ALTERNATING SUMS OVER π-SUBGROUPS

GABRIEL NAVARRO AND BENJAMIN SAMBALE

Abstract. Dade's conjecture predicts that if p is a prime, then the number of irreducible characters of a finite group of a given p-defect is determined by local subgroups. In this paper we replace p by a set of primes π and prove a π-version of Dade's conjecture for π-separable groups. This extends the (known) p-solvable case of the original conjecture and relates to a π-version of Alperin's weight conjecture previously established by the authors.

1. Introduction

One of the most general local-global counting conjecture for irreducible complex characters of finite groups is due to E. C. Dade [D]. For a finite group G, a prime p and an integer $d > 0$, the conjecture asserts that the number of irreducible characters of G of p-defect d can be computed by an alternating sum over chains of p-subgroups. (In this paper, we only deal with the group-wise ordinary conjecture; see [N, Conjecture 9.25].) Dade [D] already showed that his conjecture implies Alperin's weight conjecture. The first author has proved that McKay's conjecture is also a consequence of Dade's conjecture (see [N, Theorem 9.27]). Dade's conjecture is known to be true for p-solvable groups by work of G. R. Robinson [R] (see also Turull [T17]), and a reduction of it to simple groups has been recently conducted by B. Späth [Sp].

In previous work by Isaacs–Navarro [IN] and the present authors [NS], we have replaced p by a set of primes π in order to prove variants of Alperin's weight conjecture for π-separable groups. In this paper, we are interested in a π-version of Dade's conjecture and possible applications.

Let $\mathcal{C}(G)$ be the set of chains of π-subgroups of G, and let G_C be the stabilizer of a chain C in G. For an integer $d \geq 1$, we let $k_d(G)$ to be the number of irreducible characters $\chi \in \mathrm{Irr}(G)$ such that $|G|_\pi = d\chi(1)_\pi$, where $n_\pi = \prod_{p\in\pi} n_p$, and n_p is the largest power of p dividing the positive integer n. (Notice that this deviates slightly from the usual notation for $\pi = \{p\}$.)

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Our main result is a natural generalization of Dade's conjecture for p-solvable groups:[1]

THEOREM A. *Let G be a π-separable group, and let $d > 1$. Then*

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}|G_C|k_d(G_C) = 0.$$

As usual, the factor $|G_C|$ can be eliminated by summing over G-conjugacy classes of chains. Unlike the original conjecture in the case $\pi = \{p\}$, we cannot restrict ourselves to so-called *normal* chains in Theorem A (see [N, Theorem 9.16]). In fact, $G = \mathsf{S}_3$ with $\pi = \{2, 3\}$ is already a counterexample. (This is related to the fact that π-subgroups are not in general nilpotent!) For this reason, the known proofs of the p-solvable case cited above cannot be carried over to π. We will obtain Theorem A as a special case of a more general projective statement with respect to normal subgroups.

To state applications, we denote the number of conjugacy classes of G by $k(G)$ and the number of conjugacy classes of π'-elements by $l(G)$. Recall that $\chi \in \mathrm{Irr}(G)$ has π-defect zero if $\chi(1)_p = |G|_p$ for all $p \in \pi$. The number of those characters is $k_1(G)$, using the notation above.

COROLLARY B. *Let G be a π-separable group. Then*

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}\frac{|G_C|}{|G|}k(G_C) = \sum_{C\in\mathcal{C}(G)} (-1)^{|C|}\frac{|G_C|}{|G|}l(G_C)$$

is the number of π-defect zero characters of G.

As another application we generalize a theorem of Webb [We] (see Theorem 2.5 below).

Unlike the case where $\pi = \{p\}$, Alperin's weight conjecture cannot be deduced from Corollary B. As a matter of fact, the most natural formulation of Alperin's conjecture does not even hold in some π-separable groups as indicated in [NS] (consider $G = A_5$ and $\pi = \{2, 3, 5\}$). We now believe that the general situation can be explained by introducing *negative* weights. More precisely, we checked for many π-separable groups G that

$$l(G) = \sum_{P\in\Pi(G)} \epsilon(P)k_1(\mathbf{N}_G(P)/P)$$

where $\Pi(G)$ is a set of representatives for the conjugacy classes of π-subgroups of G, and $\epsilon(P)$ is an integer only depending on the isomorphism type of P (and not on G or π). It is easy to see that the function ϵ is uniquely determined and can be computed recursively (if it exists). Among solvable groups H we have $\epsilon(H) = 1$ if H is nilpotent and $\epsilon(H) = 0$ otherwise. This is in complete accordance with [NS].

[1]Theorem A was proposed as a conjecture in the second author's Oberwolfach talk in 2019.

On the other hand, $\epsilon(A_6) = -2$. Understanding and interpreting these mysterious coefficients remains a challenge.

2. Proofs

We fix a set of primes π for the rest of the paper. If G is a finite group, we consider chains C of π-subgroups in G of the form $1 = P_0 < P_1 < \ldots < P_n$ where $n = 0$ is allowed (the trivial chain). Let $|C| = n$ and let

$$G_C = \mathbf{N}_G(P_0) \cap \ldots \cap \mathbf{N}_G(P_n)$$

be the stabilizer of C in G. The set of all such chains of G is denoted by $\mathcal{C}(G)$.

For a normal subgroup N of G and $\theta \in \mathrm{Irr}(N)$, let $k_d(G|\theta)$ be the number of irreducible characters χ of G lying over θ with $|G|_\pi = d\chi(1)_\pi$. We denote by G_θ the stabilizer of θ in G. By the Clifford correspondence, notice that

$$k_d(G|\theta) = k_d(G_\theta|\theta).$$

We start with the following.

Lemma 2.1. *Let G be a finite group, and let f be a real-valued function on the set of subgroups of G. If $\mathbf{O}_\pi(G) > 1$, then*

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| f(G_C) = 0.$$

Proof. Let $C : 1 = P_0 < \ldots < P_n$ be a chain in $\mathcal{C}(G)$. If $N = \mathbf{O}_\pi(G) \nsubseteq P_n$, we obtain $C^* \in \mathcal{C}(G)$ from C by adding NP_n at the end which is still a π-group. Otherwise let $N \subseteq P_k$ and $N \nsubseteq P_{k-1}$. If $P_{k-1}N = P_k$, then we delete P_k, otherwise we add $P_{k-1}N$ between P_{k-1} and P_k. It is easy to see that in all cases $|C^*| = |C| \pm 1$, $(C^*)^* = C$ and $G_C = G_{C^*}$. Hence, the map $C \mapsto C^*$ is a bijection on $\mathcal{C}(G)$ such that

$$\begin{aligned}\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| f(G_C) &= \sum_{C \in \mathcal{C}(G)} (-1)^{|C^*|} |G_{C^*}| f(G_{C^*}) \\ &= -\sum_{C \in \mathcal{C}(G)} (-1)^{|C|} |G_C| f(G_C) = 0.\end{aligned}$$ □

It is obvious that G acts by conjugation on $\mathcal{C}(G)$. The set of G-orbits is denoted by $\mathcal{C}(G)/G$ in the following. If $K \lhd G$, notice that G also acts on $\mathcal{C}(G/K)$.

Lemma 2.2. *Let G be a finite group with a normal π'-subgroup K. Let $\overline{H} := HK/K$ for $H \leq G$.*

(a) The map $\mathcal{C}(G) \mapsto \mathcal{C}(\overline{G})$ given by

$$C : P_0 < \ldots < P_n \mapsto \overline{C} : \overline{P_0} < \ldots < \overline{P_n}$$

induces a bijection $\mathcal{C}(G)/G \to \mathcal{C}(\overline{G})/\overline{G}$.

(b) For $\overline{C} \in \mathcal{C}(\overline{G})$, we have that $\overline{G}_{\overline{C}} = G_{\overline{C}}/K = G_C K/K$.

(c) Let f be a real-valued function on the set of subgroups of G such that $f(H) = f(H^g)$ for all $H \leq G$ and $g \in G$. Then

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}|G_C|f(G_CK) = \sum_{\overline{C}\in\mathcal{C}(\overline{G})} (-1)^{|\overline{C}|}|G_{\overline{C}}|f(G_{\overline{C}}).$$

Proof. First, we notice that the map $\mathcal{C}(G) \to \mathcal{C}(\overline{G})$ given by $C \mapsto \overline{C}$ is surjective. Indeed, suppose that $\overline{C} : 1 = U_0/K < \ldots < U_n/K$ is a chain of π-subgroups of G/K. By the Schur–Zassenhaus theorem, we have that $U_n = KP_n$ for some π-subgroup P_n of G. Then $U_i = K(U_i \cap P_n)$, and therefore the chain $1 = U_0 \cap P_n < \ldots < P_n$ maps to $\overline{C}$.

If chains $C : P_0 < \ldots < P_n$ and $D : Q_0 < \ldots < Q_n$ are conjugate in G, then $\overline{C}$ and $\overline{D}$ are obviously conjugate in $\overline{G}$. Suppose conversely that $\overline{C}$ and $\overline{D}$ are $\overline{G}$-conjugate. Without loss of generality, we may assume that $P_iK = Q_iK$ for $i = 0, \ldots, n$. Again by the Schur–Zassenhaus theorem (this time relying on the Feit–Thompson theorem), P_n is conjugate to Q_n by some $x \in K$. We still have $P_i^xK = Q_iK$ for $i = 0, \ldots, n$. Since $P_i^x, Q_i \leq Q_n$ it follows that $P_i^x = Q_i$ for $i = 0, \ldots, n$. Hence, C and D are G-conjugate. This proves (a).

Suppose that $P_1 < P_2$ are π-subgroups of G. We claim that

$$\mathbf{N}_G(P_1)K \cap \mathbf{N}_G(P_2)K = (\mathbf{N}_G(P_1) \cap \mathbf{N}_G(P_2))K.$$

If $x \in \mathbf{N}_G(P_1)K \cap \mathbf{N}_G(P_2)K$, then $P_2^x = P_2^k$ for some $k \in K$. Therefore $xk^{-1} \in \mathbf{N}_G(P_2) \cap \mathbf{N}_G(P_1)K$. Since $P_1K \cap P_2 = P_1$, we have that $xk^{-1} \in \mathbf{N}_G(P_1)$, and therefore $x \in (\mathbf{N}_G(P_1) \cap \mathbf{N}_G(P_2))K$. This proves the claim.

Suppose now that $C : P_0 < \ldots < P_n$ is a chain of π-subgroups of G. By the Frattini argument, $\overline{\mathbf{N}_G(P_i)} = \mathbf{N}_{\overline{G}}(\overline{P_i})$ and therefore $\overline{G_C} = \overline{G}_{\overline{C}}$, using the last paragraph. Regarding the action of G on $\mathcal{C}(\overline{G})$ we also have $G_{\overline{C}}/K = \overline{G}_{\overline{C}}$.

Finally, we prove (c). The G-orbit of C has size $|G : G_C|$, while the $\overline{G}$-orbit of $\overline{C}$ has size $|\overline{G} : \overline{G}_{\overline{C}}| = |G : G_{\overline{C}}|$. Let $C_1, \ldots, C_k$ be representatives for $\mathcal{C}(G)/G$, so that $\overline{C_1}, \ldots, \overline{C_k}$ are representatives for $\mathcal{C}(\overline{G})/\overline{G}$. Then

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}|G_C|f(G_CK) = |G|\sum_{i=1}^{k}(-1)^{|C_i|}f(G_{C_i}K) = \sum_{\overline{C}\in\mathcal{C}(\overline{G})} (-1)^{\overline{C}}|G_{\overline{C}}|f(G_{\overline{C}}). \quad \square$$

The deep part of our results comes from the "above Glauberman–Isaacs correspondence" theory. If A is a solvable finite group, acting coprimely on G, recall that Glauberman discovered a natural bijection * from $\mathrm{Irr}_A(G)$, the set of A-invariant irreducible characters of G, and $\mathrm{Irr}(\mathbf{C}_G(A))$, the irreducible characters of the fixed-point subgroup. The case where A is a p-group is fundamental in the local/global counting conjectures. If A is not solvable, an important case in this paper, then G has odd order by the Feit-Thompson theorem. In this case, Isaacs [I] proved that there is also a natural bijection $\mathrm{Irr}_A(G) \to \mathrm{Irr}(\mathbf{C}_G(A))$. T. R. Wolf [Wo] proved that both correspondences agree in the intersection of their hypotheses.

Theorem 2.3. *Let G be a finite group with a normal π'-subgroup K. Let $C \in \mathcal{C}(G)$ with last subgroup $P_C = P_{|C|}$. Let $\tau \in \mathrm{Irr}(K)$ be P_C-invariant, and let $\tau^* \in \mathrm{Irr}(\mathbf{C}_K(P_C))$ be its Glauberman–Isaacs correspondent. Then*

$$k_d(G_CK|\tau) = k_d(G_C|\tau^*)$$

for every integer d.

Proof. Let $U = K(P_CG_C)$. Notice that $G_C \cap K = \mathbf{C}_K(P_C)$. Also, $KP_C \lhd U$. Thus $U = K\mathbf{N}_U(P_C)$, by the Frattini argument and the Schur–Zassenhaus theorem. Also,

$$\mathbf{N}_U(P_C) = \mathbf{N}_G(P_C) \cap (P_CG_C)K = (P_CG_C)\mathbf{N}_K(P_C) = P_CG_C\mathbf{C}_K(P_C).$$

Since G_C normalizes P_C, we have that G_C commutes with the P_C-Glauberman–Isaacs correspondence. In particular,

$$(G_C)_{\tau^*} = (G_C)_\tau\,.$$

Hence, by using the Clifford correspondence, we may assume that τ is G_C-invariant (and therefore U-invariant) and that τ^* is G_C-invariant too.

Now, we claim that the character triples (U, K, τ) and $(\mathbf{N}_U(P_C), \mathbf{C}_K(P_C), \tau^*)$ are isomorphic. If P_C is solvable, this is a well-known fact which follows from the Dade–Puig theory. (A comprehensive proof is given in [T08].) If P_C is not solvable, then $|K|$ is odd, by the Feit–Thompson theorem. Then the claim follows from the theory developed by Isaacs in [I]. (A proof is given in the last paragraphs of [L].)

Since the character triples (U, K, τ) and $(\mathbf{N}_U(P_C), \mathbf{C}_K(P_C), \tau^*)$ are isomorphic, it follows from the definition that the sub-triples (G_CK, K, τ) and $(G_C, \mathbf{C}_K(P_C), \tau^*)$ are isomorphic too. This yields a bijection $\mathrm{Irr}(G_CK|\tau) \to \mathrm{Irr}(G_C|\tau^*)$, $\chi \mapsto \chi^*$ such that $\chi(1)/\tau(1) = \chi^*(1)/\tau^*(1)$ (see [N, p. 87]). In particular, $k_d(G_CK|\tau) = k_d(G_C|\tau^*)$ (if $d_\pi \neq d$, both numbers are 0). □

Theorem A is the special case $N = 1$ of the following projective version.

Theorem 2.4. *Let G be a π-separable group with a normal π'-subgroup N. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant and $d > 1$. Then*

$$\sum_{C \in \mathcal{C}(G)} (-1)^{|C|}|G_C|k_d(G_CN|\theta) = 0.$$

Proof. We may assume that $d_\pi = d$. We argue by induction on $|G : N|$. Let $\overline{G} = G/N$. By Lemma 2.2, we may sum over $\overline{C} \in \mathcal{C}(\overline{G})$ by replacing G_C with $G_{\overline{C}}$. Recall that a character triple isomorphism $(G, N, \theta) \to (G^*, N^*, \theta^*)$ induces an isomorphism $\overline{G} \cong G^*/N^*$ and a bijection $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$, $\chi \mapsto \chi^*$ such that $\chi(1)/\theta(1) = \chi^*(1)/\theta^*(1)$. Thus, $k_d(G_{\overline{C}}|\theta) = k_d(G^*_{\overline{C}^*}|\theta^*)$ and the numbers $|G_{\overline{C}}|$, $|G^*_{\overline{C}^*}|$ differ only by a factor independent of C. This allows us to replace N by N^*. Using

[N, Corollary 5.9], we assume that N is a central π'-subgroup in the following. Now using Lemma 2.2 in the opposite direction, we sum over $C \in \mathcal{C}(G)$ again and note that $N \subseteq G_C$. Thus, it suffices to show that

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}|G_C|k_d(G_C|\theta) = 0.$$

By Lemma 2.1, we may assume that $\mathbf{O}_\pi(G) = 1$. Let $K = \mathbf{O}_{\pi'}(G)$. If $K = N$, then $N = G$ by the Hall–Higman 1.2.3 lemma. In this case, the theorem is correct because $d > 1$. So we may assume that $K > N$. Let P_C be the last member of $C \in \mathcal{C}(G)$. Observe that $G_C \cap K = \mathbf{C}_K(P_C)$. Each $\psi \in \mathrm{Irr}(G_C|\theta)$ lies over some $\mu \in \mathrm{Irr}(\mathbf{C}_K(P_C)|\theta)$. But ψ lies also over μ^g for every $g \in G_C$. Therefore,

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}|G_C||k_d(G_C|\theta) = \sum_{C\in\mathcal{C}(G)} (-1)^{|C|}|G_C|\Big(\sum_{\mu\in\mathrm{Irr}(\mathbf{C}_K(P_C)|\theta)} \frac{k_d(G_C|\mu)}{|G_C : G_{C,\mu}|}\Big)$$

where $G_{C,\mu} = G_C \cap G_\mu$. According to Theorem 2.3, we replace $\mathrm{Irr}(\mathbf{C}_K(P_C)|\theta)$ by $\mathrm{Irr}_{P_C}(K|\theta)$ and $k_d(G_C|\mu)$ by $k_d(G_CK|\mu)$. By the Clifford correspondence, $k_d(G_CK|\mu) = k_d(G_{C,\mu}K|\mu)$. Moreover, $\mu \in \mathrm{Irr}_{P_C}(K|\theta)$ implies $P_C \le G_\mu$. Thus, for a fixed μ we only need to consider chains in G_μ. Hence,

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}|G_C||k_d(G_C|\theta) = \sum_{\mu\in\mathrm{Irr}(K|\theta)} \Big(\sum_{C\in\mathcal{C}(G_\mu)} (-1)^{|C|}|G_{C,\mu}|k_d(G_{C,\mu}K|\mu)\Big).$$

Since $|G_\mu : K| < |G : N|$, the inner sum vanishes for every μ by induction. Hence, we are done. □

Finally, we come to our second result.

Proof of Corollary B. Let $C : P_0 < \ldots < P_n$ in $\mathcal{C}(G)$ such that $n > 0$. Then $P_1 \trianglelefteq G_C$. Let $\chi \in \mathrm{Irr}(G_C)$ and $\theta \in \mathrm{Irr}(P_1)$ under χ. By Clifford theory, $\chi(1)/\theta(1)$ divides $|G_C/P_1|$ (see [N, Theorem 5.12]). Since $\theta(1) < |P_1|$, it follows that $\chi(1)_\pi < |G_C|_\pi$ and $k_1(G_C) = 0$. Summing over $d \ge 1$ in Theorem A yields

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}\frac{|G_C|}{|G|}k(G_C) = k_1(G).$$

The second equality follows from a straight-forward generalization of the Knörr–Robinson argument. In fact, the proofs of [N, 9.18–9.23] go through word by word (replacing p by π, of course). □

Given the proof above, we take the opportunity to point out that a theorem of Webb [We] (see also [N, Corollary 9.20]) remains true in the π-setting:

Theorem 2.5. *Let G be an arbitrary finite group, and let π be a set of primes. Then the generalized character*

$$\sum_{C\in\mathcal{C}(G)} (-1)^{|C|}|G_C|(1_{G_C})^G$$

vanishes on all elements of G whose order is divisible by a prime in π.

It is interesting to speculate on variations of Theorem A, that is projective versions of Dade's conjecture, that might be even true for arbitrary normal subgroups of any finite group G, whenever $\pi = \{p\}$. We have not attempted a block version of Theorem A. Although π-block theory is well-developed in π-separable groups (see [Sl], for instance), Brauer's block induction does not behave well if Hall π-subgroups are not nilpotent.

Computations with chains are almost impossible to do by hand. The results of this paper would not have been discovered without the help of [GAP].

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Department of Mathematics, Universitat de València, 46100 Burjassot, València, Spain
Email address: gabriel@uv.es

Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
Email address: sambale@math.uni-hannover.de