lambda_i)^c = \lambda_{i+1}$, where the subscripts are to be read modulo n. Now if c^r stabilizes λ, then $\lambda_i = \lambda_{i+r}$ for all i. Now when the λ_i are viewed as characters of H, it follows that

$$\lambda_1 + \cdots + \lambda_r = \lambda_{1+r} + \cdots + \lambda_{2r} = \lambda_{1+2r} + \cdots + \lambda_{3r} = \cdots ,$$

and thus

$$\xi = (n/r)(\lambda_1 + \cdots + \lambda_r) ,$$

and it follows that $n/r = 1$ so $c^r = c^n = 1$, and thus the stabilizer of λ in C is trivial as claimed. It follows that B is the stabilizer of λ in G, and hence λ^G is irreducible, and we write $\chi = \lambda^G$.

Now $\chi_B = \lambda + \lambda^c + \cdots + \lambda^{c^{n-1}}$, so if we identify H_1 with H, it is not hard to see that $\chi_H = \lambda_1 + \cdots + \lambda_n = \xi$, as required. ∎

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