Blocks with defect group $Q_{2^n} \times C_{2^m}$ and $SD_{2^n} \times C_{2^m}$

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Abstract

We determine the numerical invariants of blocks with defect group $Q_{2^n} \times C_{2^m}$ and $SD_{2^n} \times C_{2^m}$, where Q_{2^n} denotes a quaternion group of order 2^n, C_{2^m} denotes a cyclic group of order 2^m, and SD_{2^n} denotes a semidihedral group of order 2^n. This generalizes Olsson's results for $m = 0$. As a consequence, we prove Brauer's $k(B)$-Conjecture, Olsson's Conjecture, Brauer's Height-Zero Conjecture, the Alperin-McKay Conjecture, Alperin's Weight Conjecture and Robinson's Ordinary Weight Conjecture for these blocks. Moreover, we show that the gluing problem has a unique solution in this case. This paper follows (and uses) (Sambale, 2012, [21, 18]).

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1 Introduction

Let R be a discrete complete valuation ring with quotient field K of characteristic 0. Moreover, let (π) be the maximal ideal of R and $F := R/(\pi)$. We assume that F is algebraically closed of characteristic 2. We fix a finite group G, and assume that K contains all $|G|$-th roots of unity. Let B be a 2-block of RG with defect group D. We denote the number of irreducible ordinary characters of B by $k(B)$. These characters split in $k_i(B)$ characters of height $i \in \mathbb{N}_0$. Here the *height* of a character χ in B is the largest integer $h(\chi) \geq 0$ such that $2^{h(\chi)}|G : D|_2 \mid \chi(1)$, where $|G : D|_2$ denotes the highest 2-power dividing $|G : D|$. Finally, let $l(B)$ be the number of irreducible Brauer characters of B.

Brauer and Olsson had determined the invariants $k(B)$, $k_i(B)$ and $l(B)$ in the case where D has maximal class, i. e. D is a dihedral group, a semidihedral group, or a quaternion group (see [1, 14]). We have seen in [21] that it is also possible to replace the dihedral group D_{2^n} by a direct product $D_{2^n} \times C_{2^m}$ with a cyclic group. The aim of this paper is to do the same with the other 2-groups of maximal class. One motivation for this comes from Theorem 2 in [19]. In particular, it was observed that defect groups D with a central, cyclic subgroup $Z \leq \mathrm{Z}(D)$ such that D/Z is metacyclic are easy to handle. More precisely in our case D/Z has maximal class, and the Cartan invariants of certain major subsections are known by Erdmann's work [5].

Despite that the proofs for $D \cong Q_{2^n} \times C_{2^m}$ and for $D \cong SD_{2^n} \times C_{2^m}$ are very similar, we give both separately for the convenience of the reader. For the induction step we have to consider blocks with defect groups $D_{2^n} \times C_{2^m}$ and $Q_{2^n} * C_{2^m} \cong SD_{2^n} * C_{2^m} \cong D_{2^n} * C_{2^m}$. Hence, we will need the results from [21, 18]. In fact most of the present paper works the same way as in [18]. However, in the proof of the main theorem we have to consider more cases for the generalized decomposition numbers.

2 The quaternion case

We write

$$D := \langle x, y, z \mid x^{2^{n-1}} = z^{2^m} = [x, z] = [y, z] = 1,\ y^2 = x^{2^{n-2}},\ yxy^{-1} = x^{-1}\rangle = \langle x, y\rangle \times \langle z\rangle \cong Q_{2^n} \times C_{2^m},$$

where $n \geq 3$ and $m \geq 0$. We allow $m = 0$, since the results are completely consistent in this case.

2.1 Subsections

The first lemma shows that the situation splits naturally in two cases according to $n = 3$ or $n \geq 4$.

Lemma 2.1. *The automorphism group* $\mathrm{Aut}(D)$ *is a* 2*-group if and only if* $n \geq 4$.

Proof. Since $\mathrm{Aut}(Q_8) \cong S_4$, the "only if"-part is easy to see. Now let $n \geq 4$. Then the subgroups $\Phi(D) < \Phi(D)\,\mathrm{Z}(D) < \langle x, z\rangle < D$ are characteristic in D. By Theorem 5.3.2 in [7] every automorphism of $\mathrm{Aut}(D)$ of odd order acts trivially on $D/\Phi(D)$. The claim follows from Theorem 5.1.4 in [7]. □

It follows that the inertial index $e(B)$ of B equals 1 for $n \geq 4$. In case $n = 3$ there are two possibilities $e(B) \in \{1, 3\}$, since $\Phi(D)\,\mathrm{Z}(D)$ is still characteristic in D. Now we investigate the fusion system $\mathcal{F}$ of the B-subpairs. For this we use the notation of [15, 11], and we assume that the reader is familiar with these articles. Let b_D be a Brauer correspondent of B in $RD\,\mathrm{C}_G(D)$. Then for every subgroup $Q \leq D$ there is a unique block b_Q of $RQ\,\mathrm{C}_G(Q)$ such that $(Q, b_Q) \leq (D, b_D)$. We denote the inertial group of b_Q in $\mathrm{N}_G(Q)$ by $\mathrm{N}_G(Q, b_Q)$.

Lemma 2.2. *Let* $Q_1 := \langle x^{2^{n-3}}, y, z\rangle \cong Q_8 \times C_{2^m}$ *and* $Q_2 := \langle x^{2^{n-3}}, xy, z\rangle \cong Q_8 \times C_{2^m}$. *Then* Q_1 *and* Q_2 *are the only candidates for proper* $\mathcal{F}$*-centric,* $\mathcal{F}$*-radical subgroups up to conjugation. In particular the fusion of subpairs is controlled by* $\mathrm{N}_G(Q_1, b_{Q_1}) \cup \mathrm{N}_G(Q_2, b_{Q_2}) \cup D$. *Moreover, one of the following cases occurs:*

(aa) $n = e(B) = 3$ *or* ($n \geq 4$ *and* $\mathrm{Out}_{\mathcal{F}}(Q_1) \cong \mathrm{Out}_{\mathcal{F}}(Q_2) \cong S_3$).

(ab) $n \geq 4$, $\mathrm{N}_G(Q_1, b_{Q_1}) = \mathrm{N}_D(Q_1)\,\mathrm{C}_G(Q_1)$ *and* $\mathrm{Out}_{\mathcal{F}}(Q_2) \cong S_3$.

(ba) $n \geq 4$, $\mathrm{Out}_{\mathcal{F}}(Q_1) \cong S_3$ *and* $\mathrm{N}_G(Q_2, b_{Q_2}) = \mathrm{N}_D(Q_2)\,\mathrm{C}_G(Q_2)$.

(bb) $\mathrm{N}_G(Q_1, b_{Q_1}) = \mathrm{N}_D(Q_1)\,\mathrm{C}_G(Q_1)$ *and* $\mathrm{N}_G(Q_2, b_{Q_2}) = \mathrm{N}_D(Q_2)\,\mathrm{C}_G(Q_2)$.

In case (bb) the block B *is nilpotent.*

Proof. Let $Q < D$ be $\mathcal{F}$-centric and $\mathcal{F}$-radical. Then $z \in \mathrm{Z}(D) \subseteq \mathrm{C}_D(Q) \subseteq Q$ and $Q = (Q \cap \langle x, y\rangle) \times \langle z\rangle$. Let us consider the case $Q = \langle x, z\rangle$. Then $m = n - 1$ (this is not important here). The group $D \subseteq \mathrm{N}_G(Q, b_Q)$ acts trivially on $\Omega(Q) \subseteq \mathrm{Z}(D)$, while a nontrivial automorphism of $\mathrm{Aut}(Q)$ of odd order acts nontrivially on $\Omega(Q)$ (see Theorem 5.2.4 in [7]). This contradicts $\mathrm{O}_2(\mathrm{Aut}_{\mathcal{F}}(Q)) = 1$. Moreover, by Lemma 5.4 in [11] we see that $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a 2-group (this will be needed later).

Now let $Q = \langle x^i y, z\rangle$ for some $i \in \mathbb{Z}$. Then we have $m = 2$, and the same argument as before leads to a contradiction.

Hence by Lemma 2.1, Q is isomorphic to $Q_8 \times C_{2^m}$, and contains an element of the form $x^i y$. After conjugation with a suitable power of x we may assume $Q \in \{Q_1, Q_2\}$. This shows the first claim.

The second claim follows from Alperin's fusion theorem. Here observe that in case $n = 3$ we have $Q_1 = Q_2 = D$.

Let $S \leq D$ be an arbitrary subgroup isomorphic to $Q_8 \times C_{2^m}$. If $z \notin S$, then for $\langle S, z\rangle = (\langle S, z\rangle \cap \langle x, y\rangle) \times \langle z\rangle$ we have $\langle S, z\rangle' = S' \cong C_2$. However, this is impossible, since $\langle S, z\rangle \cap \langle x, y\rangle$ has at least order 16. This contradiction shows $z \in S$. Thus, S is conjugate to $Q \in \{Q_1, Q_2\}$ under D. In particular Q is fully $\mathcal{F}$-normalized (see Definition 2.2 in [11]). Hence, $\mathrm{N}_D(Q)\,\mathrm{C}_G(Q)/Q\,\mathrm{C}_G(Q) \cong \mathrm{N}_D(Q)/Q \cong C_2$ is a Sylow 2-subgroup of $\mathrm{Out}_{\mathcal{F}}(Q) =$

$\mathrm{N}_G(Q, b_Q)/Q\,\mathrm{C}_G(Q)$ by Proposition 2.5 in [11]. Assume $\mathrm{N}_D(Q)\,\mathrm{C}_G(Q) < \mathrm{N}_G(Q, b_Q)$. Since $\mathrm{O}_2(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$ and $|\mathrm{Aut}(Q)| = 2^k \cdot 3$ for some $k \in \mathbb{N}$, we get $\mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$.

The last claim follows from Alperin's fusion theorem and $e(B) = 1$ (for $n \geq 4$). □

The naming of these cases is adopted from [14]. Since the cases (ab) and (ba) are symmetric, we ignore case (ba) for the rest of the paper. It is easy to see that Q_1 and Q_2 are not conjugate in D if $n \geq 4$. Hence, by Alperin's fusion theorem the subpairs (Q_1, b_{Q_1}) and (Q_2, b_{Q_2}) are not conjugate in G. It is also easy to see that Q_1 and Q_2 are always $\mathcal{F}$-centric.

Lemma 2.3. *Let $Q \in \{Q_1, Q_2\}$ such that $\mathrm{N}_G(Q, b_Q)/Q\,\mathrm{C}_G(Q) \cong S_3$. Then*

$$\mathrm{C}_Q(\mathrm{N}_G(Q, b_Q)) = \mathrm{Z}(Q) = \langle x^{2^{n-2}}, z \rangle.$$

Proof. Since $Q \subseteq \mathrm{N}_D(Q, b_Q)$, we have $\mathrm{C}_Q(\mathrm{N}_G(Q, b_Q)) \subseteq \mathrm{C}_Q(Q) = \mathrm{Z}(Q)$. On the other hand $\mathrm{N}_D(Q)$ acts trivially on $\mathrm{Z}(Q) = \mathrm{Z}(D)$. Hence, it suffices to determine the fixed points of an automorphism $\alpha \in \mathrm{Aut}(Q)$ of order 3 in $\mathrm{Z}(Q)$. Since α acts trivially on $Q' \cong C_2$ and on $\mathrm{Z}(Q)/Q' \cong C_{2^m}$, the claim follows from Theorem 5.3.2 in [7]. □

We recall a lemma from [18].

Lemma 2.4. *Let $\mathcal{R}$ be a set of representatives for the $\mathcal{F}$-conjugacy classes of elements of D such that $\langle \alpha \rangle$ is fully $\mathcal{F}$-normalized for $\alpha \in \mathcal{R}$ ($\mathcal{R}$ always exists). Then*

$$\big\{(\alpha, b_\alpha) : \alpha \in \mathcal{R}\big\}$$

is a set of representatives for the G-conjugacy classes of B-subsections, where $b_\alpha := b_{\langle \alpha \rangle}$ has defect group $\mathrm{C}_D(\alpha)$.

Proof. See [18]. □

Lemma 2.5. *A set $\mathcal{R}$ as in Lemma 2.4 is given as follows:*

(i) $x^i z^j$ ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) in case (aa).

(ii) $x^i z^j$ and yz^j ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) in case (ab).

Proof. By Lemma 2.3 in any case the elements $x^i z^j$ ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) are pairwise non-conjugate in $\mathcal{F}$. If $n = 3$, the block B is controlled and every subgroup is fully $\mathcal{F}$-normalized. Thus, assume for the moment that $n \geq 4$. Then $\langle x, z \rangle \subseteq \mathrm{C}_G(x^i z^j)$ and $|D : \mathrm{N}_D(\langle x^i z^j \rangle)| \leq 2$. Suppose that $\langle x^i y z^j \rangle \trianglelefteq D$ for some $i, j \in \mathbb{Z}$. Then we have $x^{i+2} y z^j = x(x^i y z^j)x^{-1} \in \langle x^i y z^j \rangle$ and the contradiction $x^2 \in \langle x^i y z^j \rangle$. This shows that the subgroups $\langle x^i z^j \rangle$ are always fully $\mathcal{F}$-normalized.

Assume that case (aa) occurs. Then the elements of the form $x^{2i} y z^j$ ($i, j \in \mathbb{Z}$) are conjugate to elements of the form $x^{2i} z^j$ under $D \cup \mathrm{N}_G(Q_1, b_{Q_1})$. Similarly, the elements of the form $x^{2i+1} y z^j$ ($i, j \in \mathbb{Z}$) are conjugate to elements of the form $x^{2i} z^j$ under $D \cup \mathrm{N}_G(Q_2, b_{Q_2})$. The claim follows in this case.

In case (ab) the given elements are pairwise non-conjugate, since no conjugate of yz^j lies in Q_2. As in case (aa) the elements of the form $x^{2i} y z^j$ ($i, j \in \mathbb{Z}$) are conjugate to elements of the form yz^j under D and the elements of the form $x^{2i+1} y z^j$ ($i, j \in \mathbb{Z}$) are conjugate to elements of the form $x^{2i} z^j$ under $D \cup \mathrm{N}_G(Q_2, b_{Q_2})$. Finally, the subgroups $\langle yz^j \rangle$ are fully $\mathcal{F}$-normalized, since yz^j is not conjugate to an element in Q_2. □

2.2 The numbers $k(B)$, $k_i(B)$ and $l(B)$

Lemma 2.6. *Olsson's Conjecture $k_0(B) \le 2^{m+2} = |D : D'|$ is satisfied in all cases.*

Proof. See Lemma 3.2 in [21]. $\square$

Theorem 2.7.

(i) In case (aa) and $n = 3$ we have $k(B) = 2^m \cdot 7$, $k_0(B) = 2^{m+2}$, $k_1(B) = 2^m \cdot 3$ and $l(B) = 3$.

(ii) In case (aa) and $n \ge 4$ we have $k(B) = 2^m(2^{n-2} + 5)$, $k_0(B) = 2^{m+2}$, $k_1(B) = 2^m(2^{n-2} - 1)$, $k_{n-2}(B) = 2^{m+1}$ and $l(B) = 3$.

(iii) In case (ab) we have $k(B) = 2^m(2^{n-2} + 4)$, $k_0(B) = 2^{m+2}$, $k_1(B) = 2^m(2^{n-2} - 1)$, $k_{n-2}(B) = 2^m$ and $l(B) = 2$.

(iv) In case (bb) we have $k(B) = 2^m(2^{n-2} + 3)$, $k_0(B) = 2^{m+2}$, $k_1(B) = 2^m(2^{n-2} - 1)$ and $l(B) = 1$.

In particular Brauer's $k(B)$-Conjecture, Brauer's Height-Zero Conjecture and the Alperin-McKay Conjecture hold.

Proof. Assume first that case (bb) occurs. Then B is nilpotent and $k_i(B)$ is just the number $k_i(D)$ of irreducible characters of D of degree 2^i ($i \ge 0$) and $l(B) = 1$. Since C_{2^m} is abelian, we get $k_i(B) = 2^m k_i(Q_{2^n})$. The claim follows in this case.

Now assume that case (aa) or case (ab) occurs. We determine the numbers $l(b)$ for the subsections in Lemma 2.5 and apply Theorem 5.9.4 in [13]. Let us begin with the non-major subsections. Since $\mathrm{Aut}_{\mathcal{F}}(\langle x, z\rangle)$ is a 2-group, the block $b_{\langle x,z\rangle}$ with defect group $\langle x, z\rangle$ is nilpotent. Hence, we have $l(b_{x^i z^j}) = 1$ for all $i = 1, \ldots, 2^{n-2} - 1$ and $j = 0, 1, \ldots, 2^m - 1$. The blocks b_{yz^j} ($j = 0, 1, \ldots, 2^{m-1} - 1$) have $\mathrm{C}_D(yz^j) = \langle y, z\rangle \cong C_4 \times C_{2^m}$ as defect group. In case (ab), $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_D(\langle y, z\rangle)) = \mathrm{Aut}_{\mathcal{F}}(Q_1)$ is a 2-group. Thus, by Lemma 5.4 in [11] also $\mathrm{Aut}_{\mathcal{F}}(\langle y, z\rangle)$ is a 2-group. Hence, the blocks b_{yz^j} are also nilpotent, and it follows that $l(b_{yz^j}) = 1$ for $j = 0, 1, \ldots, 2^{m-1} - 1$.

Now let (u, b_u) be a major subsection. By Lemma 2.3 the cases for B and b_u coincide. As usual, the blocks b_u dominate blocks $\overline{b_u}$ of $R\,\mathrm{C}_G(u)/\langle u\rangle$ with defect group $D/\langle u\rangle$. In case $u = z^j$ for some $j \in \mathbb{Z}$ we have $D/\langle u\rangle \cong Q_{2^n} \times C_{2^m/|\langle z^j\rangle|}$. Of course the cases for b_u and $\overline{b_u}$ coincide, and by Theorem 5.8.11 in [13] we have $l(b_{z^j}) = l(\overline{b_{z^j}})$. Thus, we can apply induction on m. The beginning of this induction ($m = 0$) is satisfied by Olsson's results (see [14]).

In case $u = x^{2^{n-2}}$ we have $D/\langle u\rangle \cong D_{2^{n-1}} \times C_{2^m}$. Then we can apply the results of [21]. Observe again that the cases for b_u and $\overline{b_u}$ coincide.

Finally, if $u = x^{2^{n-2}} z^j$ for some $j \in \{1, \ldots, 2^m - 1\}$, we have

$$D/\langle u\rangle \cong (D/\langle z^{2j}\rangle)/(\langle x^{2^{n-2}} z^j\rangle/\langle z^{2j}\rangle) \cong Q_{2^n} * C_{2^m/|\langle z^{2j}\rangle|}.$$

For $\langle z^j\rangle = \langle z\rangle$ we get $D/\langle u\rangle \cong Q_{2^n}$. Otherwise we have $Q_{2^n} * C_{2^m/|\langle z^{2j}\rangle|} \cong D_{2^n} * C_{2^m/|\langle z^{2j}\rangle|}$. Here we can apply the main theorem of [18]. Now we discuss the cases (ab) and (aa) separately.

Case (ab):
Then we have $l(b_u) = l(\overline{b_u}) = 2$ for $1 \ne u \in \mathrm{Z}(D)$. Hence, Theorem 5.9.4 in [13] implies

$$k(B) - l(B) = 2^m(2^{n-2} - 1) + 2^m + 2(2^{m+1} - 1) = 2^m(2^{n-2} + 4) - 2.$$

Since B is a centrally controlled block, we have $l(B) \ge l(b_z) = 2$ and $k(B) \ge 2^m(2^{n-2} + 4)$ (see Theorem 1.1 in [9]). In order to bound $k(B)$ from above we study the numbers $d^z_{\chi\varphi}$. Let $D^z := (d^z_{\chi\varphi_i})_{\chi \in \mathrm{Irr}(B),\, i=1,2}$. Then $(D^z)^{\mathrm{T}}\overline{D^z} = C^z$ is the Cartan matrix of b_z. Since $\overline{b_z}$ has defect group Q_{2^n}, the Cartan matrix of $\overline{b_z}$ (up to basic sets) only depends on the fusion system of $\overline{b_z}$ (see [2]). It follows that

$$C^z = 2^m \begin{pmatrix} 2^{n-2} + 2 & 4 \\ 4 & 8 \end{pmatrix}$$

up to basic sets. Hence, Lemma 1 in [20] implies $k(B) \le 2^m(2^{n-2}+6)$. In order to derive a sharper bound, we consider the generalized decomposition numbers more carefully. Here the proof follows the lines of Theorem 3.6 in [18]. However, we have to consider more cases. As in [18] we write

$$d^z_{\chi\varphi_i} = \sum_{j=0}^{2^{m-1}-1} a^i_j(\chi)\zeta^j$$

for $i = 1, 2$, where ζ is a primitive 2^m-th root of unity. Since the subsections (z^j, b_{z^j}) are pairwise non-conjugate for $j = 0, \ldots, 2^m - 1$, we get

$$(a^1_i, a^1_j) = (2^{n-1}+4)\delta_{ij}, \qquad (a^1_i, a^2_j) = 8\delta_{ij}, \qquad (a^2_i, a^2_j) = 16\delta_{ij}.$$

Since C^z is just twice as large as in [18], the contributions remain the same in terms of $d^z_{\chi\varphi}$. In particular we get

$$h(\chi) = 0 \iff \sum_{j=0}^{2^{m-1}-1} a^2_j(\chi) \equiv 1 \pmod 2. \tag{1}$$

Assume that $k(B)$ is as large as possible. Since (z, b_z) is a major subsection, no row of D^z vanishes. Hence, for $j \in \{0, 1, \ldots, 2^{m-1}-1\}$ we have essentially the following possibilities (where $\epsilon_1, \ldots, \epsilon_8 \in \{\pm 1\}$):

$$(I): \left(\begin{array}{c|cccccccccccccc} a^1_j & \pm1 & \cdots & \pm1 & \epsilon_1 & \cdots & \epsilon_8 & . & \cdots & \cdots & \cdots & \cdots & . \\ a^2_j & . & \cdots & . & \epsilon_1 & \cdots & \epsilon_8 & \pm1 & \cdots & \pm1 & . & \cdots & . \end{array}\right),$$

$$(II): \left(\begin{array}{c|ccccccccccccccc} a^1_j & \pm1 & \cdots & \pm1 & \epsilon_1 & \epsilon_2 & \cdots & \epsilon_7 & . & \cdots & \cdots & \cdots & \cdots & . \\ a^2_j & . & \cdots & . & 2\epsilon_1 & \epsilon_2 & \cdots & \epsilon_7 & \pm1 & \cdots & \pm1 & . & \cdots & . \end{array}\right),$$

$$(III): \left(\begin{array}{c|cccccccccccccccc} a^1_j & \pm1 & \cdots & \pm1 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \cdots & \epsilon_6 & . & \cdots & \cdots & \cdots & \cdots & \cdots & . \\ a^2_j & . & \cdots & . & 2\epsilon_1 & 2\epsilon_2 & \epsilon_3 & \cdots & \epsilon_6 & \pm1 & \pm1 & \pm1 & \pm1 & . & \cdots & . \end{array}\right),$$

$$(IV): \left(\begin{array}{c|cccccccccccccc} a^1_j & \pm1 & \cdots & \pm1 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & . & \cdots & \cdots & \cdots & . \\ a^2_j & . & \cdots & . & 2\epsilon_1 & 2\epsilon_2 & 2\epsilon_3 & \epsilon_4 & \epsilon_5 & \pm1 & \pm1 & . & \cdots & . \end{array}\right),$$

$$(V): \left(\begin{array}{c|cccccccccc} a^1_j & \pm1 & \cdots & \pm1 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & . & \cdots & . \\ a^2_j & . & \cdots & . & 2\epsilon_1 & 2\epsilon_2 & 2\epsilon_3 & 2\epsilon_4 & . & \cdots & . \end{array}\right).$$

The number $k(B)$ would be maximal if case (I) occurs for all j and for every character $\chi \in \mathrm{Irr}(B)$ we have $\sum_{j=0}^{2^{m-1}-1} |a^1_j(\chi)| \le 1$ and $\sum_{j=0}^{2^{m-1}-1} |a^2_j(\chi)| \le 1$. However, this contradicts Lemma 2.6 and Equation (1). This explains why we have to allow other possibilities. We illustrate with two example that the given forms (I) to (V) are the only possibilities we need. For that consider

$$(IIa): \left(\begin{array}{c|cccccccccccccc} a^1_j & \pm1 & \cdots & \pm1 & 2\epsilon_1 & \cdots & \epsilon_7 & . & \cdots & \cdots & \cdots & \cdots & . \\ a^2_j & . & \cdots & . & \epsilon_1 & \cdots & \epsilon_7 & \pm1 & \cdots & \pm1 & . & \cdots & . \end{array}\right),$$

$$(IVa): \left(\begin{array}{c|cccccccccccc} a^1_j & \pm1 & \cdots & \pm1 & \epsilon_1 & \epsilon_2 & \cdots & \epsilon_6 & . & \cdots & \cdots & \cdots & . \\ a^2_j & . & \cdots & . & 3\epsilon_1 & \epsilon_2 & \cdots & \epsilon_6 & \pm1 & \pm1 & . & \cdots & . \end{array}\right).$$

Then both (II) and (IIa) contribute $2^{n-1}+10$ to $k(B)$. However, (II) contributes 12 to $k_0(B)$, while (IIa) contributes 16 to $k_0(B)$. Hence (II) is "better" than (IIa). In the same way (IV) is "better" than (IVa). Now let α_1 (resp. $\alpha_2, \ldots, \alpha_5$) be the number of indices $j \in \{0, 1, \ldots, 2^{m-1}-1\}$ such that case (I) (resp. (II),…,(V)) occurs for a^i_j. Then obviously $\alpha_1 + \ldots + \alpha_5 = 2^{m-1}$. It is easy to see that we may assume for all $\chi \in \mathrm{Irr}(B)$ that $\sum_{j=0}^{2^{m-1}-1} |a^1_j(\chi)| \le 1$ in order to maximize $k(B)$. In contrast to that it does make sense to have $a^2_j(\chi) \ne 0 \ne a^2_k(\chi)$ for some $j \ne k$ in order to satisfy Olsson's Conjecture in view of Equation (1). Let δ be the number of pairs

$(\chi, j) \in \mathrm{Irr}(B) \times \{0, 1, \ldots, 2^{m-1} - 1\}$ such that there exists a $k \neq j$ with $a^2_j(\chi)a^2_k(\chi) \neq 0$. Then it follows that

$$\begin{aligned}
\alpha_5 &= 2^{m-1} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \\
k(B) &\leq (2^{n-1} + 12)\alpha_1 + (2^{n-1} + 10)\alpha_2 + (2^{n-1} + 8)\alpha_3 \\
&\quad + (2^{n-1} + 6)\alpha_4 + (2^{n-1} + 4)\alpha_5 - \delta/2 \\
&= 2^{m+n-2} + 12\alpha_1 + 10\alpha_2 + 8\alpha_3 + 6\alpha_4 + 4\alpha_5 - \delta/2 \\
&= 2^{m+n-2} + 2^{m+1} + 8\alpha_1 + 6\alpha_2 + 4\alpha_3 + 2\alpha_4 - \delta/2, \\
16\alpha_1 + 12\alpha_2 + 8\alpha_3 + 4\alpha_4 - \delta &\leq k_0(B) \leq 2^{m+2}.
\end{aligned}$$

This gives $k(B) \leq 2^{m+n-2} + 2^{m+2} = 2^m(2^{n-2} + 4)$. Together with the lower bound above, we have shown that $k(B) = 2^{m-1}(2^{n-2} + 4)$ and $l(B) = 2$. In particular the cases (I), …,(V) are really the only possibilities which can occur. The inequalities above imply also $k_0(B) = 2^{m+2}$. As in [18] we can show that $\delta = 0$. Moreover, as there we see that the rows of type $(\pm\zeta^j, 0)$ of D^z correspond to characters of height 1. The number of these rows is

$$(2^{n-1} - 4)\alpha_1 + (2^{n-1} - 3)\alpha_2 + (2^{n-1} - 2)\alpha_3 + (2^{n-1} - 1)\alpha_4 + 2^{n-1}\alpha_5 = 2^{n+m-2} - 2^m = 2^m(2^{n-2} - 1).$$

The remaining rows of D^z correspond to characters of height 0 or $n - 2$. This gives $k_i(B)$ for $i \in \mathbb{N}$ (recall that $n \geq 4$ in case (ab)).

Case (aa):
Here we have $l(b_u) = l(\overline{b_u}) = 3$ for $1 \neq u \in \mathrm{Z}(D)$. Hence, Theorem 5.9.4 in [13] implies

$$k(B) - l(B) = 2^m(2^{n-2} - 1) + 3(2^{m+1} - 1) = 2^m(2^{n-2} + 5) - 3.$$

Again B is a centrally controlled, $l(B) \geq l(b_z) = 3$ and $k(B) \geq 2^m(2^{n-2} + 5)$ (see Theorem 1.1 in [9]). The Cartan matrix of b_z is

$$C^z = 2^m \begin{pmatrix} 2^{n-2} + 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}$$

up to basic sets. We write $\mathrm{IBr}(b_z) = \{\varphi_1, \varphi_2, \varphi_3\}$ and define the integral columns a^i_j for $i = 1, 2, 3$ and $j = 0, 1, \ldots, 2^{m-1} - 1$ as in case (ab). Then we can calculate the scalar products (a^i_j, a^k_l). Again C^z is just twice as large as in [18] and we get

$$h(\chi) = 0 \Longleftrightarrow \sum_{j=0}^{2^{m-1}-1} \left(a^2_j(\chi) + a^3_j(\chi)\right) \equiv 1 \pmod 2. \tag{2}$$

In order to search the maximum value for $k(B)$ (in view of Lemma 2.6 and Equation (2)) we have to consider the following possibilities (where $\epsilon_1, \ldots, \epsilon_8 \in \{\pm 1\}$):

(I)

a_j^1	a_j^2	a_j^3
±1	.	.
$\vdots$	$\vdots$	$\vdots$
±1	.	.
ϵ_1	ϵ_1	.
$\vdots$	$\vdots$	$\vdots$
ϵ_4	ϵ_4	.
ϵ_5	.	ϵ_5
$\vdots$	$\vdots$	$\vdots$
ϵ_8	.	ϵ_8
.	±1	.
$\vdots$	$\vdots$	$\vdots$
.	±1	.
.	.	±1
$\vdots$	$\vdots$	$\vdots$
.	.	±1
.	.	.
$\vdots$	$\vdots$	$\vdots$
.	.	.

(II)

a_j^1	a_j^2	a_j^3
±1	.	.
$\vdots$	$\vdots$	$\vdots$
±1	.	.
ϵ_1	ϵ_1	.
ϵ_2	ϵ_2	.
ϵ_3	ϵ_3	.
ϵ_4	ϵ_4	ϵ_4
ϵ_5	.	ϵ_5
ϵ_6	.	ϵ_6
ϵ_7	.	ϵ_7
.	ϵ_8	$-\epsilon_8$
.	±1	.
.	±1	.
.	±1	.
.	.	±1
.	.	±1
.	.	±1
.	.	.
$\vdots$	$\vdots$	$\vdots$
.	.	.

(III)

a_j^1	a_j^2	a_j^3
±1	.	.
$\vdots$	$\vdots$	$\vdots$
±1	.	.
ϵ_1	ϵ_1	.
ϵ_2	ϵ_2	.
ϵ_3	ϵ_3	ϵ_3
ϵ_4	ϵ_4	ϵ_4
ϵ_5	.	ϵ_5
ϵ_6	.	ϵ_6
.	ϵ_7	$-\epsilon_7$
.	ϵ_8	$-\epsilon_8$
.	±1	.
.	±1	.
.	.	±1
.	.	±1
.	.	.
$\vdots$	$\vdots$	$\vdots$
.	.	.

(IV)

a_j^1	a_j^2	a_j^3
±1	.	.
$\vdots$	$\vdots$	$\vdots$
±1	.	.
ϵ_1	ϵ_1	.
ϵ_2	ϵ_2	ϵ_2
ϵ_3	ϵ_3	ϵ_3
ϵ_4	ϵ_4	ϵ_4
ϵ_5	.	ϵ_5
.	ϵ_6	$-\epsilon_6$
.	ϵ_7	$-\epsilon_7$
.	ϵ_8	$-\epsilon_8$
.	±1	.
.	.	±1
.	.	.
$\vdots$	$\vdots$	$\vdots$
.	.	.

(V)

a_j^1	a_j^2	a_j^3
±1	.	.
$\vdots$	$\vdots$	$\vdots$
±1	.	.
ϵ_1	ϵ_1	ϵ_1
ϵ_2	ϵ_2	ϵ_2
ϵ_3	ϵ_3	ϵ_3
ϵ_4	ϵ_4	ϵ_4
.	ϵ_5	$-\epsilon_5$
.	ϵ_6	$-\epsilon_6$
.	ϵ_7	$-\epsilon_7$
.	ϵ_8	$-\epsilon_8$
.	.	.
$\vdots$	$\vdots$	$\vdots$
.	.	.

Define $\alpha_1, \ldots, \alpha_5$ as before. Let δ be the number of triples $(\chi, i, j) \in \mathrm{Irr}(B) \times \{2, 3\} \times \{0, 1, \ldots, 2^{m-1} - 1\}$ such that there exists a $k \neq j$ with $a_j^i(\chi)a_k^2(\chi) \neq 0$ or $a_j^i(\chi)a_k^3(\chi) \neq 0$. Then the following holds:

$$\begin{aligned}
\alpha_5 &= 2^{m-1} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \\
k(B) &\leq (2^{n-1} + 12)\alpha_1 + (2^{n-1} + 11)\alpha_2 + (2^{n-1} + 10)\alpha_3 \\
&\quad + (2^{n-1} + 9)\alpha_4 + (2^{n-1} + 8)\alpha_5 - \delta/2 \\
&= 2^{m+n-2} + 12\alpha_1 + 11\alpha_2 + 10\alpha_3 + 9\alpha_4 + 8\alpha_5 - \delta/2 \\
&= 2^{m+n-2} + 2^{m+2} + 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 - \delta/2, \\
16\alpha_1 + 12\alpha_2 + 8\alpha_3 + 4\alpha_4 - \delta &\leq k_0(B) \leq 2^{m+2}.
\end{aligned}$$

This gives $k(B) \leq 2^{n+m-2} + 2^{m+2} + 2^m = 2^m(2^{n-2} + 5)$. Together with the lower bound we have shown that $k(B) = 2^m(2^{n-2}+5)$, $k_0(B) = 2^{m+2}$, and $l(B) = 3$. In particular the maximal value for $k(B)$ is indeed attended. Moreover, $\delta = 0$. As in [18] we see that the rows of D^z of type $(\pm\zeta^j, 0, 0)$ correspond to characters of height 1. The number of these rows is

$$(2^{n-1} - 4)\alpha_1 + (2^{n-1} - 3)\alpha_2 + (2^{n-1} - 2)\alpha_3 + (2^{n-1} - 1)\alpha_4 + 2^{n-1}\alpha_5 = 2^{n+m-2} - 2^m = 2^m(2^{n-2} - 1).$$

The remaining rows of D^z correspond to characters of height 0 or $n - 2$. This gives $k_i(B)$ for $i \in \mathbb{N}$. Observe that we have to add $k_1(B)$ and $k_{n-2}(B)$ in case $n = 3$. □

We add some remarks. The principal block of D gives an example for case (bb). For $n = 3$ the principal block of $D \rtimes C_3$ gives an example for case (aa). If $n = 4$, the principal blocks of $\mathrm{SL}(2, 7) \times C_{2^m}$ and $2.S_4 \times C_{2^m}$ show that also the cases (aa) resp. (ab) can occur. Here $2.S_4 =$ `SmallGroup(48,28)` denotes the double cover of S_4 which is not isomorphic to $\mathrm{GL}(2, 3)$ (this can be seen with GAP). If $\widetilde{B}$ is a block with defect group $Q_{2^n} * C_{2^{m+1}}$, then the invariants of B and $\widetilde{B}$ coincide in the corresponding cases (see [18]). However, it was shown in [20] (for $n = 3$ and $m = 1$) that the numbers of 2-rational characters of B resp. $\widetilde{B}$ are different.

2.3 Alperin's Weight Conjecture

Theorem 2.8. *Alperin's Weight Conjecture holds for B.*

Proof. Just copy the proof of Theorem 4.1 in [18]. $\square$

2.4 Ordinary weight conjecture

In this section we prove Robinson's Ordinary Weight Conjecture (OWC) for B (see [17]). If OWC holds for all groups and all blocks, then also Alperin's Weight Conjecture holds. However, for our particular block B this implication is not known. In the same sense OWC is equivalent to Dade's Projective Conjecture (see [3]). Uno has proved Dade's Invariant Conjecture in the case $m = 0$ (see [22]). For $\chi \in \mathrm{Irr}(B)$ let $d(\chi) := n + m - h(\chi)$ be the *defect* of χ. We set $k^i(B) = |\{\chi \in \mathrm{Irr}(B) : d(\chi) = i\}|$ for $i \in \mathbb{N}$.

Theorem 2.9. *The ordinary weight conjecture holds for B.*

Proof. We prove the version in Conjecture 6.5 in [8]. We may assume that B is not nilpotent, and thus case (bb) does not occur. Suppose that $n = 3$ and case (aa) occurs. Then D is the only $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup of D. Since $\mathrm{Out}_{\mathcal{F}}(D) \cong C_3$, the set $\mathcal{N}_D$ consists only of the trivial chain (with the notations of [8]). We have $\mathbf{w}(D, d) = 0$ for $d \notin \{m + 2, m + 3\}$, since then $k^d(D) = 0$. For $d = m + 2$ we get $\mathbf{w}(D, d) = 3 \cdot 2^m$, since the irreducible characters of D of degree 2 are stable under $\mathrm{Out}_{\mathcal{F}}(D)$. In case $d = m + 3$ it follows that $\mathbf{w}(D, d) = 3 \cdot 2^m + 2^m = 2^{m+2}$. Hence, OWC follows from Theorem 2.7.

Now let $n \geq 4$ and assume that case (aa) occurs. Then there are three $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups up to conjugation: Q_1, Q_2 and D. Since $\mathrm{Out}_{\mathcal{F}}(D) = 1$, it follows easily that $\mathbf{w}(D, d) = k^d(D)$ for all $d \in \mathbb{N}$. By Theorem 2.7 it suffices to show

$$\mathbf{w}(Q, d) = \begin{cases} 2^m & \text{if } d = m + 2 \\ 0 & \text{otherwise} \end{cases}$$

for $Q \in \{Q_1, Q_2\}$, because $k^{m+2}(B) = k_{n-2}(B) = 2^{m+1}$. We already have $\mathbf{w}(Q, d) = 0$ unless $d \in \{m+2, m+3\}$. W. l. o. g. let $Q = Q_1$.

Let $d = m + 2$. Up to conjugation $\mathcal{N}_Q$ consists of the trivial chain $\sigma : 1$ and the chain $\tau : 1 < C$, where $C \leq \mathrm{Out}_{\mathcal{F}}(Q)$ has order 2. We consider the chain σ first. Here $I(\sigma) = \mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$ acts trivially on the characters of Q or defect $m + 2$. This contributes 2^m to the alternating sum of $\mathbf{w}(Q, d)$. Now consider the chain τ. Here $I(\tau) = C$ and $z(FC) = 0$ (notation from [8]). Hence, the contribution of τ vanishes and we get $\mathbf{w}(Q, d) = 2^m$ as desired.

Let $d = m + 3$. Then we have $I(\sigma, \mu) \cong S_3$ for every character $\mu \in \mathrm{Irr}(Q)$ with $\mu(x^{2^{n-3}}) = \mu(y) = 1$. For the other characters of Q with defect d we have $I(\sigma, \mu) \cong C_2$. Hence, the chain σ contributes 2^m to the alternating sum. There are 2^{m+1} characters $\mu \in \mathrm{Irr}(D)$ which are not fixed under $I(\tau) = C$. Hence, they split in 2^m orbits of length 2. For these characters we have $I(\tau, \mu) = 1$. For the other irreducible characters μ of D of defect d we have $I(\tau, \mu) = C$. Thus, the contribution of τ to the alternating sum is -2^m. This shows $\mathbf{w}(Q, d) = 0$.

In case (ab) we have only two $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups: Q_2 and D. Since $k_{n-2}(B) = 2^m$ in this case, the calculations above imply the result. $\square$

2.5 The gluing problem

Finally we show that the gluing problem (see Conjecture 4.2 in [10]) for the block B has a unique solution. This was done for $m = 0$ in [16]. We will not recall the very technical statement of the gluing problem. Instead we refer to [16] for most of the notations. Observe that the field F is denoted by k in [16].

Theorem 2.10. *The gluing problem for B has a unique solution.*

Proof. Let σ be a chain of $\mathcal{F}$-centric subgroups of D, and let Q be the largest subgroup occurring in σ. Then $Q = (Q \cap \langle x, y \rangle) \times \langle z \rangle$. If $Q \cap \langle x, y \rangle$ is abelian, then $\mathrm{Aut}_{\mathcal{F}}(Q)$ and $\mathrm{Aut}_{\mathcal{F}}(\sigma)$ are 2-groups. So we have $\mathrm{H}^i(\mathrm{Aut}_{\mathcal{F}}(\sigma), F^\times) = 0$ for $i = 1, 2$.

Now assume that $Q \cap \langle x, y \rangle$ is nonabelian. Then again $\mathrm{Aut}_{\mathcal{F}}(\sigma)$ is a 2-group unless $Q \in \{Q_1, Q_2\}$ (up to conjugation). W. l. o. g. assume $Q = Q_1$ and $\mathrm{Aut}_{\mathcal{F}}(Q) \cong S_4$. If Q is the only subgroup occurring in σ, we get $\mathrm{Aut}_{\mathcal{F}}(\sigma) = \mathrm{Aut}_{\mathcal{F}}(Q) \cong S_4$. If σ consists of another subgroup, $\mathrm{Aut}_{\mathcal{F}}(\sigma)$ must be a 2-group, since an automorphism of $\mathrm{Aut}_{\mathcal{F}}(Q)$ of order 3 permutes the three maximal subgroups of $\langle x^{2^{n-3}}, y \rangle$ transitively. So in both cases we have $\mathrm{H}^i(\mathrm{Aut}_{\mathcal{F}}(\sigma), F^\times) = 0$ for $i = 1, 2$.

Hence, $\mathcal{A}^i_{\mathcal{F}} = 0$ and $\mathrm{H}^0([S(\mathcal{F}^c)], \mathcal{A}^2_{\mathcal{F}}) = \mathrm{H}^1([S(\mathcal{F}^c)], \mathcal{A}^1_{\mathcal{F}}) = 0$. Now by Theorem 1.1 in [16] the gluing problem has only the trivial solution. □

3 The semidihedral case

Now we study blocks with defect groups $SD_{2^n} \times C_{2^m}$. As usual the situation is a mixture of the dihedral and quaternion case. For this reason we will skip some details in the proofs, and refer to [21] or to the last section instead. Let

$$D := \langle x, y, z \mid x^{2^{n-1}} = y^2 = z^{2^m} = [x, z] = [y, z] = 1,\ yxy^{-1} = x^{-1+2^{n-2}} \rangle = \langle x, y \rangle \times \langle z \rangle \cong SD_{2^n} \times C_{2^m},$$

with $n \geq 4$ and $m \geq 0$.

3.1 Subsections

Lemma 3.1. *The automorphism group* $\mathrm{Aut}(D)$ *is a 2-group.*

Proof. Follows as in Lemma 2.1, because the maximal subgroups of the semidihedral group are pairwise non-isomorphic. □

The last lemma implies that the inertial index of B is $e(B) = 1$.

Lemma 3.2. *Let* $Q_1 := \langle x^{2^{n-2}}, y, z \rangle \cong C_2^2 \times C_{2^m}$ *and* $Q_2 := \langle x^{2^{n-3}}, xy, z \rangle \cong Q_8 \times C_{2^m}$*. Then* Q_1 *and* Q_2 *are the only candidates for proper* $\mathcal{F}$*-centric,* $\mathcal{F}$*-radical subgroups up to conjugation. In particular the fusion of subpairs is controlled by* $\mathrm{N}_G(Q_1, b_{Q_1}) \cup \mathrm{N}_G(Q_2, b_{Q_2}) \cup D$*. Moreover, one of the following cases occurs:*

(aa) $\mathrm{Aut}_{\mathcal{F}}(Q_1) \cong \mathrm{Out}_{\mathcal{F}}(Q_2) \cong S_3$.

(ab) $\mathrm{Aut}_{\mathcal{F}}(Q_1) \cong S_3$ *and* $\mathrm{N}_G(Q_2, b_{Q_2}) = \mathrm{N}_D(Q_2)\,\mathrm{C}_G(Q_2)$.

(ba) $\mathrm{N}_G(Q_1, b_{Q_1}) = \mathrm{N}_D(Q_1)\,\mathrm{C}_G(Q_1)$ *and* $\mathrm{Out}_{\mathcal{F}}(Q_2) \cong S_3$.

(bb) $\mathrm{N}_G(Q_1, b_{Q_1}) = \mathrm{N}_D(Q_1)\,\mathrm{C}_G(Q_1)$ *and* $\mathrm{N}_G(Q_2, b_{Q_2}) = \mathrm{N}_D(Q_2)\,\mathrm{C}_G(Q_2)$.

In case (bb) the block B *is nilpotent.*

Proof. Let $Q < D$ be $\mathcal{F}$-centric and $\mathcal{F}$-radical. Then $z \in \mathrm{Z}(D) \subseteq \mathrm{C}_D(Q) \subseteq Q$ and $Q = (Q \cap \langle x, y \rangle) \times \langle z \rangle$. Since $\mathrm{Aut}(Q)$ is not a 2-group, only the following cases are possible: $Q \cong C_{2^m}^2$, $C_2^2 \times C_{2^m}$, $Q_8 \times C_{2^m}$. In the first case we have $Q = \langle x, z \rangle$ or $Q = \langle x^i y, z \rangle$ for some odd i. Then $m = n - 1$ or $m = 2$ respectively (this is not important here). The group $D \subseteq \mathrm{N}_G(Q, b_Q)$ (resp. $\langle x^{2^{n-3}} \rangle Q$) acts trivially on $\Omega(Q) \subseteq \mathrm{Z}(D)$, while a nontrivial automorphism of $\mathrm{Aut}(Q)$ of odd order acts nontrivially on $\Omega(Q)$ (see Theorem 5.2.4 in [7]). This contradicts $\mathrm{O}_2(\mathrm{Aut}_{\mathcal{F}}(Q)) = 1$. Moreover, by Lemma 5.4 in [11] we see that $\mathrm{Aut}_{\mathcal{F}}(\langle x, z \rangle)$ is a 2-group (this will be needed later).

If $Q \cong C_2^2 \times C_{2^m}$, then Q contains an element of the form $x^{2i} y$. After conjugation with a suitable power of x we may assume $Q = Q_1$. Similarly, Q is conjugate to Q_2 if $Q \cong Q_8 \times C_{2^m}$. This shows the first claim.

It remains to show that one of the given cases occurs. For the subgroup Q_1 this can be done as in Lemma 2.2 of [21]. For the subgroup Q_2 we can copy the proof of Lemma 2.2 of the present paper. In particular both Q_1 and Q_2 are fully $\mathcal{F}$-normalized. The last claim follows from Alperin's fusion theorem and $e(B) = 1$. □

Again the naming of these cases is adopted from Olsson's paper [14], but in contrast to the dihedral and quaternion case, the cases (ab) and (ba) are *not* symmetric, since $Q_1 \not\cong Q_2$. Moreover, it is easy to see that Q_1 and Q_2 are always $\mathcal{F}$-centric.

Lemma 3.3. *Let $Q \in \{Q_1, Q_2\}$ such that $\mathrm{N}_G(Q, b_Q)/Q\,\mathrm{C}_G(Q) \cong S_3$. Then*

$$\mathrm{C}_Q(\mathrm{N}_G(Q, b_Q)) = \begin{cases} \langle z \rangle & \text{if } Q = Q_1, \\ \langle x^{2^{n-2}}, z \rangle & \text{if } Q = Q_2. \end{cases}$$

Proof. For Q_2 this follows as in the quaternion case. For Q_1 we can consult [21]. Observe that we may have to replace z by $x^{2^{n-2}} z$ here. However, this does not affect $\mathrm{C}_{Q_2}(\mathrm{N}_G(Q_2, b_{Q_2}))$. □

Lemma 3.4. *A set $\mathcal{R}$ as in Lemma 2.4 is given as follows:*

(i) $x^i z^j$ ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) in case (aa).

(ii) $x^i z^j$ and xyz^j ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) in case (ab).

(iii) $x^i z^j$ and yz^j ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) in case (ab).

Proof. By Lemma 3.3 in any case the elements $x^i z^j$ ($i = 0, 1, \ldots, 2^{n-2}$, $j = 0, 1, \ldots, 2^m - 1$) are pairwise non-conjugate in $\mathcal{F}$. Moreover, $\langle x, z \rangle \subseteq \mathrm{C}_G(x^i z^j)$ and $|D : \mathrm{N}_D(\langle x^i z^j \rangle)| \le 2$. Suppose that $\langle x^i y z^j \rangle \trianglelefteq D$ for some $i, j \in \mathbb{Z}$. Then we have $x^{i+2+2^{n-2}} y z^j = x(x^i y z^j)x^{-1} \in \langle x^i y z^j \rangle$ and the contradiction $x^{2+2^{n-2}} \in \langle x^i y z^j \rangle$. This shows that the subgroups $\langle x^i z^j \rangle$ are always fully $\mathcal{F}$-normalized.

Assume that case (aa) occurs. Then the elements of the form $x^{2i} y z^j$ ($i, j \in \mathbb{Z}$) are conjugate to elements of the form $x^{2i} z^j$ under $D \cup \mathrm{N}_G(Q_1, b_{Q_1})$. Similarly, the elements of the form $x^{2i+1} y z^j$ ($i, j \in \mathbb{Z}$) are conjugate to elements of the form $x^{2i} z^j$ under $D \cup \mathrm{N}_G(Q_2, b_{Q_2})$. The claim follows in this case.

In case (ab) the given elements are pairwise non-conjugate, since no conjugate of xyz^j lies in Q_1. As in case (aa) the elements of the form $x^{2i} y z^j$ ($i, j \in \mathbb{Z}$) are conjugate to elements of the form $x^{2i} z^j$ under $D \cup \mathrm{N}_G(Q_1, b_{Q_1})$ and the elements of the form $x^{2i+1} y z^j$ ($i, j \in \mathbb{Z}$) are conjugate to elements of the form xyz^j under D. Finally, the subgroups $\langle xyz^j \rangle$ are fully $\mathcal{F}$-normalized, since xyz^j is not conjugate to an element in Q_1.

The situation in case (ba) is very similar. We omit the details. □

3.2 The numbers $k(B)$, $k_i(B)$ and $l(B)$

Lemma 3.5. *Olsson's Conjecture $k_0(B) \le 2^{m+2} = |D : D'|$ is satisfied in all cases.*

Proof. See Lemma 3.2 in [21]. □

Theorem 3.6.

(i) In case (aa) we have $k(B) = 2^m(2^{n-2} + 4)$, $k_0(B) = 2^{m+2}$, $k_1(B) = 2^m(2^{n-2} - 1)$, $k_{n-2}(B) = 2^m$ and $l(B) = 3$.

(ii) In case (ab) we have $k(B) = 2^m(2^{n-2} + 3)$, $k_0(B) = 2^{m+2}$, $k_1(B) = 2^m(2^{n-2} - 1)$ and $l(B) = 2$.

(iii) In case (ba) we have $k(B) = 2^m(2^{n-2} + 4)$, $k_0(B) = 2^{m+2}$, $k_1(B) = 2^m(2^{n-2} - 1)$, $k_{n-2}(B) = 2^m$ and $l(B) = 2$.

(iv) In case (bb) we have $k(B) = 2^m(2^{n-2} + 3)$, $k_0(B) = 2^{m+2}$, $k_1(B) = 2^m(2^{n-2} - 1)$ and $l(B) = 1$.

In particular Brauer's $k(B)$-Conjecture, Brauer's Height-Zero Conjecture and the Alperin-McKay Conjecture hold.

Proof. Assume first that case (bb) occurs. Then B is nilpotent, and the result follows as in Theorem 2.7.

Now assume that case (aa), (ab) or (ba) occurs. We determine the numbers $l(b)$ for the subsections in Lemma 3.4 and apply Theorem 5.9.4 in [13]. Let us begin with the non-major subsections. Since $\operatorname{Aut}_{\mathcal{F}}(\langle x,z\rangle)$ is a 2-group, the block $b_{\langle x,z\rangle}$ with defect group $\langle x,z\rangle$ is nilpotent. Hence, we have $l(b_{x^iz^j})=1$ for all $i=1,\ldots,2^{n-2}-1$ and $j=0,1,\ldots,2^m-1$. The blocks b_{xyz^j} $(j=0,1,\ldots,2^{m-1}-1)$ have $\mathrm{C}_D(xyz^j)=\langle xy,z\rangle\cong C_4\times C_{2^m}$ as defect group. In case (ab), $\operatorname{Aut}_{\mathcal{F}}(\mathrm{N}_D(\langle xy,z\rangle))=\operatorname{Aut}_{\mathcal{F}}(Q_2)$ is a 2-group. Hence, Lemma 5.4 in [11] implies that also $\operatorname{Aut}_{\mathcal{F}}(\langle xy,z\rangle)$ is a 2-group. This gives $l(b_{xyz^j})=1$ for $j=0,1,\ldots,2^m-1$. Similarly, in case (ba) we have $l(b_{yz^j})=1$.

Now we consider the major subsections. By Lemma 3.3 the cases for B and b_{z^j} coincide. As usual, the blocks b_{z^j} dominate blocks $\overline{b_{z^j}}$ of $R\,\mathrm{C}_G(z^j)/\langle z^j\rangle$ with defect group $D/\langle z^j\rangle\cong SD_{2^n}\times C_{2^m/|\langle z^j\rangle|}$. Of course the cases for b_{z^j} and $\overline{b_{z^j}}$ coincide, and by Theorem 5.8.11 in [13] we have $l(b_{z^j})=l(\overline{b_{z^j}})$. Thus, we can apply induction on m. The beginning of this induction ($m=0$) is satisfied by Olsson's results (see [14]).

Let $u:=x^{2^{n-1}}z^j$ for a $j\in\{0,1,\ldots,2^m-1\}$. If case (ab) occurs for B, then case (bb) occurs for b_u by Lemma 3.3. Thus, $l(b_u)=1$ in this case. If case (ba) or (aa) occurs for B, then case (ba) occurs for b_u. In case $j=0$, b_u dominates a block $\overline{b_u}$ with defect group $D/\langle u\rangle\cong D_{2^{n-1}}\times C_{2^m}$. Then we can apply the results of [21]. Observe again that the cases for b_u and $\overline{b_u}$ coincide.

Finally, if $j\in\{1,\ldots,2^m-1\}$, we have

$$D/\langle u\rangle\cong(D/\langle z^{2j}\rangle)/(\langle x^{2^{n-2}}z^j\rangle/\langle z^{2j}\rangle)\cong SD_{2^n}*C_{2^m/|\langle z^{2j}\rangle|}.$$

For $\langle z^j\rangle=\langle z\rangle$ we get $D/\langle u\rangle\cong SD_{2^n}$. Otherwise we have $SD_{2^n}*C_{2^m/|\langle z^{2j}\rangle|}\cong D_{2^n}*C_{2^m/|\langle z^{2j}\rangle|}$. Here we can apply the main theorem of [18]. Now we discuss the cases (ab), (ba) and (aa) separately.

Case (ab):
Then we have $l(b_{z^j})=l(\overline{b_{z^j}})=2$ for $j=1,\ldots,2^m-1$ by induction on m. As explained above, we also have $l(b_u)=1$ for $u=x^{2^{n-1}}z^j$ and $j=0,1,\ldots,2^m-1$. Hence, Theorem 5.9.4 in [13] implies

$$k(B)-l(B)=2^m(2^{n-2}-1)+2^m+2(2^m-1)+2^m=2^m(2^{n-2}+3)-2.$$

Since B is a centrally controlled block, we have $l(B)\ge l(b_z)=2$ and $k(B)\ge 2^m(2^{n-2}+3)$ (see Theorem 1.1 in [9]).

Let $u:=x^{2^{n-2}}\in\mathrm{Z}(D)$. Lemma 3.1(ii) in [21] implies $2^{h(\chi)}\mid d^u_{\chi\varphi_u}$ and $2^{h(\chi)+1}\nmid d^u_{\chi\varphi_u}$ for $\chi\in\operatorname{Irr}(B)$. In particular $d^u_{\chi\varphi_u}\ne 0$. Lemma 3.5 gives

$$2^{n+m}\le k_0(B)+4(k(B)-k_0(B))\le\sum_{\chi\in\operatorname{Irr}(B)}\left(d^u_{\chi\varphi_u}\right)^2=(d(u),d(u))=|D|=2^{n+m}.$$

Hence, we have

$$d^u_{\chi\varphi_u}=\begin{cases}\pm1 & \text{if } h(\chi)=0\\ \pm2 & \text{otherwise}\end{cases},$$

and the claim follows in case (ab).

Case (ba):
Here we have $l(b_u)=2$ for all $1\ne u\in\mathrm{Z}(D)$ by induction on m. This gives

$$k(B)-l(B)=2^m(2^{n-2}-1)+2^m+2(2^{m+1}-1)=2^m(2^{n-2}+4)-2.$$

Since B is a centrally controlled block, we have $l(B)\ge l(b_z)=2$ and $k(B)\ge 2^m(2^{n-2}+4)$ (see Theorem 1.1 in [9]). Now the proof works as in the quaternion case by studying the numbers $d^z_{\chi\varphi}$. Since $\overline{b_z}$ has defect group SD_{2^n}, the Cartan matrix of $\overline{b_z}$ (up to basic sets) only depends on the fusion system of $\overline{b_z}$ (see [2]). It follows that the Cartan matrix of b_z is given by

$$2^m\begin{pmatrix}2^{n-2}+2 & 4\\ 4 & 8\end{pmatrix}$$

up to basic sets. This is exactly the same matrix as in the quaternion case. So we omit the details.

Case (aa):
We have $l(b_{z^j}) = 3$ for $j = 1, \ldots, 2^m - 1$ by induction on m. Moreover, for $u = x^{2^{n-1}} z^j$ we get $l(b_u) = 2$. Hence,

$$k(B) - l(B) = 2^m(2^{n-2} - 1) + 3(2^m - 1) + 2^{m+1} = 2^m(2^{n-2} + 4) - 3.$$

Again B is centrally controlled which implies $l(B) \geq l(b_z) = 3$ and $k(B) \geq 2^m(2^{n-2} + 4)$. In contrast to case (ba) we study the generalized decomposition numbers of the element $u := x^{2^{n-2}} z$. Then case (ba) occurs for b_u and the Cartan matrix of b_u is given by

$$2^m \begin{pmatrix} 2^{n-2} + 2 & 4 \\ 4 & 8 \end{pmatrix}$$

up to basic sets. Hence, the proof works as above. □

The principal blocks of D, $M_{10} \times C_{2^m}$, $\mathrm{GL}(2,3) \times C_{2^m}$ and $2.S_5 \times C_{2^m}$ give examples for case (bb), (ab), (ba) and (aa) respectively. Here M_{10} is the (non-simple) Mathieu group of degree 10 and order 720 and $2.S_5 =$ `SmallGroup(240,90)` is one of the two double covers of the symmetric group S_5 (this can be seen with GAP).

3.3 Alperin's Weight Conjecture

Theorem 3.7. *Alperin's Weight Conjecture holds for B.*

Proof. Just copy the proof of Theorem 4.1 in [18]. □

3.4 Ordinary weight conjecture

Here we use the same notations as in Section 2.4.

Theorem 3.8. *The ordinary weight conjecture holds for B.*

Proof. We prove the version in Conjecture 6.5 in [8]. We may assume that B is not nilpotent, and thus case (bb) does not occur.

Assume first that case (aa) occurs. Then there are three $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups up to conjugation: Q_1, Q_2 and D. Since $\mathrm{Out}_{\mathcal{F}}(D) = 1$, it follows easily that $\mathbf{w}(D, d) = k^d(D)$ for all $d \in \mathbb{N}$. By Theorem 3.6 it suffices to show $\mathbf{w}(Q_1, d) = 0$ for all d and

$$\mathbf{w}(Q_2, d) = \begin{cases} 2^m & \text{if } d = m + 2, \\ 0 & \text{otherwise,} \end{cases}$$

because $k^{m+2}(B) = k_{n-2}(B) = 2^m$. For the group Q_1 this works exactly as in [21] and for Q_2 we can copy the proof of Theorem 2.9.

In the cases (ab) and (ba) we have only two $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups: Q_1 (resp. Q_2) and D. In case (ab) Theorem 3.6 implies $k^d(B) = k^d(D)$ for all $d \in \mathbb{N}$ while in case (ba) we still have $k^{m+2}(B) = 2^m$. So the calculations above imply the result. □

3.5 The gluing problem

Finally we show that the gluing problem (see Conjecture 4.2 in [10]) for the block B has a unique solution. This was done for $m = 0$ in [16]. We use the notation of Section 2.5.

Theorem 3.9. *The gluing problem for B has a unique solution.*

Proof. Let σ be a chain of $\mathcal{F}$-centric subgroups of D, and let Q be the largest subgroup occurring in σ. Then $Q = (Q \cap \langle x, y\rangle) \times \langle z\rangle$. If $Q \cap \langle x, y\rangle$ is abelian, then $\mathrm{Aut}_{\mathcal{F}}(Q)$ and $\mathrm{Aut}_{\mathcal{F}}(\sigma)$ are 2-groups unless $Q = Q_1$ (up to conjugation). In case $Q = Q_1$, σ only consists of Q, and we can also have $\mathrm{Aut}_{\mathcal{F}}(\sigma) = \mathrm{Aut}_{\mathcal{F}}(Q) \cong S_3$. So in all these cases we have $\mathrm{H}^i(\mathrm{Aut}_{\mathcal{F}}(\sigma), F^{\times}) = 0$ for $i = 1, 2$.

Now assume that $Q \cap \langle x, y\rangle$ is nonabelian. Then again $\mathrm{Aut}_{\mathcal{F}}(\sigma)$ is a 2-group unless $Q = Q_2$ (up to conjugation). Now the claim follows as in Theorem 2.10. □

4 Summary

In this section we provide a theorem which summarizes the results of [21, 18] and the present paper. Here we also consider the less-known conjectures by Eaton [3], Eaton-Moretó [4] and Malle-Navarro [12]. The verification of these is only a matter of elementary calculations.

Theorem 4.1. *Let M be a 2-group of maximal class, and let C be a cyclic group. Then for every block B with defect group $M \times C$ or $M * C$ the following conjectures are satisfied:*

- *Alperin's Weight Conjecture*
- *Brauer's $k(B)$-Conjecture*
- *Brauer's Height-Zero Conjecture*
- *Olsson's Conjecture*
- *Alperin-McKay Conjecture*
- *Robinson's Ordinary Weight Conjecture*
- *Eaton's Conjecture*
- *Eaton-Moretó Conjecture*
- *Malle-Navarro Conjecture*

Moreover, the gluing problem for B has a unique solution.

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