A solution to Brauer's Problem 14

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Abstract

It is well known that the number of real irreducible characters of a finite group G coincides with the number of real conjugacy classes of G. Richard Brauer has asked if the number of irreducible characters with Frobenius–Schur indicator 1 can also be expressed in group theoretical terms. We show that this can done by counting solutions of $g_1^2 \dots g_n^2 = 1$ with $g_1, \dots, g_n \in G$.

Keywords: Frobenius–Schur indicator; real characters; Brauer's Problem 14
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1 Introduction

Since Brauer's survey [2], there has been some interest in expressing representation theoretical invariants of a finite group G in terms of group theoretical descriptions. The most basic observation of this kind is probably that the number of irreducible characters of G equals the number of conjugacy classes of G. In principle the whole character table of G is implicitly determined via the class multiplication constants

$$|\{(x, y) \in C \times D : xy = z\}|,$$

where C, D, E are conjugacy classes and $z \in E$ is fixed (this is the basis of the Dixon–Schneider algorithm; see [5, Corollary 2.4.3]). It is often desirable to have more direct relations. Brauer was particularly interested in the existence of *p-defect zero* characters, i. e. $\chi \in \mathrm{Irr}(G)$ such that the degree $\chi(1)$ is divisible by the p-part $|G|_p$. Strunkov [13] (see also [7, Theorem 4.12]) showed that such characters exist if and only if there exists $g \in G$ such that p does not divide

$$|\{(x, y) \in G^2 : [x, y] = g\}|,$$

where $[x, y] = xyx^{-1}y^{-1}$ denotes the commutator of x and y. Similar criteria were obtained by Barker [1], Broué [3], Qian [8] and Shi [12]. Robinson [9] (see also [6, Theorem 4.20]) has expressed the precise number of p-defect zero characters as the rank of a certain matrix defined in group theoretical terms. This has answered Brauer's Problem 19 of [2].

Brauer's permutation lemma on the character table implies that the number of real irreducible characters coincides with the number of real conjugacy classes. Here, a conjugacy class C of G is called *real* if

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$C = \{x^{-1} : x \in C\}$. Brauer's Problem 14 asks whether one can describe the number of characters which arise from real irreducible *representations* (i. e. $\chi \in \mathrm{Irr}(G)$ with Frobenius–Schur indicator $\epsilon(\chi) = 1$) group theoretically. Note that this number is not encoded in the character table (compare D_8 and Q_8). We answer this question positively as follows.

Theorem A. *Let G be a finite group with $k_r(G) = |\mathrm{Irr}_{\mathbb{R}}(G)|$ real conjugacy classes. Then the multiset $\{\chi(1)\epsilon(\chi) : \chi \in \mathrm{Irr}_{\mathbb{R}}(G)\}$ is determined by the sequence*

$$s(n) := |\{(g_1, \dots, g_n) \in G^n : g_1^2 \dots g_n^2 = 1\}|$$

for $n = 1, \dots, k_r(G) + 1$. In particular, $k_r(G) = \frac{s(2)}{|G|}$ and the number of irreducible characters of G with Frobenius–Schur indicator 1 can be described purely in group theoretical terms.

Robinson [10] has shown that the multiset of irreducible character degrees of G is determined by the group theoretical sequence

$$|\{(a_1, b_1, \dots, a_n, b_n) \in G^{2n} : [a_1, b_1] \dots [a_n, b_n] = 1\}| \qquad (n \in \mathbb{N}).$$

He has shown further that the number of real characters of a given degree is computable from the sequence $\{s(2n) : n = 1, \dots, |G|\}$ where $s(n)$ is as in Theorem A. Our result improves this.

2 Proofs

We start with a combinatorial lemma.

Lemma 1. *Let $a_1, \dots, a_n \in \mathbb{C}$. Then the multiset $\{a_1, \dots, a_n\}$ is uniquely determined by the power sums $\sum_{i=1}^n a_i^k$ for $k = 0, 1, \dots, n$.*

Proof. Let $\sigma_0 := 1, \sigma_1, \dots, \sigma_n$ be the elementary symmetric functions in n variables. By the Girard–Newton identities (see [11, Theorem 8.7]), the values $\sigma_k(a_1, \dots, a_n)$ can be computed from the power sums. Hence, the polynomial

$$(X - a_1) \dots (X - a_n) = \sum_{k=0}^{n} (-1)^{n-k} \sigma_{n-k}(a_1, \dots, a_n) X^k$$

is uniquely determined and so are its roots. $\square$

Remark: Suppose that $a_1, \dots, a_n$ are non-zero. Let $\rho_k(a_1, \dots, a_n) := \sum_{i=1}^n a_i^k$ for $k \in \mathbb{Z}$. With the notation of the proof above, we have

$$\sigma_n(a_1, \dots, a_n)\rho_{-1}(a_1, \dots, a_n) = a_1 \dots a_n \sum_{i=1}^{n} a_i^{-1} = \sigma_{n-1}(a_1, \dots, a_n).$$

Hence, if $\rho_{-1}(a_1, \dots, a_n)$ is known and non-zero, then $\rho_n(a_1, \dots, a_n)$ is not required to compute $a_1, \dots, a_n$. This will be used in the following proof.

Recall that the Frobenius–Schur indicator of $\chi \in \mathrm{Irr}(G)$ is defined by

$$\epsilon(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2) \in \{0, 1, -1\}. \tag{2.1}$$

Proof of Theorem A. Consider $S := \frac{1}{|G|}\sum_{g\in G} g^2 \in \mathrm{Z}(\mathbb{C}G)$. For a fixed $n \in \mathbb{N}$ we may write $S^n = \sum_{g\in G}\alpha_g g$ with $\alpha_g \in \mathbb{Z}$ for $g \in G$. Note that $s(n) = \alpha_1|G|^n$. Let $\chi \in \mathrm{Irr}(G)$. Recall that the function $\frac{\chi}{\chi(1)}$ extends to an algebra homomorphism $\omega_\chi\colon \mathrm{Z}(\mathbb{C}G) \to \mathbb{C}$. By (2.1), $\omega_\chi(S) = \frac{\epsilon(\chi)}{\chi(1)}$. By the orthogonality relation,

$$\begin{aligned}\frac{s(n)}{|G|^{n-1}} &= \sum_{g\in G}\alpha_g \sum_{\chi\in\mathrm{Irr}(G)}\chi(g)\chi(1) = \sum_{\chi\in\mathrm{Irr}(G)}\chi(1)^2\sum_{g\in G}a_g\frac{\chi(g)}{\chi(1)}\\ &= \sum_{\chi\in\mathrm{Irr}(G)}\chi(1)^2\omega_\chi(S)^n = \sum_{\chi\in\mathrm{Irr}(G)}\frac{\epsilon(\chi)^n}{\chi(1)^{n-2}} = \sum_{\chi\in\mathrm{Irr}_{\mathbb{R}}(G)}\Bigl(\frac{\epsilon(\chi)}{\chi(1)}\Bigr)^{n-2}\end{aligned}$$

for every $n \in \mathbb{N}$. In particular, $s(2) = |G|k_r(G)$. We apply Lemma 1 with the non-zero numbers $\{a_1,\ldots,a_{k_r(G)}\} = \{\frac{\epsilon(\chi)}{\chi(1)} : \chi \in \mathrm{Irr}_{\mathbb{R}}(G)\}$. The power sums are given in terms of the $s(n)$. By the remark after Lemma 1, the multiset $\{\chi(1)\epsilon(\chi) : \chi \in \mathrm{Irr}_{\mathbb{R}}(G)\}$ is determined by $s(n)$ for $n = 1,\ldots,k_r(G)+1$ since $s(1) > 0$. $\square$

If all $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ have $\epsilon(\chi) = 1$, then the proof shows that $\frac{s(n)}{|G|^n}$ is a non-increasing sequence. Hence, if there exists some (odd) n such that $s(n)|G| < s(n+1)$, then some $\chi \in \mathrm{Irr}(G)$ has $\epsilon(\chi) = -1$. This criterion applies to the simple group $G = \mathrm{PSU}(3,3)$ with $n = 5$ as one can check with GAP [4]. It does not, however, apply to the McLaughlin group McL, which is the only sporadic group with some $\epsilon(\chi) = -1$. In general it is easy to show that

$$\lim_{n\to\infty}\frac{s(n)}{|G|^{n-1}} = |G/N|$$

where $N := \langle g^2 : g \in G\rangle$ is the smallest normal subgroup of G with elementary abelian 2-quotient.

Since $k_r(G)$ and $s(2)$ are group theoretical invariants, one may ask if the equality $k_r(G)|G| = s(2)$ from Theorem A can be proved without characters. To this end, let C be a real conjugacy class and $x \in C$. Then there exist exactly $|\mathrm{C}_G(x)|$ elements $y \in G$ such that $y^{-1}xy = x^{-1}$. Now there are $|C||\mathrm{C}_G(x)| = |G|$ pairs $(x,y) \in G^2$ such that $y^{-1}xy = x^{-1}$ and $x \in C$. Consequently, it suffices to verify that the maps

$$\begin{gathered}\{(x,y)\in G^2 : y^{-1}xy = x^{-1}\} \longleftrightarrow \{(g,h)\in G^2 : g^2h^2 = 1\},\\ (x,y) \longmapsto (xy^{-1}, y),\\ (gh,h) \longleftarrow\!\shortmid (g,h)\end{gathered}$$

are mutually inverse bijections (we leave this to the reader).

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