On the Brauer-Feit bound for abelian defect groups

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Abstract

We improve the Brauer-Feit bound on the number of irreducible characters in a p-block for abelian defect groups by making use of [Halasi-Podoski, 2012] and [Kessar-Malle, 2011]. We also prove Brauer's $k(B)$-Conjecture for 2-blocks with abelian defect groups of rank at most 5 and 3-blocks and 5-blocks with abelian defect groups of rank at most 3.

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1 Introduction

Let B be a p-block of a finite group G with defect d. Then the so-called $k(B)$-Conjecture proposed by Richard Brauer in 1954 [1] asserts that the number $k(B)$ of irreducible ordinary characters of B can be bounded by p^d. In 1959, Brauer and Feit [2] proved the following weaker bound.

Theorem (Brauer-Feit). *If $d > 2$, then $k(B) < p^{2d-2}$.*

In this paper we are interested in the case where B has an abelian defect group D. Brauer himself already verified his conjecture if D is abelian of rank at most 2. For abelian defect groups of rank 3, he obtained $k(B) < p^{5d/3}$ (see for example Theorem VII.10.13 in [4]; observe that $<$ and $\leq$ are mixed up there).

In the first part of the paper we substantially improve the Brauer-Feit bound under the assumption that D is abelian (see Theorem 1). Here we use a recent result by Halasi and Podoski [7] about large orbits under coprime actions, and one implication of Brauer's Height Zero Conjecture proved by Kessar and Malle [11]. In particular, our result relies on the classification of the finite simple groups. At first sight one might think that the condition on the defect group to be abelian is very restrictive. In fact, many open conjectures in modular representation theory are hard to verify *especially* for abelian defect groups (see e. g. [10]).

In the second part we verify Brauer's $k(B)$-Conjecture for abelian defect groups of small rank (and small primes p). The proof makes use of results by Usami and Puig [22, 16, 15, 23] about perfect isometries, and a result by Külshammer [12] about normal defect groups. We also take the opportunity to investigate "small" groups with regular orbits via computer computations.

We denote the number of irreducible Brauer characters of B by $l(B)$. Let us choose a Brauer correspondent b_D of B in $D\,\mathrm{C}_G(D)$. Then the *inertial quotient* of B is given by $I(B) := \mathrm{N}_G(D, b_D)/D\,\mathrm{C}_G(D)$. Its order $e(B) := |I(B)|$ is the *inertial index* of B. For $x \in D$ there is always a Brauer correspondent b_x of B in $\mathrm{C}_G(x)$. The pair (x, b_x) is called *(B-)subsection.* If $x \in \mathrm{Z}(D)$, then b_x also has defect group D and one may choose $(b_x)_D = b_D$. It follows that $I(b_x) \cong \mathrm{C}_{I(B)}(x)$. This fact will be used often.

We denote a cyclic group of order $n \in \mathbb{N}$ by C_n. For convenience we set $C_n^m := C_n \times \ldots \times C_n$ (m factors).

2 The Brauer-Feit bound for abelian defect groups

Theorem 1. *Let B be a p-block of a finite group with abelian defect group of order $p^d > p$. Then*

$$k(B) < p^{3d/2-1/2}. \tag{1}$$

Proof. Let D be a defect group of B. By Corollary 1.2 in [7] there exist elements $x, y \in D$ such that $\mathrm{C}_{I(B)}(x) \cap \mathrm{C}_{I(B)}(y) = 1$. Without loss of generality $x \neq 1$. Consider a B-subsection (x, b_x). As usual, b_x dominates a block $\overline{b_x}$ with defect group $\overline{D} := D/\langle x\rangle$ and $I(\overline{b_x}) \cong \mathrm{C}_{I(B)}(x)$ (see Theorem 5.8.11 in [13]). We write $\overline{y} := y\langle x\rangle \in \overline{D}$. Choose a $\overline{b_x}$-subsection $(\overline{y}, \beta_{\overline{y}})$ and $\alpha \in I(\beta_{\overline{y}})$. We may regard α as an element of $\mathrm{C}_{I(B)}(x)$. Hence, α acts trivially on $\langle x\rangle$ and on $\langle x, y\rangle/\langle x\rangle$. Since α is a p'-element, it must act trivially on $\langle x, y\rangle$ (see for example Theorem 5.3.2 in [6]). This shows $\alpha = 1$ and $e(\beta_{\overline{y}}) = 1$. Thus, $\overline{b_x}$ satisfies the $k(B)$-Conjecture. In particular, $l(b_x) = l(\overline{b_x}) < k(\overline{b_x}) \leq |\overline{D}| \leq p^{d-1}$ (or $l(\overline{b_x}) = k(\overline{b_x}) = 1 < p^{d-1}$). Since B has abelian defect groups, [11] shows $k(B) = k_0(B)$. Now Theorem V.9.17(ii) in [4] implies

$$k(B) \leq p^d\sqrt{l(b_x)} < p^{3d/2-1/2}. \qquad \square$$

Robinson [18, Theorem 2.1(iii)] gave a proof of Eq. (1) under the hypothesis that p does not belong to a finite set of primes which depends on the rank of D. For $p = 2$, Theorem 1 can be improved further by invoking Theorem 2.4 in [19]. In special situations one may choose $x \in D$ in the proof above such that the order of x is large. We illustrate this by an example. Suppose $D \cong C_{p^n}^m$ for some $n, m \in \mathbb{N}$. Then $I(B)$ acts faithfully on $D/\Phi(D)$. Thus, by [7] we may assume that x has order p^n. Then Eq. (1) becomes $k(B) \leq p^{3d/2-n/2}$.

3 Abelian defect groups of small rank

Theorem 1 already improves Brauer's bound for abelian defect groups of rank 3 (see Introduction). We give an even better bound by using [11] only.

Proposition 2. *Let B be a p-block of a finite group with abelian defect group of rank 3 and order p^d. Then*

$$k(B) < p^{4d/3}.$$

Proof. Let D be a defect group of B, and let $x \in D$ be an element of maximal order p^c. Then for the B-subsection (x, b_x) the block b_x dominates a block $\overline{b_x}$ with defect group $D/\langle x\rangle$ of rank 2. Hence, $l(b_x) = l(\overline{b_x}) < k(\overline{b_x}) \leq |D/\langle x\rangle| = p^{d-c}$. Since D has rank 3, it follows that $p^{d-c} \leq p^{2d/3}$. By [11], we have $k(B) = k_0(B)$. Thus, Theorem V.9.17(ii) in [4] implies

$$k(B) \leq p^d\sqrt{l(b_x)} < p^{4d/3}. \qquad \square$$

In the following we improve Proposition 2 for small primes.

Lemma 3. *Let D be an abelian p-group, let F be an algebraically closed field of characteristic p, and let $A \leq \mathrm{Aut}(D)$ a p'-group such that $|A| \leq 4$ or $A \cong S_3$. Then for the Cartan matrix $C = (c_{ij})$ of $F[D \rtimes A]$ there exists a positive definite, integral quadratic form $q = \sum_{1\leq i\leq j\leq k(A)} q_{ij}x_ix_j$ such that*

$$\sum_{1\leq i\leq j\leq k(A)} q_{ij}c_{ij} \leq |D|.$$

Proof. Let $H := D \rtimes A$. After going over to $H/\mathrm{Z}(H)$, we may assume $\mathrm{Z}(H) = 1$ and $A \neq 1$. Now we determine the decomposition matrix of FH by discussing the various isomorphism types of A. Assume first that $|A| = 2$. The irreducible Brauer characters of H are just the inflations of $H/D \cong C_2$. Since $D = [D, A] \subseteq H' \subseteq D$ (see Theorem 5.2.3 in [6]), we see that H has just two linear characters. Hence, the character group $\widehat{D} := \mathrm{Irr}(D) \cong D$ splits under the action of A into one orbit of length 1 (containing the trivial character) and $(|D| - 1)/2$ orbits of length 2. We compute the irreducible (ordinary) characters of H via induction. The trivial character of D extends to two irreducible characters of H whose rows in the decomposition matrix are $(1, 0)$ and $(0, 1)$.

Inducing a character of $\widehat{D}$ which is not stable under A yields an irreducible character of H whose row in the decomposition matrix is $(1,1)$. For $\chi \in \operatorname{Irr}(H)$ we denote the corresponding row in the decomposition matrix by r_χ. Let $q = x_1^2 + x_2^2 - x_1x_2$ the positive definite quadratic form corresponding to the Dynkin diagram of type A_2. Then we have

$$\sum_{1\le i\le j\le 2} q_{ij}c_{ij} = \sum_{\chi\in\operatorname{Irr}(H)} q(r_\chi) = k(H) \le |D|.$$

Here the last inequality holds by the affirmative solution of Brauer's $k(B)$-Conjecture for solvable groups, but one could certainly use more elementary arguments. Exactly the same proof works for $|A| = 3$.

Suppose next that $A \cong C_4$. Here the action of A on $\widehat{D}$ gives one orbit of length 1, α orbits of length 2, and β orbits of length 4. As before we get rows of the form $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ and $(1,1,1,1)$ in the decomposition matrix. Let $\chi \in \widehat{D}$ be a character in an orbit of length 2. Then χ extends to $D \rtimes \Phi(A)$. Hence, if we arrange the Brauer characters of H suitably, χ contributes two rows $(1,1,0,0)$ and $(0,0,1,1)$ to the decomposition matrix. Again we have $q(r_\chi) = 1$ for all $\chi \in \operatorname{Irr}(H)$, and the claim follows.

The case $A \cong C_2^2$ is slightly more complicated. First note that $p > 2$. Again $\widehat{D}$ splits into one orbit of length 1, α orbits of length 2, and β orbits of length 4. Suppose first that there is an element $1 \ne g \in A$ which acts freely on $\widehat{D}$. In this case we may arrange the four irreducible Brauer characters of H in such a way that every row of the decomposition matrix has the form $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,1,0,0)$, $(0,0,1,1)$, $(1,0,0,1)$, $(0,1,1,0)$ or $(1,1,1,1)$. Let q be the quadratic form corresponding to the positive definite matrix

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & -1 & . \\ 1 & -1 & 2 & -1 \\ -1 & . & -1 & 2 \end{pmatrix}.$$

Then it can be seen that $q(r_\chi) = 1$ for every $\chi \in \operatorname{Irr}(H)$. The claim follows as above. Now we treat the case where every non-trivial element of A has a non-trivial fixed point on $\widehat{D}$. We write $A = \{1, g_1, g_2, g_3\}$, $A_i := \mathrm{C}_{\widehat{D}}(g_i)$ and $\alpha_i := |A_i| > 1$ for $i = 1,2,3$. Without loss of generality, $\alpha_1 \le \alpha_2 \le \alpha_3$. Since A acts faithfully on $\widehat{D}$, we have $A_2 \cap A_3 = 1$ and $A_2 \times A_3 \le \widehat{D}$. Moreover, $\alpha = (\alpha_1 + \alpha_2 + \alpha_3 - 3)/2$ and $\beta = (|D| - \alpha_1 - \alpha_2 - \alpha_3 + 2)/4 \ge (\alpha_2\alpha_3 - \alpha_1 - \alpha_2 - \alpha_3 + 2)/4$. Now the inequality

$$\alpha \le 3(\beta - 1)$$

reduces to $\alpha_1 + \alpha_2 + \alpha_3 \le 3\alpha_3 \le \alpha_2\alpha_3$ which is true since $\alpha_2 \ge p > 2$. We may arrange the irreducible Brauer characters of H such that the decomposition matrix consists of $(\alpha_1 - 1)/2$ pairs of rows $(1,0,1,0)$, $(0,1,0,1)$, $(\alpha_2 - 1)/2$ pairs of the form $(1,0,0,1)$, $(0,1,1,0)$, and $(\alpha_3 - 1)/2$ pairs of the form $(1,1,0,0)$, $(0,0,1,1)$. Let q be the quadratic form corresponding to the Dynkin diagram of type A_4. Then $q(1,0,1,0) = q(0,1,0,1) = q(1,0,0,1) = 2$ and $q(r) = 1$ for all other types of rows r. Since $(\alpha_3 - 1)/2 \ge \alpha/3$ and $(\alpha_1 - 1)/2 \le \alpha/3$, it follows that

$$\sum_{1\le i\le j\le 4} q_{ij}c_{ij} = \sum_{\chi\in\operatorname{Irr}(H)} q(r_\chi) \le 4 + \frac{2}{3}\alpha + \alpha + \frac{4}{3}\alpha + \beta = 4 + 3\alpha + \beta \le 1 + 2\alpha + 4\beta = |\widehat{D}| = |D|.$$

Finally assume that $A \cong S_3$. Then $p \ge 5$. We may arrange the three irreducible Brauer characters of H such that their degrees are $(1,2,1)$. As above we get three rows in the decomposition matrix $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Again we consider the action of A on $\widehat{D}$. Let α be the number of orbits of length 2, let β the number of orbits of length 3, and let γ be the number of regular orbits. Then we get α triples of rows $(0,1,0)$, $(0,1,0)$, $(1,0,1)$, β pairs of rows $(1,1,0)$, $(0,1,1)$, and γ rows of the form $(1,2,1)$ in the decomposition matrix of H. Let q be the quadratic form corresponding to the Dynkin diagram of type A_3. We discuss some special cases separately. In case $\alpha = 0$ we obtain with the notation introduced above:

$$\sum_{1\le i\le j\le 3} q_{ij}c_{ij} = \sum_{\chi\in\operatorname{Irr}(H)} q(r_\chi) = 3 + 2\beta + 2\gamma \le 1 + 3\beta + 6\gamma = |D|.$$

Thus, in the following we suppose that $\alpha > 0$. Let $h \in A$ an element of order 3 and $A_1 := \mathrm{C}_{\widehat{D}}(h)$. Obviously, $\alpha = (|A_1| - 1)/2 \ge 2$, since $p \ge 5$. We denote the three involutions in A by g_1, g_2 and g_3. Moreover, let

$B_i := \mathrm{C}_{\widehat{D}}(g_i)$. It is easy to see that h permutes the sets B_1, B_2 and B_3 transitively. In particular, $\beta = |B_i| - 1$. Also, $A_1 \cap B_1 = 1$ and $A_1 \times B_1 \leq \widehat{D}$. We conclude that

$$\gamma = \frac{|D| - 2\alpha - 3\beta - 1}{6} \geq \frac{(2\alpha+1)(\beta+1) - 2\alpha - 3\beta - 1}{6} = \frac{\alpha\beta - \beta}{3}.$$

In case $\beta > 0$ we even have $\beta \geq p - 1 \geq 4$ and $\gamma \geq 2$. Then it follows that $\alpha \leq 3\gamma/\beta + 1 \leq 3\gamma - 2$. For $\beta = 0$ we still have $|D| \geq (2\alpha + 1)p$ and $\gamma \geq 2(2\alpha + 1)/3$. So in any case the inequality

$$\alpha \leq 3\gamma - 2$$

holds. Now we change the ordering of the Brauer characters such that their degrees are $(1, 1, 2)$. Then as above

$$\sum_{1 \leq i \leq j \leq 3} q_{ij} c_{ij} = \sum_{\chi \in \mathrm{Irr}(H)} q(r_\chi) = 3 + 3\alpha + 3\beta + 3\gamma \leq 1 + 2\alpha + 3\beta + 6\gamma = |D|.$$

This finishes the proof. □

By Section 3 in [21] it is known that Lemma 3 fails for example for $A \cong C_3^2$. Our next lemma is quite technical, but powerful.

Lemma 4. *Let B be a p-block of a finite group with defect group D. If there exists an element $x \in \mathrm{Z}(D)$ such that $D/\langle x\rangle$ is abelian, and $|\mathrm{C}_{I(B)}(x)| \leq 4$ or $\mathrm{C}_{I(B)}(x) \cong S_3$, then Brauer's $k(B)$-Conjecture holds for B.*

Proof. We consider a B-subsection (x, b_x). The aim of the proof is to apply Theorem 2.4 in [9] in connection with Lemma 3. Let C be the Cartan matrix of b_x. As usual, b_x dominates a block $\overline{b_x}$ with abelian defect group $\overline{D} := D/\langle x\rangle$, Cartan matrix $\overline{C} := \frac{1}{|\langle x\rangle|} C = (c_{ij})$, and $I(\overline{b_x}) \cong \mathrm{C}_{I(B)}(x)$. By work of Usami and Puig [22, 16, 15, 23] there exists a perfect isometry between $\overline{b_x}$ and its Brauer correspondent with normal defect group. By Theorem 4.11 in [3] the Cartan matrices are preserved under perfect isometries up to basic sets. Thus, we may assume that $\overline{b_x}$ has normal defect group $\overline{D}$. By [12], $\overline{b_x}$ is Morita equivalent to the group algebra $F[\overline{D} \rtimes I(\overline{b_x})]$ (where F is an algebraically closed field of characteristic p) except possibly if $I(\overline{b_x}) \cong C_2^2$ (which has non-trivial Schur multiplier $\mathrm{H}^2(C_2^2, F^\times) \cong C_2$). Let us first handle this exceptional case. Here $\overline{b_x}$ is Morita equivalent to a (non-trivial) twisted group algebra $F_\gamma[\overline{D} \rtimes C_2^2]$ where the 2-cocycle γ is uniquely determined. By Lemma 5.5 and Proposition 5.15 in [14] we can treat the twisted group algebra as a block algebra. More precisely, the Cartan matrix of $\overline{b_x}$ is the same as the Cartan matrix of a non-principal block of a group of type $\overline{D} \rtimes D_8$ (note that D_8 is a covering group of C_2^2; the other covering group Q_8 would lead to the same conclusion). The group algebra of $\overline{D} \rtimes D_8$ has $k(D_8) = 5$ irreducible Brauer characters. Four of them lie in the principal block. Therefore, the Cartan matrix of $\overline{b_x}$ is a 1×1 matrix. Hence, we are done in the exceptional case.

Now assume that $\overline{b_x}$ is Morita equivalent to FH where $H := \overline{D} \rtimes I(\overline{b_x})$. Then by Lemma 3 there is a positive definite quadratic form $q = \sum_{1 \leq i \leq j \leq k(\overline{b_x})} q_{ij} x_i x_j$ such that

$$\sum_{1 \leq i \leq j \leq k(\overline{b_x})} q_{ij} c_{ij} \leq |\overline{D}|.$$

The result follows easily by Theorem 2.4 in [9]. □

The following lemma generalizes Corollary 1.2(ii) in [18].

Lemma 5. *Let B be a block of a finite group with abelian defect group D. If $I(B)$ contains an abelian subgroup of index at most 4, then Brauer's $k(B)$-Conjecture holds for B.*

Proof. Let $A \leq I(B)$ be abelian such that $|I(B) : A| \leq 4$. It is well-known that A has a regular orbit on D, i. e. there exists an element $x \in D$ such that $\mathrm{C}_A(x) = 1$. Hence, $|\mathrm{C}_{I(B)}(x)| \leq 4$, and the claim follows from Lemma 4. □

We remark that Lemma 5 also holds under the more general hypothesis that $I(B)$ contains a subgroup R of index at most 4 such that R has a regular orbit on D. For example, if R is nilpotent, one can use [25]. Since many non-abelian groups also guarantee regular orbits, it is worthwhile to study small groups with this property in detail. In the following Lemma we make use of the "small group library" available in GAP [5].

Lemma 6. *Let A act faithfully on the finite group P such that $(|A|,|P|) = 1$. If $|A| < 100$ and A is* not *isomorphic to* `SmallGroup`(n,i) *where (n,i) is one of the pairs given in the appendix, then A has a regular orbit on P.*

Proof. The proof is computer assisted. By Lemma 2.6.2 in [8] we may assume that P is an elementary abelian p-group, i. e. a vector space over $\mathbb{F}_p$. By Maschke's Theorem, P decomposes into a direct sum $P = P_1 \times \ldots \times P_n$ of irreducible A-invariant subgroups P_i. Assume that we have already found elements $x_i \in P_i$ such that $\mathrm{C}_A(x_i) \subseteq \mathrm{C}_A(P_i)$ for $i = 1,\ldots,n$. Then the element $x := x_1 \ldots x_n$ fulfills $\mathrm{C}_A(x) = 1$ and we are done. Hence, we may replace A by $A/\mathrm{C}_A(P_1)$ and P by P_1, i. e. the action is faithful and irreducible. Let $\mathcal{M}$ be the set of subgroups of A of prime order. If A has no regular orbit on P, we have

$$P = \bigcup_{M \in \mathcal{M}} \mathrm{C}_P(M).$$

Since P cannot be the union of p proper subgroups, it follows that $p < |\mathcal{M}| < |A|$. Now there are only finitely many possibilities for the action of A on P and we compute these with GAP (which in turn uses Meataxe routines).

In few cases (namely $A \cong D_m$ where $m \in \{46, 50, 58, 74, 82, 86, 92, 94, 98\}$) not all irreducible representations are available immediately, since certain Conway polynomials are unknown. However, for most of these cases (except D_{50}, D_{92} and D_{98}) we can use a simpler argument described below: We have $|A| = 2q$ for some odd prime q. Let $S \in \mathrm{Syl}_q(A)$ and $\mathrm{Syl}_2(A) = \{T_1,\ldots,T_q\}$. Then S permutes the $\mathrm{C}_P(T_i)$ transitively. If A has no regular orbit, we obtain $P = \mathrm{C}_P(S) \cup \mathrm{C}_P(T_1) \cup \ldots \cup \mathrm{C}_P(T_q)$ and

$$|P| = p^a + qp^b - q$$

where $|\mathrm{C}_P(S)| =: p^a$ and $|\mathrm{C}_P(T_i)| =: p^b$. Since S has a regular orbit, we have $b \geq 1$. Evaluating the equation modulo p implies $a = 0$ and $p \mid q - 1$. Now it is easy to see that this cannot hold.

For $A \cong D_{92}$ things are a bit more complicated. We have $p < |\mathcal{M}| = 48$, and GAP shows that there is a regular orbit provided $p \notin \{17, 19, 37, 43\}$ (in these four cases the order of p modulo 23 is 22). We may also assume that A acts faithfully and irreducibly on P, since we already know that proper quotients of A have regular orbits. Suppose first that there exists $1 \neq x \in P$ such that $23 \mid |\mathrm{C}_A(x)|$. Then the orbit of x has at most 4 elements $x_1,\ldots,x_4$. Moreover, $x_1 \ldots x_4 \in \mathrm{C}_P(A) = 1$. Thus, $x_1,\ldots,x_4$ are not linearly independent. Since A acts irreducibly, $P = \langle x_1,\ldots,x_4\rangle \leq \mathbb{F}_p^3$. It follows that $23 \mid (p-1)(p^2-1)(p^3-1)$ and $p = 47$. This was already excluded. Hence, $|\mathrm{C}_A(x)| \in \{2,4\}$ for all $1 \neq x \in P$. Let $\{g_1,\ldots,g_{23}\}$, $\{h_1,\ldots,h_{23}\}$ and $\{z\}$ be the three conjugacy classes of involutions in A. Define $|\mathrm{C}_P(g_i)| =: p^a$, $|\mathrm{C}_P(h_i)| =: p^b$ and $|\mathrm{C}_P(z)| =: p^c$. Then we have $|\mathrm{C}_P(g_i) \cap \mathrm{C}_P(z)| = |\mathrm{C}_P(h_j) \cap \mathrm{C}_P(z)| =: p^d$ for $i,j = 1,\ldots,23$. Moreover $\mathrm{C}_P(g_i) \cap \mathrm{C}_P(g_j) = \mathrm{C}_P(h_i) \cap \mathrm{C}_P(h_j) = 1$ for $i \neq j$, and

$$|\mathrm{C}_P(g_i) \cap \mathrm{C}_P(h_j)| = \begin{cases} p^d & \text{if } g_i = h_j z, \\ 1 & \text{otherwise.} \end{cases}$$

Now the principle of inclusion and exclusion implies

$$\begin{aligned} P &= 23p^a + 23p^b + p^c - 3 \cdot 23p^d - \left(\binom{47}{2} - 3 \cdot 23\right) + 23p^d + \left(\binom{47}{3} - 23\right) + \sum_{i=4}^{47} (-1)^{i-1} \binom{47}{i} \\ &= 23p^a + 23p^b + p^c - 2 \cdot 23p^d. \end{aligned}$$

Obviously, $d \leq \min\{a,b,c\}$. After dividing by p^d, we may assume that $d = 0$. Since $p \notin \{2, 23\}$, at least one of a, b or c also vanishes. In fact we must have $c = 0$. Evaluating modulo p gives $p \in \{3, 5, 11\}$. All these cases are already checked by GAP.

Next, let $A \cong D_{98}$. Only the primes $p \in \{17, 23, 37, 47\}$ cause problems. Suppose first that there is an element $x \in P$ such that $|\mathrm{C}_A(x)| = 14$. Then we may assume $|P| \leq p^6$. We may also assume that A acts faithfully.

However, $\mathrm{GL}(6,p)$ does not contain a cyclic subgroup of order 49. Hence, we may assume that $|\mathrm{C}_A(x)| \in \{2,7\}$ for all $1 \neq x \in P$. It follows easily that

$$|P| = p^a + 49p^b - 49$$

for some $a, b \in \mathbb{Z}$ and $b \geq 1$. This gives $p = 3$ which was already excluded. The proof for $A \cong D_{50}$ is completely similar.

We give some improvements of the algorithm above. If $\mathrm{Z}(A)$ is not cyclic, it is well-known that A has no faithful, irreducible representation. Hence, it suffices to consider proper quotients here. Conversely, if A has only one minimal normal subgroup, then it is enough to look at the faithful irreducible representations of A. If A has no regular orbit, then P is usually pretty small (we use $|P| \leq 10^5$ in our implementation). If on the other hand P is large, then there are usually many regular orbits. In this case we pick elements $x \in P$ randomly and check $\mathrm{C}_A(x) = 1$. This is much faster than going through P as a list. Since we only check a sufficient condition we may miss some groups which also have regular orbits. In order to find more groups we do the following. Make a list of all subgroups of A which have regular orbits (i. e. groups we have already found). For each subgroup H on this list check if there is another subgroup $K \neq 1$ such that $H \cap K = 1$. If not, A must have a regular orbit. This gives us new groups with regular orbits and we can even repeat the procedure. □

Notice that we have not proved the converse of Lemma 6. For example, we do not know whether the group `SmallGroup`$(32, 30)$ must have regular orbits or not (although it can probably be figured out if needed). The problem here is that there are non-faithful irreducible representations without regular orbits.

One can show that more than two thirds of the groups of order less than 100 provide regular orbits in the situation above (for this reason we list the complementary set in the appendix). Lemma 6 will be applied later in Proposition 10, but we need to settle a special case for $p = 2$ first.

Lemma 7. *Let A be a p'-automorphism group of an abelian p-group $P \cong \prod_{i=1}^n C_{p^i}^{m_i}$. Then A is isomorphic to a subgroup of*

$$\prod_{i=1}^n \mathrm{GL}(m_i, p)$$

where $\mathrm{GL}(0,p) := 1$.

Proof. As a p'-group, A acts faithfully on $P/\Phi(P)$. Hence, the canonical homomorphism

$$A \longrightarrow \prod_{i=1}^n \mathrm{Aut}(\Omega_{n-i+1}(P)\Phi(P)/\Omega_{n-i}(P)\Phi(P)) \tag{2}$$

where $\Omega_i(P) := \langle x \in P : x^{p^i} = 1\rangle$ for $i \geq 0$ is injective. Since $\Omega_i(P)\Phi(P)/\Omega_{i-1}(P)\Phi(P)$ is elementary abelian of rank m_i for $i = 1, \ldots, n$, the claim follows. □

Combining Lemma 5 and Lemma 7 gives the following result which is probably not new.

Corollary 8. *Let B be a p-block of a finite group with abelian defect group $D \cong \prod_{i=1}^n C_{p^i}^{m_i}$ such that $m_i \leq 1$ for $i = 1, \ldots, n$. Then Brauer's $k(B)$-Conjecture holds for B.*

Now we turn to abelian p-groups with homocyclic factors. Here it is necessary to restrict p.

Theorem 9. *Let B be a 2-block of a finite group with abelian defect group $D \cong \prod_{i=1}^n C_{2^i}^{m_i}$. Assume that one of the following holds:*

(i) For some $i \in \{1, \ldots, n\}$ we have $m_i \leq 4$ and $m_j \leq 2$ for all $j \neq i$.

(ii) D has rank 5.

Then Brauer's $k(B)$-Conjecture holds for B.

Proof.

(i) For each $k \in \{1, \ldots, n\}$ we define A_k to be the image of the canonical map

$$I(B) \longrightarrow \operatorname{Aut}(\Omega_{n-k+1}(D)\Phi(D)/\Omega_{n-k}(D)\Phi(D)) \cong \operatorname{GL}(m_k, p).$$

Then we can refine the monomorphism from Eq. (2) to $I(B) \to \prod_{k=1}^n A_k$. Since $\operatorname{GL}(2,2) \cong S_3$, we have $A_j \le C_3$ for $j \ne i$. In order to apply Lemma 5, it suffices to show that $A_i \le \operatorname{GL}(4,2)$ contains an abelian subgroup of index at most 4. Since A_i has odd order, we have $|A_i| \mid (2^4-1)(2^3-1)(2^2-1) = 3^2 \cdot 5 \cdot 7$. It can be seen further that $|A_i| \in \{1, 3, 5, 7, 9, 15, 21\}$. The claim follows.

(ii) Now assume that D has rank 5. The case $|D| = 32$ was already handled in Corollary 2 in [21]. Thus, by part (i) we may assume that $C_4^5 \le D$ and $I(B) \le \operatorname{GL}(5,2)$. As usual, $e(B)$ is a divisor of $3^2 \cdot 5 \cdot 7 \cdot 31$. Suppose first that $31 \mid e(B)$. One can show that every group whose order divides $3^2 \cdot 5 \cdot 7 \cdot 31$ has a normal Sylow 31-subgroup. Therefore $I(B)$ lies in the normalizer of a Sylow 31-subgroup of $\operatorname{GL}(5,2)$. Thus, we may assume $e(B) = 31 \cdot 5$. Here Lemma 5 does not apply. However, we can still show the existence of a regular orbit. Obviously, $I(B)$ cannot have a regular orbit on $D/\Phi(D) \cong C_2^5$. However, using GAP one can show that $I(B)$ has a regular orbit on $\Omega_2(D) \cong C_4^5$. So we can find a subsection (u, b_u) such that $l(b_u) = 1$. The claim follows in this case.

Now we can assume that $31 \nmid e(B)$. In case $7 \mid e(B)$ we see again that $I(B)$ has a normal Sylow 7-subgroup and $e(B) = 3^2 \cdot 7$ without loss of generality. It is easy to see that every group of order $3^2 \cdot 7$ has an abelian subgroup of index 3. Thus, we may finally suppose that $7 \nmid e(B)$. Then $I(B)$ is abelian itself. This completes the proof. □

Theorem 9 improves an unpublished result by Robinson [17]. In the next proposition we investigate how far we can go only by restricting the inertial index.

Proposition 10. *Let B be a block of a finite group with abelian defect group and $e(B) \le 255$. Then the $k(B)$-Conjecture is satisfied for B.*

Proof. Let $I(B)$ be an arbitrary group of order at most 255, and let D be a defect group of B. We compute with GAP the set $\mathcal{L}$ of subgroups of $I(B)$ which have order less than 100 and are not on the list in the appendix. For every $H \in \mathcal{L}$ we check the following condition:

$$\forall K \le I(B) : K \cap H = 1 \Longrightarrow |K| \le 4 \lor K \cong S_3. \qquad (*)$$

By Lemma 6 there is $x \in D$ such that $\mathrm{C}_{I(B)}(x) \cap H = \mathrm{C}_H(x) = 1$. Hence, if Condition $(*)$ is true for some $H \in \mathcal{L}$, we get $|\mathrm{C}_{I(B)}(x)| \le 4$ or $\mathrm{C}_{I(B)}(x) \cong S_3$. Then the $k(B)$-Conjecture follows from Lemma 4. It turns out that $(*)$ is false for only a few groups which will be handled case by case.

For $I(B) \cong \operatorname{SL}(2,5)$ the algorithm from Lemma 6 shows that $I(B)$ has in fact a regular orbit on D. In case $I(B) \cong 5_+^{1+2}$ (extraspecial of order 125 and exponent 5) the same is true by the main result of [25].

Now assume $I(B) \cong C_{31} \rtimes C_5$. Then one can show that we have a regular orbit unless $p = 2$. Thus, let $p = 2$. We study the (faithful) action of $I(B)$ on $\Omega(D)$. By Theorem 9 we may assume $|\Omega(D)| \ge 2^6$. A GAP calculation shows that $I(B)$ has eight irreducible representations over $\mathbb{F}_2$ and their degrees are $1, 4, 5, \ldots, 5$. Moreover, the image of the second representation has order 5 while the last six representations are faithful. In particular the action of $I(B)$ on $\Omega(D)$ is not irreducible. So we decompose $\Omega(D) = V_1 \times \ldots \times V_n$ into irreducible $I(B)$-invariant subgroups V_i. Without loss of generality, V_1 is faithful. Hence, we find an element $v_1 \in V_1$ such that $\mathrm{C}_{I(B)}(v_1)$ has order 5. If there is at least one more non-trivial summand, say V_2, we find another element $v_2 \in V_2$ such that $\mathrm{C}_{I(B)}(v_1) \nsubseteq \mathrm{C}_{I(B)}(v_2)$. It follows that $\mathrm{C}_{I(B)}(v) = 1$ for $v := v_1 v_2$. Therefore, we may assume that $I(B)$ acts trivially on $V_2 \times \ldots \times V_n$. By Theorem 5.2.3 in [6], also D decomposes as $D = \mathrm{C}_D(I(B)) \times [D, I(B)]$. It follows that $[D, I(B)] \cong C_{2^a}^5$ for some $a \ge 1$. In case $a \ge 2$ we have seen in the proof of Theorem 9 that $I(B)$ has a regular orbit on $[D, I(B)]$. Hence, $[D, I(B)]$ is elementary abelian of order 32. Define $|\mathrm{C}_D(I(B))| =: 2^k$. Then B has 2^{k+1} subsections up to conjugation. Half of them have inertial index 155 while the other half have inertial index 5. Let (u, b_u) be one of the B-subsections with $I(b_u) \cong I(B)$. In order to determine $l(b_u)$ we may suppose that $\mathrm{C}_D(I(B)) = 1$ by a theorem of Watanabe [24] (applied inductively). Now take a non-trivial b_u-subsection (v, β_v). Then the Cartan matrix of β_v is given by

$$2\begin{pmatrix}4&3&3&3&3\\3&4&3&3&3\\3&3&4&3&3\\3&3&3&4&3\\3&3&3&3&4\end{pmatrix}$$

(see Case 2 in the proof of Proposition 1 in [20]). Theorem 2.4 in [9] gives $k(b_u) \le 16$. Since (v, β_v) is the only non-trivial b_u-subsection up to conjugation, we obtain $l(b_u) \le 11$. Similarly we can show that $l(b_u) \le 5$ if (u, b_u) is a B-subsection such that $e(b_u) = 5$. Now we get $k(B) \le 2^k \cdot 11 + 2^k \cdot 5 = 2^{k+4} \le |D|$, because $k(B)$ is the sum over the numbers $l(b_u)$ (see Theorem 5.9.4 in [13]). This completes the case $e(B) = 155$.

The next exceptional group is $I(B) \cong \texttt{SmallGroup}(160, 199)$. Here $\mathrm{Z}(I(B))$ is the unique minimal normal subgroup of $I(B)$. In particular every faithful representation contains a faithful, irreducible representation as a direct summand. Using GAP we show that only the prime $p = 3$ is "interesting". If $I(B)$ acts faithfully and irreducibly on D, then one can find an element $x \in D$ such that $|\mathrm{C}_{I(B)}(x)| \le 2$. Therefore, the $k(B)$-Conjecture follows from Lemma 4. We also need to discuss another group of the same order, namely $I(B) \cong \texttt{SmallGroup}(160, 207) \cong D_8 \times (C_5 \rtimes C_4)$. Here we may assume again that $p = 3$. Let us consider the representations of the subgroup $H \cong \texttt{SmallGroup}(80, 30) \cong C_4 \times (C_5 \rtimes C_4)$. This subgroup has just one irreducible faithful representation, and this representation provides regular orbits. Now assume that H acts faithfully on D without regular orbits. Then D decomposes into irreducible H-invariant subgroups $V_1, \ldots, V_n$ where $n \ge 2$. Without loss of generality, the V_i are distinct as $\mathbb{F}_3H$-modules. A calculation shows that there is only one index i such that no element $x \in V_i$ with $\mathrm{C}_H(x) = \mathrm{C}_H(V_i)$ exists (this is the reason why H appears on the list in the appendix). We may assume that $i = 1$. Then $|\mathrm{C}_H(V_1)| = 2$. We conclude that the situation can be reduced to the case $n = 2$. However, then it can be shown that H has a regular orbit on D. Now the $k(B)$-Conjecture for $I(B)$ follows from Lemma 4 as before.

We continue with $I(B) \cong \texttt{SmallGroup}(168, 42) \cong \mathrm{GL}(3, 2)$. Here the algorithm of Lemma 6 shows that $I(B)$ has regular orbits. The next interesting groups $I(B)$ are non-abelian of order $203 = 29 \cdot 7$ and $205 = 41 \cdot 5$. Here the arguments for the dihedral groups in Lemma 6 work. Then we have four groups $I(B) \cong \texttt{SmallGroup}(240, i)$ for $i = 89, 90, 93, 94$. For $i = 89, 90, 94$ there are always regular orbits. Now let $i = 93$. Then $I(B) \le \mathrm{GL}(2, 5)$ and the subgroup $\mathrm{SL}(2, 5)$ provides a regular orbit as we have seen above. In the same way we handle some groups of order 250 which have 5_+^{1+2} as Sylow 5-subgroup. Finally, the non-abelian group of order $253 = 23 \cdot 11$ is also easy to handle. This finishes the whole proof. □

For $e(B) = 256$ the arguments in Proposition 10 fail as one can see by the following example. There is a subgroup $A \le \mathrm{GL}(4, 3)$ of order 256 such that C_3^4 splits under the action of A into orbits of lengths 1, 16, 32 and 32. Hence, the corresponding stabilizers have order at least 8.

4 Odd primes

In this section we focus on odd primes p. The next theorem handles the $k(B)$-Conjecture for 3-blocks with abelian defect groups of rank at most 3 as a special case.

Theorem 11. *Let B be a 3-block of a finite group with defect group $D \cong \prod_{i=1}^n C_{3^i}^{m_i}$ such that for two $i, j \in \{1, \ldots, n\}$ we have $m_i, m_j \le 3$, and $m_k \le 1$ for all $i \ne k \ne j$. Then Brauer's $k(B)$-Conjecture holds for B.*

Proof. As in the proof of Theorem 9 we may assume that $I(B) \le \mathrm{GL}(3, 3) \times \mathrm{GL}(3, 3)$. By Lemma 5, it suffices to show that every $3'$-subgroup of $\mathrm{GL}(3, 3)$ has an abelian subgroup of index at most 2. In order to do so, we may assume $I(B) \le \mathrm{GL}(3, 3)$. Then $e(B)$ is a divisor of $(3^3 - 1)(3^2 - 1)(3 - 1) = 2^5 \cdot 13$. In case $13 \mid e(B)$, Sylow's Theorem shows that $I(B)$ has a normal Sylow 13-subgroup. Hence, $I(B)$ lies in the normalizer of the Sylow 13-subgroup in $\mathrm{GL}(3, 3)$. Thus, $e(B) = 2 \cdot 13$ without loss of generality. The claim holds. Suppose next that $I(B)$ is a 2-group. It can be shown that a Sylow 2-subgroup of $\mathrm{GL}(3, 3)$ is isomorphic to $SD_{16} \times C_2$; so it contains an abelian maximal subgroup. Obviously the same holds for $I(B)$ and the claim follows. □

For $p = 5$ it is necessary to restrict the rank of the defect group.

Theorem 12. *Let B be a 5-block of a finite group with abelian defect group of rank 3. Then Brauer's $k(B)$-Conjecture holds for B.*

Proof. We consider the (faithful) action of $I(B)$ on $\Omega(D) \cong C_5^3$. In particular, $I(B) \leq \mathrm{GL}(3,5)$. Fortunately, GAP is able to compute a set of representatives for the conjugacy classes of $5'$-subgroups of $\mathrm{GL}(3,5)$. In particular we obtain $e(B) \mid 2^7 \cdot 3$ or $e(B) \mid 2^2 \cdot 3 \cdot 31$. A further analysis shows that there is an element $x \in \Omega(D)$ such that $|\mathrm{C}_{I(B)}(x)| \leq 4$ or $\mathrm{C}_{I(B)}(x) \cong S_3$. The claim follows by Lemma 4. □

For the defect group C_7^3 the proof above would not work. More precisely, it is possible here that $I(B)$ has order 6^4, the largest orbit on D has length 6^3, and the corresponding stabilizer is isomorphic to C_6. Hence, the existence of a perfect isometry for b_x is unknown.

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Appendix

The following table is needed in Lemma 6.

size	id	size	id	size	id	size	id	size	id	size	id	size	id	size	id
8	3	48	7	64	95	64	174	64	251	80	36	96	79	96	136
12	4	48	14	64	97	64	176	64	253	80	37	96	80	96	137
16	7	48	15	64	98	64	177	64	254	80	38	96	81	96	138
16	8	48	17	64	99	64	178	64	255	80	39	96	82	96	139
16	11	48	25	64	101	64	186	64	258	80	40	96	83	96	144
16	13	48	29	64	115	64	187	64	261	80	41	96	87	96	145
20	4	48	33	64	116	64	189	64	263	80	42	96	88	96	146
21	1	48	35	64	117	64	196	64	265	80	44	96	89	96	147
24	5	48	36	64	118	64	198	72	5	80	46	96	90	96	148
24	6	48	37	64	119	64	201	72	6	80	50	96	91	96	149
24	8	48	38	64	121	64	202	72	8	80	51	96	92	96	153
24	14	48	39	64	123	64	203	72	17	81	7	96	93	96	154
28	3	48	40	64	124	64	205	72	20	84	8	96	98	96	155
32	9	48	41	64	128	64	206	72	21	84	12	96	99	96	156
32	11	48	43	64	129	64	207	72	22	84	13	96	100	96	157
32	19	48	47	64	130	64	210	72	23	84	14	96	101	96	158
32	25	48	48	64	131	64	211	72	25	88	5	96	102	96	160
32	27	48	51	64	133	64	213	72	27	88	7	96	103	96	168
32	28	52	4	64	134	64	215	72	28	88	9	96	104	96	179
32	30	56	4	64	137	64	216	72	30	93	1	96	105	96	186
32	31	56	5	64	138	64	217	72	32	96	4	96	106	96	187
32	34	56	7	64	140	64	218	72	33	96	5	96	107	96	189
32	39	56	9	64	141	64	219	72	35	96	6	96	108	96	192
32	40	56	12	64	142	64	220	72	46	96	7	96	109	96	195
32	42	60	12	64	144	64	221	72	48	96	12	96	110	96	200
32	43	63	3	64	145	64	223	72	49	96	13	96	111	96	206
32	46	64	6	64	146	64	226	76	3	96	16	96	113	96	207
32	48	64	8	64	147	64	227	80	4	96	27	96	114	96	208
32	50	64	10	64	149	64	228	80	5	96	28	96	115	96	209
36	4	64	12	64	150	64	229	80	6	96	30	96	116	96	210
36	10	64	32	64	152	64	230	80	7	96	32	96	117	96	211
36	12	64	34	64	155	64	231	80	14	96	33	96	118	96	212
36	13	64	38	64	157	64	232	80	15	96	34	96	119	96	213
40	5	64	41	64	159	64	233	80	16	96	35	96	120	96	214
40	6	64	52	64	161	64	234	80	17	96	44	96	121	96	215
40	8	64	67	64	162	64	235	80	25	96	54	96	122	96	216
40	10	64	71	64	163	64	236	80	26	96	61	96	123	96	217
40	12	64	73	64	167	64	237	80	28	96	62	96	124	96	219
40	13	64	75	64	169	64	240	80	29	96	64	96	125	96	223
48	4	64	89	64	170	64	243	80	30	96	67	96	126	96	226
48	5	64	90	64	171	64	244	80	31	96	68	96	134	96	230
48	6	64	92	64	173	64	250	80	34	96	78	96	135		

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