

Cartan matrices and Brauer's $k(B)$ -conjecture

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Abstract

It is well known that the Cartan matrix of a block of a finite group cannot be arranged as a direct sum of smaller matrices. In this paper we address the question if this remains true for equivalent matrices. The motivation for this question comes from the work [10], which contains certain bounds for the number of ordinary characters in terms of Cartan invariants. As an application we prove such a bound in the special case, where the determinant of the Cartan matrix coincides with the order of the defect group. In the second part of the paper we show that Brauer's $k(B)$ -conjecture holds for 2-blocks under some restrictions on the defect group. For example, the $k(B)$ -conjecture holds for 2-blocks if the corresponding defect group is a central extension of a metacyclic group by a cyclic group. The same is true if the defect group contains a central cyclic subgroup of index 8. In particular the $k(B)$ -conjecture holds for 2-blocks with defect at most 4. The paper is a part of the author's PhD thesis.

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Let G be a finite group and let B be a p -block of G . We denote the number of ordinary irreducible characters of B by $k(B)$. Similarly, $l(B)$ is the number of irreducible Brauer characters of B . Moreover, let d be the defect of B .

It is well known that the Cartan matrix C of B is indecomposable as integer matrix, i. e. there is no arrangement of the indecomposable projective modules such that C splits into a direct sum of smaller matrices (recall that C is symmetric).

We call two matrices $A, B \in \mathbb{Z}^{l \times l}$ *equivalent* if there exists a matrix $S \in \text{GL}(l, \mathbb{Z})$ with $A = S^T B S$, where S^T denotes the transpose of S . Every symmetric matrix gives rise to a quadratic form. In this sense equivalent matrices describe equivalent quadratic forms. Richard Brauer describes equivalence of Cartan matrices via so called "basic sets". He also studied Cartan matrices by applying the theory of quadratic forms (see [2]). In general the property "being indecomposable" is not shared among equivalent matrices. For example $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is indecomposable, but $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^T A \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not. However, we were not able to find a Cartan matrix of a block which provides an equivalent decomposable matrix. So we raise the question:

Question. *Do there exist a Cartan matrix C of a block B and a matrix $S \in \text{GL}(l(B), \mathbb{Z})$ such that $S^T C S$ is decomposable?*

The motivation for this question comes from the fact that $k(B)$ can be bounded in terms of Cartan invariants (see [10]). These bounds are usually invariant under equivalence of matrices. The point is that the inequalities

are significantly sharper for indecomposable matrices. We illustrate this fact with an example. Let $l(B) = 2$ and assume that the elementary divisors of C are 2 and 16. Then C has the form

$$\begin{pmatrix} 2 & 0 \\ 0 & 16 \end{pmatrix} \text{ or } \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$$

up to equivalence. In the first case one can deduce $k(B) \leq 18$, while in the second case $k(B) \leq 10$ holds.

We give an affirmative answer to the question in two special cases.

Lemma 1. *Let G be p -solvable and $l := l(B) \geq 2$. Then there is no matrix $S \in \text{GL}(l, \mathbb{Z})$ such that $S^T C S = \begin{pmatrix} p^d & 0 \\ 0 & C_1 \end{pmatrix}$ with $C_1 \in \mathbb{Z}^{(l-1) \times (l-1)}$. In particular C is not equivalent to a diagonal matrix.*

Proof. Assume the contrary, i. e. there is a matrix $S = (s_{ij}) \in \text{GL}(l, \mathbb{Z})$ such that

$$C = (c_{ij}) = S^T \begin{pmatrix} p^d & 0 \\ 0 & C_1 \end{pmatrix} S$$

with $C_1 \in \mathbb{Z}^{(l-1) \times (l-1)}$. Let $s_i := (s_{2i}, s_{3i}, \dots, s_{li})$ for $i = 1, \dots, l$. By Theorem (3H) in [6] we have

$$p^d s_{i1}^2 + s_i C_1 s_i^T = c_{ii} \leq p^d$$

for $i = 1, \dots, l$. Since S is invertible, there exists i such that $s_{1i} \neq 0$. We may assume $s_{11} \neq 0$. Then $s_{11} = \pm 1$ and $s_1 = (0, \dots, 0)$, because C_1 is positive definite. Now all other columns of S are linearly independent of the first column. This gives $s_{1i} = 0$ for $i = 2, \dots, l$. Hence, S has the form $S = \begin{pmatrix} \pm 1 & 0 \\ 0 & S_1 \end{pmatrix}$ with $S_1 \in \text{GL}(l-1, \mathbb{Z})$. But then C also has the form $\begin{pmatrix} p^d & 0 \\ 0 & C_2 \end{pmatrix}$ with $C_2 \in \mathbb{Z}^{(l-1) \times (l-1)}$, a contradiction. The second claim follows at once, since p^d is always an elementary divisor of C . \square

Unfortunately the bound for the Cartan invariants used in the proof does not hold for arbitrary groups (see [11]).

Lemma 2. *If $\det C = p^d$, then for every $S \in \text{GL}(l(B), \mathbb{Z})$ the matrix $S^T C S$ is indecomposable.*

Proof. Again assume the contrary, i. e. there is a matrix $S \in \text{GL}(l(B), \mathbb{Z})$ such that

$$C = S^T \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} S$$

with $C_1 \in \mathbb{Z}^{m \times m}$ and $C_2 \in \mathbb{Z}^{(l-m) \times (l-m)}$, where $l := l(B)$ and $1 \leq m < l$. In particular $l < k(B) =: k$, because $l \geq 2$. Since $\det C = p^d$, the elementary divisors of C are 1 and p^d , where p^d occurs with multiplicity one. W.l.o.g. we may assume $\det C_1 = 1$. Let $Q = (q_{ij})$ be the corresponding part of the decomposition matrix, i. e. $Q^T Q = C_1$. By the Binet-Cauchy formula we have

$$1 = \det C_1 = \sum_{\substack{V \subseteq \{1, \dots, k\}, \\ |V|=m}} \det Q_V^T Q_V,$$

where Q_V is the $m \times m$ submatrix consisting of the entries $\{q_{ij} : i \in V, j \in \{1, \dots, m\}\}$. Since $\det Q_V^T Q_V \geq 0$, one summand is 1 while the others are all 0. Thus we may assume, that the first m rows q_1, \dots, q_m of Q are linearly independent. Now consider a row q_i for $i \in \{m+1, \dots, k\}$. Then q_i is a rational linear combination of q_2, \dots, q_m , because q_2, \dots, q_m, q_i are linearly dependent. By the same argument, q_i is also a linear combination of $q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_m$ for $j = 2, \dots, m$. This forces $q_i = (0, \dots, 0)$. Hence, all the rows q_{m+1}, \dots, q_k vanish. Now consider a column $d(u)$ of generalized decomposition numbers, where u is a nontrivial element of a defect group of B . By the orthogonality relations the scalar product of $d(u)$ and an arbitrary column of Q vanishes. This means the first m entries of $d(u)$ must be zero. Since this holds for all columns $d(u)$ with $u \neq 1$, there exists an irreducible character of B which vanishes on the p -singular elements of G . It is well-known that this is equivalent to $d = 0$. But this contradicts $l \geq 2$. \square

As an application, we prove an upper bound for $k(B)$ in the case $\det C = p^d$. In the proof we will use the reduction theory of quadratic forms.

Theorem 1. *If $l(B) \leq 4$ and $\det C = p^d$, then*

$$k(B) \leq \frac{p^d - 1}{l(B)} + l(B).$$

Moreover, this bound is sharp.

Proof. For $l := l(B) = 1$ the assertion is clear (see e.g. Corollary 5 in [13]). So let $l \geq 2$. Let $A = (a_{ij})$ be a reduced matrix in the sense of Minkowski which is equivalent to C (see e.g. [15]). In particular we have $2|a_{ij}| \leq \min\{a_{ii}, a_{jj}\}$ and $1 \leq a_{11} \leq a_{22} \leq \dots \leq a_{ll}$. For convenience we write $\alpha := a_{11}$, $\beta := a_{22}$ and so on.

We are going to apply equation (**) in [10]. In order to do so, we will bound the trace of A from above and the sum $a_{12} + a_{23} + \dots + a_{l-1,l}$ from below.

Let $l(B) = 2$. By Lemma 2 we have $a_{12} \neq 0$ and $a_{12} > 0$ after a suitable change of signs (i.e. replacing A by an equivalent matrix). By [1] we have $4\alpha\beta - \alpha^2 \leq 4p^d$, so that

$$\alpha + \beta \leq \frac{5}{4}\alpha + \frac{p^d}{\alpha} =: f(\alpha). \quad (1)$$

Since $2|a_{ij}| \leq \min\{a_{ii}, a_{jj}\}$, we have $2 \leq \alpha$, and $\alpha \leq \beta$ yields $\alpha \leq 2\sqrt{p^d/3}$. The convex function $f(\alpha)$ takes its maximal value in the interval $[2, 2\sqrt{p^d/3}]$ on one of the two borders. An easy calculation shows $(p^d + 5)/2 = f(2) > f(2\sqrt{p^d/3})$ for $p^d \geq 9$. In case $p^d \leq 6$ only $\alpha = 2$ is possible. In the remaining cases we have $\alpha + \beta \leq f(2)$ for all feasible pairs (α, β) (we call a pair (α, β) feasible if it satisfies inequality (1)). Equation (**) in [10] yields

$$k(B) \leq \alpha + \beta - a_{12} \leq f(2) - 1 = \frac{p^d - 1}{l(B)} + l(B).$$

Let $l(B) = 3$. The same discussion leads to $a_{12} + a_{23} \geq 2$ after a suitable (simultaneous) permutation of rows and columns (i.e. replacing A by $P^T A P$ with a permutation matrix P). It is not always possible to achieve $\alpha \leq \beta \leq \gamma$ additionally. But since the trace of A is symmetric in α , β and γ , we may assume $2 \leq \alpha \leq \beta \leq \gamma$ nevertheless. The inequality in [1] reads

$$4\alpha\beta\gamma - \alpha\beta^2 - \alpha^2\gamma = 2\alpha\beta\gamma + \alpha\beta(\gamma - \beta) + \alpha\gamma(\beta - \alpha) \leq 4p^d,$$

so that

$$\alpha + \beta + \gamma \leq \alpha + \beta + \frac{4p^d + \alpha\beta^2}{4\alpha\beta - \alpha^2} =: f(\alpha, \beta).$$

We describe a set which contains all feasible points. Since $2\alpha^3 \leq 2\alpha\beta\gamma + \alpha\beta(\gamma - \beta) + \alpha\gamma(\beta - \alpha) \leq 4p^d$ we get $2 \leq \alpha \leq \sqrt[3]{2p^d}$. Similarly $4\beta^2 \leq 4p^d$ and $\alpha \leq \beta \leq \sqrt{p^d}$. Thus all feasible points are contained in the convex polygon

$$\mathcal{F} := \{(\alpha, \beta) : 2 \leq \alpha \leq \sqrt[3]{2p^d}, \alpha \leq \beta \leq \sqrt{p^d}\}.$$

It can be shown (maybe with the help of a computer) that f is convex on \mathcal{F} . Hence, the maximal value of f on \mathcal{F} will be attained on one of the 3 vertices:

$$\begin{aligned} V_1 &= (2, 2), \\ V_2 &= (2, \sqrt{p^d}), \\ V_3 &= (\sqrt[3]{2p^d}, \sqrt[3]{2p^d}). \end{aligned}$$

One can check that $(p^d + 14)/3 = f(V_1) \geq f(V_2)$ for $p^d \geq 10$ and $f(V_1) \geq f(V_3)$ for $p^d \geq 12$. If $p^d \leq 10$, then V_1 is the only feasible point. In the remaining case $p^d = 11$ there is only one more feasible pair $(\alpha, \beta) = (2, 3)$. Then $\gamma = 3$ and $\alpha + \beta + \gamma \leq f(V_1)$. Now (**) in [10] takes the form

$$k(B) \leq \alpha + \beta + \gamma - a_{12} - a_{23} \leq f(V_1) - 2 = \frac{p^d - 1}{l(B)} + l(B).$$

Finally, let $l(B) = 4$. By permuting rows and columns and changing signs, we can reach (using Lemma 2) at least one of the two arrangements

$$(i) \quad a_{12} + a_{23} + a_{34} \geq 3,$$

$$(ii) \quad a_{12} + a_{13} + a_{14} \geq 3.$$

In case (i) we can use equation (**) as before. Since the matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

is positive definite, we can use Theorem A in [10] for case (ii). Thus, for the rest of the proof we will assume that case (i) occurs. As before, we will also assume $2 \leq \alpha \leq \beta \leq \gamma \leq \delta$ and

$$\begin{aligned} & 4\alpha\beta\gamma\delta - \alpha^2\gamma\delta - \alpha\beta^2\delta - \alpha\beta\gamma^2 + \frac{1}{4}\alpha^2(\gamma - \beta)^2 \\ &= \alpha\beta\gamma\delta + \alpha\gamma\delta(\beta - \alpha) + \alpha\beta\delta(\gamma - \beta) + \alpha\beta\gamma(\delta - \gamma) + \frac{1}{4}\alpha^2(\gamma - \beta)^2 \leq 4p^d \end{aligned} \quad (2)$$

by [1]. We search for the maximum of the function

$$f(\alpha, \beta, \gamma) := \alpha + \beta + \gamma + \frac{4p^d + \alpha\beta\gamma^2 - \frac{1}{4}\alpha^2(\gamma - \beta)^2}{4\alpha\beta\gamma - \alpha^2\gamma - \alpha\beta^2}$$

on a suitable convex polyhedron. Since $\alpha^4 \leq 4p^d$ we have $2 \leq \alpha \leq \sqrt[4]{4p^d}$. In a similar way, we obtain the set

$$\mathcal{F} := \{(\alpha, \beta, \gamma) : 2 \leq \alpha \leq \sqrt[4]{4p^d}, \alpha \leq \beta \leq \sqrt[3]{2p^d}, \beta \leq \gamma \leq \sqrt{p^d}\},$$

which contains all feasible points. It can be shown that f is in fact convex on \mathcal{F} . The vertices of \mathcal{F} are

$$\begin{aligned} V_1 &:= (2, 2, 2), \\ V_2 &:= (2, 2, \sqrt{p^d}), \\ V_3 &:= (2, \sqrt[3]{2p^d}, \sqrt[3]{2p^d}), \\ V_4 &:= (\sqrt[4]{4p^d}, \sqrt[4]{4p^d}, \sqrt[4]{4p^d}). \end{aligned}$$

We fix the value $m := (p^d + 27)/4$. A calculation shows $f(V_2) \leq m$ for $p^d \geq 22$, $f(V_3) \leq m$ for $p^d \geq 20$, and $f(V_4) \leq m$ for $p^d \geq 23$. If $p^d \leq 12$, then V_1 is the only feasible point. If $p^d \leq 17$, there is only one other feasible point $(\alpha, \beta, \gamma) = (2, 2, 3)$ beside V_1 . In this case $f(2, 2, 3) \leq m$ for $p^d \geq 14$. For $p^d = 13$ we have

$$\alpha + \beta + \gamma + \delta - a_{13} - a_{14} - a_{34} \leq 7 = \frac{13 - 1}{4} + 4.$$

For $p^d \leq 20$ there is one additional point $(\alpha, \beta, \gamma) = (2, 3, 3)$, which satisfies $f(2, 3, 3) \leq m$. In the remaining cases there is another additional point $(\alpha, \beta, \gamma) = (3, 3, 3)$. For this we get $f(3, 3, 3) \leq m$ if $p^d \geq 22$. Since 21 is no prime power, we can consider $f(V_1) = p^d/4 + 7$ now. If $p > 2$, then $p^d/4$ is no integer. In this case

$$\alpha + \beta + \gamma + \delta - a_{13} - a_{14} - a_{34} \leq [f(V_1)] - 3 = \frac{p^d - 1}{4} + 4,$$

where $[f(V_1)]$ is the largest integer below $f(V_1)$. Thus, let us assume $\delta = p^d/4 + 1$ (and $p = 2$). With the help of a computer one can show that up to equivalence only the possibility

$$A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & \delta \end{pmatrix} \quad (3)$$

has the right determinant (see also the remark following the proof). By considering the corresponding decomposition matrix, one can easily deduce:

$$k(B) \leq \delta + 2 \leq \frac{p^d - 1}{l(B)} + l(B).$$

Now it remains to check, that f does not exceed m on other points of \mathcal{F} (this is necessary, since $f(V_1) > m$). For that, we exclude V_1 from \mathcal{F} and form a smaller polyhedron. Since only integral values for α, β, γ are allowed, we get three new vertices:

$$\begin{aligned} V_5 &:= (2, 2, 3), \\ V_6 &:= (2, 3, 3), \\ V_7 &:= (3, 3, 3) \end{aligned}$$

But these points were already considered. This finishes the first part of the proof. The second part follows easily, since for blocks with cyclic defect groups equality holds. \square

In the case $l(B) = 5$ there is no inequality like (2). However, one can use the so called ‘‘fundamental inequality’’ of quadratic forms

$$\alpha\beta\gamma\delta\epsilon \leq 8p^d$$

(see [1]). Of course, the complexity increases rapidly with $l(B)$. For example, the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & -1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 & \epsilon \end{pmatrix}$$

with $\epsilon = p^d/4 + 9$ ($p = 2$) has to be considered. We will demonstrate that such matrices cannot occur. For this let $l := l(B)$ arbitrary, $a_{ii} = 2$, and $a_{i,i+1} = 1$ for $i = 1, \dots, l - 1$. In the following we will speak of Cartan matrices and decomposition matrices always with respect to an arbitrary basic set.

The first two columns of the decomposition matrix Q can be arranged in the form

$$\begin{pmatrix} 1 & \cdot \\ 1 & 1 \\ \cdot & 1 \\ \cdot & \cdot \\ \vdots & \vdots \\ \cdot & \cdot \end{pmatrix}.$$

By the orthogonality relations, the first three columns cannot have the form

$$\begin{pmatrix} 1 & \cdot & \pm 1 \\ 1 & 1 & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \cdot & -1 \\ 1 & 1 & 1 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

That means they have the form

$$\begin{pmatrix} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \cdot & \cdot \\ 1 & 1 & 1 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

However, both forms give rise to equivalent matrices A . Similarly, we may assume that the first $l - 1$ columns of Q have the form

$$\begin{pmatrix} 1 & . & \cdots & . \\ 1 & 1 & \ddots & \vdots \\ . & 1 & \ddots & . \\ . & . & \ddots & 1 \\ . & . & . & 1 \\ \vdots & \vdots & \vdots & \vdots \\ . & . & . & . \end{pmatrix}.$$

(Now one can see that the 5×5 matrix above cannot occur.) If we add suitable multiples of the first $l - 1$ columns to the last column, Q becomes

$$\begin{pmatrix} 1 & . & \cdots & \cdots & . \\ 1 & 1 & \ddots & & \vdots \\ . & 1 & \ddots & \ddots & \vdots \\ . & . & \ddots & 1 & . \\ . & . & . & 1 & * \\ . & . & . & . & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & * \end{pmatrix}.$$

Thus, up to equivalence A has the form

$$\begin{pmatrix} 2 & 1 & . & \cdots & . \\ 1 & \ddots & \ddots & \ddots & \vdots \\ . & \ddots & \ddots & 1 & . \\ \vdots & \ddots & 1 & 2 & a \\ . & \cdots & . & a & \epsilon \end{pmatrix}$$

with $a \geq 1$ (notice that this matrix does not have to be reduced). This yields

$$\epsilon = \frac{p^d + a^2(l-1)}{l} \quad \text{and} \quad k(B) \leq l + \epsilon - a^2 = \frac{p^d - a^2}{l} + l \leq \frac{p^d - 1}{l} + l.$$

It seems likely that this configuration allows the largest value for $k(B)$ in general.

Fujii gives some sufficient conditions for $\det C = p^d$ in [7]. We remark that $\det C$ can be determined locally with the notion of lower defect groups.

The knowledge of the Cartan matrix implies that $l(B)$ is already known. Since $k(B) - l(B)$ is determined locally, it might seem absurd to bound $k(B)$ in terms of Cartan invariants. Instead, it would be more useful if one can apply these bounds to blocks of subsections. In this sense the next lemma is an extension of Theorem A in [10].

Lemma 3. *Let (u, b) be a major subsection associated with the 2-block B . Let $C_b = (c_{ij})$ be the Cartan matrix of b up to equivalence. Then for every positive definite integer quadratic form*

$$q(x_1, \dots, x_{l(b)}) = \sum_{1 \leq i \leq j \leq l(b)} q_{ij} x_i x_j$$

we have

$$k(B) \leq \sum_{1 \leq i \leq j \leq l(b)} q_{ij} c_{ij}.$$

Proof. Let us consider the generalized decomposition numbers d_{ij}^u associated with the subsection (u, b) . If 2^n is the order of u , then d_{ij}^u is an integer of the 2^n -th cyclotomic field \mathbb{Q}_{2^n} . It is known that $1, \zeta := e^{2\pi i/2^n}, \zeta^2, \dots, \zeta^d$ with $d = 2^{n-1} - 1$ form a basis for the ring of integers of \mathbb{Q}_{2^n} . For $i = 1, \dots, k(B)$ we write $(d_{ij}^u)_{j=1}^{l(b)} = d_i = a_i^0 + a_i^1 \zeta + \dots + a_i^d \zeta^d$ with $a_i^0, \dots, a_i^d \in \mathbb{Z}^{l(b)}$. Since (u, b) is major, for every i at least one row a_i^r does not vanish. Let $Q = (\tilde{q}_{ij})_{i,j=1}^{l(b)}$ with

$$\tilde{q}_{ij} := \begin{cases} q_{ij} & \text{if } i = j, \\ q_{ij}/2 & \text{if } i \neq j. \end{cases}$$

Then

$$\begin{aligned} \sum_{1 \leq i \leq j \leq l(b)} q_{ij} c_{ij} &= \sum_{1 \leq i \leq j \leq l(b)} \sum_{r=1}^{k(B)} q_{ij} d_{ri}^u \overline{d_{rj}^u} = \sum_{r=1}^{k(B)} d_r Q \overline{d_r}^T \\ &= \sum_{r=1}^{k(B)} \sum_{s=0}^d \left(\sum_{i-j=s} a_r^i Q(a_r^j)^T - \sum_{i-j=s-2^{n-1}} a_r^i Q(a_r^j)^T \right) \zeta^s = \sum_{r=1}^{k(B)} \sum_{i=0}^d a_r^i Q(a_r^i)^T \geq k(B). \quad \square \end{aligned}$$

Landrock has shown that Brauer's $k(B)$ -conjecture holds for 2-blocks with defect 3 (see [12]). The next theorem will generalize this.

Theorem 2. *Brauer's $k(B)$ -conjecture holds for defect groups which are central extensions of metacyclic 2-groups by cyclic 2-groups. In particular the $k(B)$ -conjecture holds for abelian defect 2-groups of rank at most 3.*

Proof. By Lemma 3 it suffices to show

$$\sum_{i=1}^{l(B)} c_{ii} - \sum_{i=1}^{l(B)-1} c_{i,i+1} \leq |D|$$

for every 2-block B with metacyclic defect groups D and Cartan matrix $C = (c_{ij})$. If D is dihedral, then $\det C = |D|$ and $l(B) \leq 3$ (see [4]). Thus, in this case the claim follows from the proof of Theorem 1. If D is a semidihedral or quaternion group, one can use the tables in [5] to show the claim (this case can also be done by the method of the proof of Theorem 1 and the fact that the elementary divisors of C are contained in $\{1, 2, |D|\}$). The author has shown (using the methods of Usami and Puig) that $\det C = |D|$ and $l(B) \in \{1, 3\}$ also holds for $D \cong C_{2^s} \times C_{2^s}$ with $s \in \mathbb{N}$. By the result of [14], we are done. \square

We note that Brauer has proved the $k(B)$ -conjecture for abelian defect groups of rank 2 and arbitrary primes p (see (7D) in [3]). The smallest 2-group which does not satisfy the hypothesis of Theorem 2 is the elementary abelian group of order 16. However, this group can be handled as well.

Theorem 3. *Brauer's $k(B)$ -conjecture holds for defect groups which are central extensions of the elementary abelian group of order 8 by a cyclic group. In particular the $k(B)$ -conjecture holds for every defect group with a central cyclic subgroup of index 8.*

Proof. Let B be a block with elementary abelian defect group of order 8 and Cartan matrix $C = (c_{ij})$. It suffices to show

$$\sum_{i=1}^{l(B)} c_{ii} - \sum_{i=1}^{l(B)-1} c_{i,i+1} \leq 8. \quad (4)$$

If the inertial index $e(B)$ is 1, then also $l(B) = 1$, and the claim follows.

Now let $e(B) = 3$. It is easy to show that there are four subsections $(1, B)$, (u_1, b_1) , (u_2, b_2) and (u_3, b_3) associated with B . Moreover, we may assume $l(b_1) = 3$ and $l(b_2) = l(b_3) = 1$. As usual, b_1 dominates a block of $C_G(u_1)/\langle u_1 \rangle$ with Klein four defect group. It follows that the Cartan matrix of b_1 is equivalent to

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

Using this, it is easy to see that there is a basic set such that the generalized decomposition numbers associated with u_i ($i = 1, 2, 3$) have form

$$\begin{pmatrix} 1 & . & . & 1 & 1 \\ 1 & . & . & 1 & -1 \\ 1 & 1 & . & -1 & 1 \\ 1 & 1 & . & -1 & -1 \\ . & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & -1 \\ . & . & 1 & -1 & 1 \\ . & . & 1 & -1 & -1 \end{pmatrix}.$$

By the orthogonality relations of generalized decomposition numbers there exists a matrix $S \in \text{GL}(3, \mathbb{Q})$ such that the ordinary decomposition matrix Q satisfies

$$Q = \begin{pmatrix} 1 & . & . \\ -1 & . & . \\ . & 1 & . \\ . & -1 & . \\ . & . & 1 \\ . & . & -1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} S$$

Moreover, it is easy to see that all entries of S are integral. It is well-known that there exists a matrix $\tilde{Q} \in \mathbb{Z}^{3 \times 8}$ such that $\tilde{Q}Q = 1_3$. This shows $S \in \text{GL}(3, \mathbb{Z})$. Hence C has the form $S^{-\text{T}}Q^{\text{T}}QS^{-1}$ up to equivalence. Thus, the claim follows in this case.

Let $e(B) = 7$. Then there are two subsections $(1, B)$ and (u, b) with $k(B) - l(B) = l(b) = 1$. Since 8 is the sum of $k(B)$ integer squares, we must have $k(B) \in \{5, 8\}$. By Corollary 1 in [7], we have $\det C = 8$. Thus in the case $l(B) = 4$, the claim follows from the proof of Theorem 1 (notice that this case contradicts Brauer's height zero conjecture). So we may assume $l(B) = 7$. Then the generalized decomposition numbers corresponding to u can be arranged in the form $(1, \dots, 1)^{\text{T}}$. Hence the ordinary decomposition matrix has the form

$$\begin{pmatrix} 1 & . & . & . & . & . & . \\ -1 & -1 & . & . & . & . & . \\ . & 1 & 1 & . & . & . & . \\ . & . & -1 & -1 & . & . & . \\ . & . & . & 1 & 1 & . & . \\ . & . & . & . & -1 & -1 & . \\ . & . & . & . & . & 1 & 1 \\ . & . & . & . & . & . & -1 \end{pmatrix},$$

and the claim follows.

Let $e(B) = 21$. Then there are two subsections $(1, B)$ and (u, b) with $k(B) - l(B) = l(b) = 3$. In particular $l(B) \leq 5$ (using Theorem 2). The theory of lower defect groups reveals that 2 occurs at least twice as elementary divisor of C . This gives $l(B) \geq 3$. The case $l(B) = 3$ contradicts Corollary 1.3 in [12]. Now let $l(B) = 4$ (again this case contradicts the height zero conjecture). Then the generalized decomposition numbers corresponding to u have the form

$$\begin{pmatrix} 1 & . & 1 \\ 1 & . & . \\ 1 & 1 & 1 \\ 1 & 1 & . \\ . & 1 & 1 \\ . & 1 & . \\ . & . & 1 \end{pmatrix}.$$

That means the ordinary decomposition matrix becomes

$$\begin{pmatrix} 1 & . & . & . \\ -1 & -1 & . & . \\ . & . & -1 & . \\ . & 1 & 1 & . \\ . & . & . & -1 \\ . & -1 & . & 1 \\ -1 & . & 1 & 1 \end{pmatrix},$$

and the Cartan matrix has the form

$$C = \begin{pmatrix} 3 & 1 & -1 & -1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{pmatrix}.$$

Unfortunately, this matrix does not satisfy inequality (4). However, we can use Lemma 3 with the quadratic form q corresponding to the positive definite matrix

$$\frac{1}{2} \begin{pmatrix} 2 & -1 & 1 & . \\ -1 & 2 & -1 & . \\ 1 & -1 & 2 & -1 \\ . & . & -1 & 2 \end{pmatrix}.$$

Finally let $l(B) = 5$. Then the generalized decomposition numbers corresponding to u have the form

$$\begin{pmatrix} 1 & . & . \\ 1 & . & . \\ 1 & 1 & . \\ 1 & 1 & . \\ . & 1 & 1 \\ . & 1 & 1 \\ . & . & 1 \\ . & . & 1 \end{pmatrix}$$

and the ordinary decomposition matrix becomes

$$\begin{pmatrix} 1 & . & . & . & . \\ -1 & . & . & . & -1 \\ . & 1 & . & . & 1 \\ . & -1 & . & . & . \\ . & . & -1 & . & -1 \\ . & . & 1 & . & . \\ . & . & . & 1 & 1 \\ . & . & . & -1 & . \end{pmatrix}.$$

Thus, the Cartan matrix is

$$\begin{pmatrix} 2 & . & . & . & 1 \\ . & 2 & . & . & 1 \\ . & . & 2 & . & 1 \\ . & . & . & 2 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}.$$

In this case we can use Lemma 3 with the quadratic form q corresponding to the positive definite matrix

$$\frac{1}{2} \begin{pmatrix} 2 & 1 & . & . & -1 \\ 1 & 2 & . & . & -1 \\ . & . & 2 & . & -1 \\ . & . & . & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

□

In connection with Theorem 2 the second assertion is also clear.

Recently, Kessar, Koshitani and Linckelmann have proven that the cases $k(B) = 5$ and $k(B) = 7$ in the proof above cannot occur (see [8]). However, their proof uses the classification of finite simple groups. By [9] one can replace the elementary abelian group of order 8 by $C_4 \wr C_2$ in Theorem 3.

We deduce a corollary.

Theorem 4. *Brauer's $k(B)$ -conjecture holds for 2-blocks of defect at most 4.*

For odd primes it is only known that the $k(B)$ -conjecture holds for blocks of defect at most 2.

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