Cartan matrices and Brauer's $k(B)$-Conjecture V

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We prove Brauer's $k(B)$-Conjecture for the 3-blocks with abelian defect groups of rank at most 5 and for all 3-blocks of defect at most 4. For this purpose we develop a computer algorithm to construct isotypies based on a method of Usami and Puig. This leads further to some previously unknown perfect isometries for the 5-blocks of defect 2. We also investigate basic sets which are compatible under the action of the inertial group.

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1 Introduction

This work continues a series of articles the last one being [27]. Before stating the main theorems we briefly explain the strategy behind all papers in this series.

Let B be a block of a finite group G with respect to an algebraically closed field F of characteristic $p > 0$. Let D be a defect group B, and let $k(B) := |\mathrm{Irr}(B)|$ and $l(B) := |\mathrm{IBr}(B)|$. To prove Brauer's $k(B)$-Conjecture, that $k(B) \le |D|$, we investigate Brauer correspondents of B in local subgroups. More precisely, let $z \in \mathrm{Z}(D)$ and let b_z be a Brauer correspondent of B in $\mathrm{C}_G(z)$. If we can determine the Cartan matrix C_z of b_z up to basic sets (i. e. up to transformations $C_z \to SC_zS^{\mathrm{t}}$ where $S \in \mathrm{GL}(l(b_z), \mathbb{Z})$), then Brauer's Conjecture usually follows from [26, Theorem 4.2] or from the much stronger result [31, Theorem A]. Now b_z dominates a unique block $\overline{b_z}$ of $\mathrm{C}_G(z)/\langle z\rangle$ with Cartan matrix $\overline{C_z} = \frac{1}{|\langle z\rangle|}C_z$. Hence, it suffices to consider $\overline{b_z}$. By [27, Lemma 3], $\overline{b_z}$ has defect group $\overline{D} := D/\langle z\rangle$ and the fusion system of $\overline{b_z}$ is uniquely determined by the fusion system of B. This means that we have full information on $\overline{b_z}$ on the local level. The inertial quotient of B is denoted by $I(B)$ in the following.

In the present paper we deal mostly with situations where $\overline{D}$ is abelian. Then the fusion system of $\overline{b_z}$ is essentially determined by the inertial quotient $I(\overline{b_z}) = I(b_z) \cong \mathrm{C}_{I(B)}(z)$ and by the action of $I(\overline{b_z})$ on $\overline{D}$. In the next section we will revisit a method developed by Usami and Puig to construct perfect isometries between $\overline{b_z}$ and its Brauer first main theorem correspondent in certain situations. Since perfect isometries preserve Cartan matrices (up to basic sets), it suffices to determine the Cartan matrix of a block β_z with normal defect group $\overline{D}$ and $I(\beta_z) = I(b_z)$. By a theorem of Külshammer [17], we may even assume that β_z is a twisted group algebra of $L := \overline{D} \rtimes I(b_z)$. Finally, we can regard β_z

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as a faithful block of a certain central extension $\widehat{L}$ of L by a cyclic p'-group. It is then straight-forward to compute the desired Cartan matrix.

In the third section we apply a novel computer implementation of the Usami–Puig method to construct many new isotypies for 5-blocks of defect 2. This verifies Broué's Abelian Defect Group Conjecture [4] on the level of characters in those cases. When this approach fails, it is often still possible to determine a short list of all potential Cartan matrices of b_z. To do so, we improve the Cartan method introduced in [26, Section 4.2]. As a new ingredient we investigate in Proposition 7 the existence of basic sets which are compatible with the action of the inertial group. Eventually, we combine both methods to verify Brauer's $k(B)$-Conjecture for all 3-blocks with abelian defect groups of rank at most 5. This extends the corresponding results from [27, Proposition 21] and [28, Corollary 3] for p-blocks with abelian defect groups of rank at most 3 (respectively 7 if $p = 2$). Afterwards we turn the focus to non-abelian defect groups. In the last section we prove Brauer's Conjecture under the hypothesis that $\overline{D}$ is metacyclic. This result relies on a recent paper by Tasaka–Watanabe [35]. Finally, a careful analysis shows that Brauer's Conjecture holds for all defect groups of order 3^4. We remark that Brauer's Conjecture for p-blocks of defect 3 has been verified previously in [28, Theorem B] for arbitrary p.

Although our methods are of elementary nature they crucially rely on one direction of Brauer's Height Zero Conjecture proven by Kessar–Malle [13] via the classification of finite simple groups.

2 A method of Usami and Puig

In addition to the notation already introduced we follow mostly [26]. To distinguish cyclic groups from Cartan matrices and centralizers we denote them by Z_n. The symmetric and alternating groups of degree n as well as the dihedral, semidihedral and quaternion groups of order n are denoted by S_n, A_n, D_n, SD_n and Q_n respectively. Moreover, we make use of the Mathieu group $M_9 \cong Z_3^2 \rtimes Q_8$ (a sharply 2-transitive group of degree 9). A central product of groups G and H is denoted by $G * H$. The Kronecker δ_{ij} (being 1 if $i = j$ and 0 otherwise) is often used to write matrices in a concise form. Finally, a *basic set* of a block B is a $\mathbb{Z}$-basis of the Grothendieck group $\mathbb{Z}\mathrm{IBr}(B)$ of generalized Brauer characters of B.

In this section we assume that B is a block with *abelian* defect group D. Let b be a Brauer correspondent of B in $\mathrm{C}_G(D)$. We regard the inertial quotient $E := I(B) = \mathrm{N}_G(D, b)/\mathrm{C}_G(D)$ as a subgroup of $\mathrm{Aut}(D)$. Let $L := D \rtimes E$. It is well-known that b is nilpotent and $\mathrm{IBr}(b) = \{\varphi\}$. Now φ gives rise to a projective representation Γ of the inertial group $\mathrm{N}_G(D, b)$ (see [20, Theorem 8.14]). Moreover, Γ is associated to a 2-cocycle γ of E with values in $F^\times$ (see [20, Theorem 8.15]). Külshammer's result mentioned above states that $b^{\mathrm{N}_G(D)}$ is Morita equivalent to the twisted group algebra $F_\gamma L$ (note that $b^{\mathrm{N}_G(D)}$ and $b^{\mathrm{N}_G(D,b)}$ are Morita equivalent by the Fong–Reynolds theorem).

In several papers (starting perhaps with [36]), Usami and Puig developed an inductive method to establish an isotypy between B and $F_\gamma L$. This is a family of compatible perfect isometries between $b^{\mathrm{C}_G(Q)}$ and $F_\gamma \mathrm{C}_L(Q)$ for every $Q \le D$. In particular, for $Q = 1$ we obtain a perfect isometry between B and $F_\gamma L$. In the following we introduce the necessary notation. For $Q \le D$ and $H \le \mathrm{N}_G(Q)$ we write $\overline{H} := HQ/Q$. Let $b_Q := b^{\mathrm{C}_G(Q)}$ and let $\overline{b_Q}$ be the unique block of $\overline{\mathrm{C}_G(Q)}$ dominated by b_Q. For any (twisted) group algebra or block A let $\mathbb{Z}\mathrm{Irr}(A)$ be the Grothendieck group of generalized characters of A. Let $\mathbb{Z}\mathrm{Irr}^0(A)$ be the subgroup of $\mathbb{Z}\mathrm{Irr}(A)$ consisting of the generalized characters which vanish on the p-regular elements of the corresponding group. For class functions χ and ψ on G we use the usual scalar product

$$[\chi, \psi] := \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}. \tag{2.1}$$

We note that $\mathrm{N}_E(Q)$ acts naturally on $\mathbb{Z}\mathrm{Irr}^0(F_\gamma\overline{\mathrm{C}_L(Q)})$ and $\mathbb{Z}\mathrm{Irr}(F_\gamma\overline{\mathrm{C}_L(Q)})$ as well as on $\mathbb{Z}\mathrm{Irr}^0(\overline{b_Q})$ and $\mathbb{Z}\mathrm{Irr}(\overline{b_Q})$. A map f between any two of these sets is called $\mathrm{N}_E(Q)$-*equivariant* if $f(\chi)^e = f(\chi^e)$ for every $e \in \mathrm{N}_E(Q)$ and every generalized character χ in the respective set. By [22, Proposition 3.11 and Section 4.3] (compare with [37, Section 3.4]), it suffices to show that a given $\mathrm{N}_E(Q)$-equivariant bijective isometry (with respect to (2.1))

$$\Delta_0 : \mathbb{Z}\mathrm{Irr}^0(F_\gamma\overline{\mathrm{C}_L(Q)}) \to \mathbb{Z}\mathrm{Irr}^0(\overline{b_Q})$$

extends to an $\mathrm{N}_E(Q)$-equivariant isometry

$$\Delta : \mathbb{Z}\mathrm{Irr}(F_\gamma\overline{\mathrm{C}_L(Q)}) \to \mathbb{Z}\mathrm{Irr}(\overline{b_Q}) \tag{2.2}$$

(Δ will automatically be surjective). To prove this, we may replace (G, B, E) by

$$\left(\mathrm{N}_G(Q, b_Q), b_Q^{\mathrm{N}_G(Q,b_Q)}, \mathrm{N}_E(Q)\right)$$

in order to argue by induction on $|E|$. For example, if $|\mathrm{N}_E(Q)| \le 4$ or $\mathrm{N}_E(Q) \cong S_3$, then the claim holds by the main theorems of [22, 23, 36, 37]. Furthermore, the claim holds for $Q = D$ as shown in [22, 3.4.2].

In the following we assume that $Q < D$ is given. It is straight-forward to determine from the character table a $\mathbb{Z}$-basis $\rho_1, \ldots, \rho_m$ of $\mathbb{Z}\mathrm{Irr}^0(F_\gamma\overline{\mathrm{C}_L(Q)})$. Let $\chi_1, \ldots, \chi_k \in \mathrm{Irr}(F_\gamma\overline{\mathrm{C}_L(Q)})$ and $A = (a_{ij}) \in \mathbb{Z}^{k\times m}$ such that $\rho_i = \sum_{j=1}^k a_{ji}\chi_j$ for $i = 1, \ldots, m$. Let $\widehat{\rho_i} := \Delta_0(\rho_i)$ for $i = 1, \ldots, m$. Since Δ_0 is an isometry, we have

$$C_* := A^{\mathrm{t}}A = (\rho_i, \rho_j)_{1\le i,j\le m} = (\widehat{\rho_i}, \widehat{\rho_j})_{1\le i,j\le m}$$

where A^{t} denotes the transpose of A. The matrix equation $Q_*^{\mathrm{t}}Q_* = C_*$ can be solved with an algorithm of Plesken [21] which is implemented in GAP [7] (command `OrthogonalEmbeddings`). We will see that in many situations there is only one solution up to permutations and signs of the rows of Q_*. This implies that there exist $\widehat{\chi}_1, \ldots, \widehat{\chi}_k \in \pm\mathrm{Irr}(\overline{b_Q})$ such that $\widehat{\rho_i} = \sum_{j=1}^k a_{ji}\widehat{\chi}_j$ for $i = 1, \ldots, m$. Then the isometry Δ defined by $\Delta(\chi_i) := \widehat{\chi}_i$ for $i = 1, \ldots, k$ clearly extends Δ_0. If $\mathrm{N}_E(Q) = \mathrm{C}_E(Q)$, then Δ is always $\mathrm{N}_E(Q)$-equivariant. This holds in particular if $Q = 1$. In several other cases we can show that the rows of Q_* are pairwise linearly independent (i. e. $r \ne \pm s$ for distinct rows r, s). It follows that Δ is in fact the only extension of Δ_0 (note that $-\Delta$ does not extend Δ_0 since we are assuming $Q < D$). Now for every $e \in \mathrm{N}_E(Q)$ the map $\widetilde{\Delta} : \mathbb{Z}\mathrm{Irr}(F_\gamma\overline{\mathrm{C}_L(Q)}) \to \mathbb{Z}\mathrm{Irr}(\overline{b_Q})$, $\chi \mapsto {}^{e^{-1}}(\Delta({}^e\chi))$ is also an isometry extending Δ_0. Therefore, $\widetilde{\Delta} = \Delta$ and Δ is $\mathrm{N}_E(Q)$-equivariant.

Since the generalized characters $\widehat{\rho_i}$ vanish on the p-regular elements, these characters are orthogonal to the projective indecomposable characters of $\overline{b_Q}$. In other words, the columns of Q_* are orthogonal to the columns of the decomposition matrix of $\overline{b_Q}$. In order to reduce the number of possible solutions of the equation $Q_*^{\mathrm{t}}Q_* = C_*$, we prove the following result.

Lemma 1. *Let B be a p-block of a finite group G with abelian defect group $D \ne 1$ and decomposition matrix $Q_1 \in \mathbb{Z}^{k\times l}$. Let $Q_* \in \mathbb{Z}^{k\times(k-l)}$ be of rank $k - l$ such that $Q_1^{\mathrm{t}}Q_* = 0$. Let $C_* := Q_*^{\mathrm{t}}Q_*$. Then for every row r of Q_* we have $|D|rC_*^{-1}r^{\mathrm{t}} \in \{1, \ldots, |D|\} \setminus p\mathbb{Z}$.*

Proof. Let $C := Q_1^{\mathrm{t}}Q_1$ be the Cartan matrix of B. Since Q_1 and Q_* have full rank, the matrix $R := (Q_1, Q_*) \in \mathbb{Z}^{k\times k}$ is invertible. We compute

$$1_k = R(R^{\mathrm{t}}R)^{-1}R^{\mathrm{t}} = (Q_1, Q_*)\begin{pmatrix} C^{-1} & 0 \\ 0 & C_*^{-1} \end{pmatrix}\begin{pmatrix} Q_1^{\mathrm{t}} \\ Q_*^{\mathrm{t}} \end{pmatrix} = Q_1C^{-1}Q_1^{\mathrm{t}} + Q_*C_*^{-1}Q_*^{\mathrm{t}}.$$

It is well-known that $|D|C^{-1}$ and $|D|Q_1C^{-1}Q_1^{\mathrm{t}}$ are integer matrices (see [20, Theorem 3.26]). Hence, $|D|Q_*C_*^{-1}Q_*^{\mathrm{t}}$ is also an integer matrix and the (non-negative) diagonal entries are bounded by $|D|$. By Kessar–Malle [13], all irreducible characters in B have height 0. By a result of Brauer (see [26, Proposition 1.36]), it follows that the diagonal entries of $|D|Q_1C^{-1}Q_1^{\mathrm{t}}$ are not divisible by p. Hence, the same must hold for the diagonal entries of $|D|Q_*C_*^{-1}Q_*^{\mathrm{t}}$. The claim follows. □

Sometimes we know a priori that $l(F_\gamma\overline{\mathrm{C}_L(Q)}) = l(\overline{b_Q})$ (for instance, if $\overline{D}$ is cyclic or $|\mathrm{C}_E(Q)| \le 4$ by the results of Usami–Puig cited above). Since Δ_0 is an isomorphism, we also have

$$k(F_\gamma\overline{\mathrm{C}_L(Q)}) - l(F_\gamma\overline{\mathrm{C}_L(Q)}) = k(\overline{b_Q}) - l(\overline{b_Q}).$$

Hence, we can restrict Plesken's algorithm to those Q_* which have exactly $k(F_\gamma\overline{\mathrm{C}_L(Q)})$ rows. In this favorable situation the Grothendieck groups $\mathbb{Z}\,\mathrm{PIM}(F_\gamma\overline{\mathrm{C}_L(Q)})$ and $\mathbb{Z}\,\mathrm{PIM}(\overline{b_Q})$ spanned by the projective indecomposable characters have the same rank. Since $\mathbb{Z}\,\mathrm{PIM}(.)$ is the orthogonal complement of $\mathbb{Z}\mathrm{Irr}^0(.)$ in $\mathbb{Z}\mathrm{Irr}(.)$, it suffices to construct an $\mathrm{N}_E(Q)$-equivariant isometry $\mathbb{Z}\,\mathrm{PIM}(F_\gamma\overline{\mathrm{C}_L(Q)}) \to \mathbb{Z}\,\mathrm{PIM}(\overline{b_Q})$ which can then be combined with Δ_0 to obtain Δ. This alternative strategy is pursued in Proposition 2 below.

The entire procedure can be executed by GAP without human intervention. In fact, hand calculations of this kind become very tedious and are prone to errors. We summarize our algorithm under the assumption that $L := D \rtimes E$ is given.

(1) Determine the Schur multiplier $H := \mathrm{H}^2(E, \mathbb{C}^\times)$.

(2) For every cyclic subgroup $Z \le H$ do the following

  (a) Construct a stem extension $\widehat{L}$ of L such that $\widehat{L}/Z \cong L$.

  (b) Determine a set $\mathcal{Q}$ of representatives for the L-conjugacy classes of subgroups $Q < D$ such that $|\mathrm{N}_E(Q)| > 4$ and $\mathrm{N}_E(Q) \not\cong S_3$.

  (c) For every $Q \in \mathcal{Q}$ and every faithful block β of $Y := \mathrm{C}_{\widehat{L}}(Q)/Q$ do the following:

    (i) Determine the matrix $A := (\chi_i(y_j))_{i,j}$ where $\mathrm{Irr}(\beta) = \{\chi_1, \ldots, \chi_k\}$ and $y_1, \ldots, y_l$ are representatives for the conjugacy classes of p'-elements of Y.

    (ii) Compute a $\mathbb{Z}$-basis $u_1, \ldots, u_{k-l}$ of the orthogonal space $\{v :\in \mathbb{Z}^k : vA = 0\}$ (using the Smith normal form for instance).

    (iii) Compute $C_* = (u_iu_j^{\mathrm{t}})_{i,j=1}^{k-l}$.

    (iv) Determine the (finite) set $\mathcal{R}$ of rows $r \in \mathbb{Z}^{k-l}$ such that

$$|D/Q|rC_*^{-1}r^{\mathrm{t}} \in \{1, \ldots, |D/Q|\} \setminus p\mathbb{Z}.$$

    (v) Apply Plesken's algorithm to solve $C_* = Q_*^{\mathrm{t}}Q_*$ such that every row of Q_* belongs to $\mathcal{R}$.

    (vi) If there is a unique solution Q_* up to permutations and signs of rows, then Δ_0 extends to some isometry Δ.

    (vii) If $\mathrm{N}_E(Q) = \mathrm{C}_E(Q)$, then Δ is $\mathrm{N}_E(Q)$-equivariant.

    (viii) If the rows of Q_* are pairwise linearly independent, then Δ is $\mathrm{N}_E(Q)$-equivariant.

    (ix) Deal with the exceptions.

In view of the fact that Kessar–Malle's result was not available to Usami and Puig, it is not surprising that our approach goes beyond their results. For instance, the isotypies for $D \cong Z_2 \times Z_2 \times Z_2$ constructed by Kessar–Koshitani–Linckelmann [10] (also relying on the classification of finite simple groups) can now be obtained by pressing a button. Our algorithm in combination with [26, Proposition 13.4] also applies to $L \cong A_4 \times A_4$ ($p = 2$) and therefore simplifies and improves the main result of [18]. In fact, an extension of this case was recently settled by the first author in [2, Proposition 3.3]. We should however also mention that the computational complexity of Plesken's algorithm grows rapidly with the size of the involved matrices.

3 Blocks of defect 2

One approach to classify blocks B with a given defect group D is to distribute them into families such that each family corresponds to a Morita equivalence class of the Brauer correspondent B_D of B in $\mathrm{N}_G(D)$ (there are only finitely many choices for these classes). If D is cyclic, this has been accomplished by using the Brauer tree. Also the blocks with Klein four defect group $D \cong Z_2 \times Z_2$ are completely classified. The group $D \cong Z_3 \times Z_3$ has first been investigated by Kiyota [14] in 1984 and is still not fully understood today. We will recap the details and further study $D \cong Z_5 \times Z_5$ in this section.

We begin by showing that only the subgroup $Q = 1$ in Usami–Puig's methods needs to be considered. This fact is related to the existence of a stable equivalence of Morita type stated in [24, Section 6.2].

Proposition 2. *Let B be a block of a finite group G with defect group $D \cong C_p \times C_p$ and cocycle γ as in the previous section. Let $L := D \rtimes I(B)$. Suppose that every $I(B)$-equivariant isometry $\Delta_0 : \mathbb{Z}\mathrm{Irr}^0(F_\gamma L) \to \mathbb{Z}\mathrm{Irr}^0(B)$ extends to an $I(B)$-equivariant isometry $\Delta : \mathbb{Z}\mathrm{Irr}(F_\gamma L) \to \mathbb{Z}\mathrm{Irr}(B)$. Then B is isotypic to its Brauer correspondent in $\mathrm{N}_G(D)$.*

Proof. Let $Q \le D$ be of order p. We need to show the existence of Δ with respect to Q as in (2.2). To this end we may assume that $G = \mathrm{N}_G(Q, b_Q)$ and $E := I(B)$ normalizes Q. Let $\widehat{L}$ be a suitable stem extension such that $F_\gamma \overline{\mathrm{C}_L(Q)}$ is isomorphic to a block β_Q of $\mathrm{C}_{\widehat{L}}(Q)/Q$. Observe that β_Q and $\overline{b_Q}$ have defect 1 and inertial quotient $\mathrm{C}_E(Q)$. By Brauer's theory of blocks of defect 1, we have $l := l(\beta_Q) = |\mathrm{C}_E(Q)| = l(\overline{b_Q})$. Since $G/\mathrm{C}_G(Q) \cong E/\mathrm{C}_E(Q)$ is cyclic, [29, Lemma 3.3] (or Proposition 7 below) implies the existence of a basic set Φ of $\overline{b_Q}$ such that $\mathrm{IBr}(\overline{b_Q})$ and Φ are isomorphic E-sets and the Cartan matrix of $\overline{b_Q}$ with respect to Φ is $C := (m + \delta_{ij})_{i,j=1}^{l}$ where $m := (p-1)/l$. This is also the Cartan matrix of β_Q (with respect to $\mathrm{IBr}(\beta_Q)$). Let $Q = (d_{\chi\varphi})$ be the decomposition matrix of $\overline{b_Q}$ with respect to Φ. For $\varphi \in \Phi$ we define the projective character $\widehat{\varphi} := \sum_{\chi \in \mathrm{Irr}(\overline{b_Q})} d_{\chi\varphi}\chi$. By the shape of the matrix C, every bijection between $\mathrm{PIM}(\beta_Q)$ and $\{\widehat{\varphi} : \varphi \in \Phi\}$ induces an isometry $\mathbb{Z}\,\mathrm{PIM}(\beta_Q) \to \mathbb{Z}\,\mathrm{PIM}(\overline{b_Q})$. Since $\mathbb{Z}\,\mathrm{PIM}(\beta_Q)$ is the orthogonal complement of $\mathbb{Z}\mathrm{Irr}^0(\beta_Q)$, we can extend Δ_0 in this way to an isometry $\Delta : \mathbb{Z}\mathrm{Irr}(\beta_Q) \to \mathbb{Z}\mathrm{Irr}(\overline{b_Q})$. In order to make Δ E-equivariant, it suffices to show that $\mathrm{IBr}(\beta_Q)$ and $\mathrm{IBr}(\overline{b_Q})$ are isomorphic E-sets.

By [22, Proposition 3.14] there exists a bijection between the set of blocks of $\widehat{L}/Q$ covering β_Q and the set of blocks of $\overline{G}$ covering $\overline{b_Q}$. Moreover, this bijection preserves defect groups and inertial quotients. Since the blocks in both sets (still) have defect 1, the number of irreducible Brauer characters is uniquely determined by the respective inertial indices. Consequently, the number of Brauer characters of $\widehat{L}/Q$ lying over β_Q coincides with the number of Brauer characters of $\overline{G}$ lying over $\overline{b_Q}$. We claim that this number uniquely determines the action of E on $\mathrm{IBr}(\beta_Q)$ and on $\mathrm{IBr}(\overline{b_Q})$. Since $\widehat{L}/\mathrm{C}_{\widehat{L}}(Q) \cong G/\mathrm{C}_G(Q) \cong E/\mathrm{C}_E(Q)$ is cyclic, every $\varphi \in \mathrm{IBr}(\beta_Q) \cup \mathrm{IBr}(\overline{b_Q})$ extends to its inertial group (see [20, Theorem 8.12]). Moreover by [29, Proposition 3.2], E acts $\frac{1}{2}$-transitively on $\mathrm{IBr}(\beta_Q)$ and on $\mathrm{IBr}(\overline{b_Q})$. This means that all orbits on $\mathrm{IBr}(\beta_Q)$ have a common length, say d_L, and similarly all orbits on $\mathrm{IBr}(\overline{b_Q})$ have length, say d_G. By Clifford theory, there are exactly $l(\beta_Q)|E/\mathrm{C}_E(Q)|/d_L^2$ irreducible Brauer characters in $\widehat{L}/Q$ lying over β_Q. Similarly, there are $l(\overline{b_Q})|E/\mathrm{C}_E(Q)|/d_G^2$ Brauer characters in $\overline{G}$ lying over $\overline{b_Q}$. Since $l(\beta_Q) = l(\overline{b_Q})$ we conclude that $d_L = d_G$. Thus, $\mathrm{IBr}(\beta)$ and $\mathrm{IBr}(\overline{b_Q})$ are isomorphic E-sets. □

The following result on the case $p = 3$ is mostly well-known, but hard to find explicitly in the literature. The column *group* in Theorem 3 refers to the small group library in GAP. If this group has an easy structure, then it is described in the *comments* column. If the comment is *non-principal*, then the group is a double cover of the preceding group in the list and the block is the unique non-principal block. In the remaining cases, the group has only one block (the principal block). Finally the column *isotypy* indicates if an isotypy between B and B_D is known to exist.

Theorem 3. *Let B be a block of a finite group G with defect group $D \cong Z_3 \times Z_3$. Then the Brauer correspondent B_D of B in $\mathrm{N}_G(D)$ is Morita equivalent to exactly one of the following blocks:*

no.	$I(B)$	*group*	$k(B_D)$	$l(B_D)$	*isotypy*	*comments*
1	1	$9:2$	9	1	✓	D, *nilpotent*
2	Z_2	$18:3$	9	2	✓	$S_3 \times Z_3$
3	Z_2	$18:4$	6	2	✓	*Frobenius group*
4	Z_2^2	$36:10$	9	4	✓	S_3^2
5	Z_2^2	$72:23$	6	1	✓	*non-principal*
6	Z_4	$36:9$	6	4	✓	*Frobenius group*
7	Z_8	$72:39$	9	8		$\mathrm{AGL}(1,9)$
8	Q_8	$72:41$	6	5		M_9
9	D_8	$72:40$	9	5	✓	$S_3 \wr Z_2$
10	D_8	$144:117$	6	2	✓	*non-principal*
11	SD_{16}	$144:182$	9	7	✓	$\mathrm{A\Gamma L}(1,9)$

Proof. Since $\mathrm{Aut}(D) \cong \mathrm{GL}(2,3)$ has order $16 \cdot 3$, $E := I(B)$ is a subgroup of the semidihedral group $SD_{16} \cong \Gamma\mathrm{L}(1,9) \in \mathrm{Syl}_2(\mathrm{GL}(2,3))$. As explained above, B_D is Morita equivalent to a twisted group algebra $F_\gamma[D \rtimes E]$. If $E \notin \{Z_2^2, D_8\}$, then E has trivial Schur multiplier and $\gamma = 1$. In this case we list $L := D \rtimes E$ in the group column and compute $k(B_D) = k(L)$ and $l(B_D) = k(E)$. If, on the other hand, $E \in \{Z_2^2, D_8\}$, then the Schur multiplier of E is Z_2. Thus, there is exactly one non-trivial twisted group algebra in each case. Here we compute $l(B_D) = k(\widehat{E}) - k(E)$ where $\widehat{E}$ is a double cover of E. The isotypies can be obtained with our algorithm from the last section (cf. [32, Proposition 6.3]).

It remains to show that each two different cases in our list are not Morita equivalent. This is clear from the computed invariants except for the cases 3 and 10. By [5, Corollary 3.5], a Morita equivalence preserves the isomorphism type of the stable center $\overline{\mathrm{Z}}(B_D)$. In case 3, this algebra is symmetric by [11, Theorem 1.1]. Now we use [11, Theorem 3.1] in order to show that the stable center is not symmetric in case 10. The group $E \cong D_8$ has two orbits on $D \setminus \{1\}$. Hence, there exists two non-trivial B_D-subsections (u, β_u) and (v, β_v) up to conjugation. Brauer's formula (see [26, Theorem 1.35]) gives $4 = k(B_D) - l(B_D) = l(\beta_u) + l(\beta_v)$. Hence, we may assume that $l(\beta_u) > 1$ and the claim follows from [11, Theorem 3.1]. □

The reason why the Usami–Puig method fails for $I(B) \in \{Z_8, Q_8\}$ is because in both cases one gets $C_* = (9)$. For $I(B) \cong Z_8$, C_* factors into $Q_* = (\pm 1, \ldots, \pm 1)^{\mathrm{t}}$ and for $I(B) \cong Q_8$ we have $Q_* = (\pm 2, \pm 1, \ldots, \pm 1)^{\mathrm{t}}$. Once we know that $k(B) = k(B_D)$, then B and B_D must be isotypic. Even worse, it seems to be open whether $Q_* = (\pm 2, \pm 2, \pm 1)^{\mathrm{t}}$ can actually occur. Equivalently, does there exist a block B with defect group $D \cong Z_3 \times Z_3$ and $k(B) = 3$?

We remark that B is usually not Morita equivalent to B_D. For principal blocks the possible Morita equivalence classes for B were obtained by Koshitani [15]. The column *type of* B_D in the following table refers to the numbering in Theorem 3.

Theorem 4 (Koshitani)**.** *Let B be the principal block of a finite group G with defect group $D \cong Z_3 \times Z_3$ and B_D be the principal 3-block of $\mathrm{N}_G(D)$. Then B is (splendid) Morita equivalent either to one of the nine principal cases in Theorem 3 or to exactly one of the following principal blocks:*

no.	*type of* B_D	*group*
12	6	A_6
13	6	A_7
14	7	$\mathrm{PGL}(2,9)$
15	8	M_{10}
16	8	$\mathrm{PSL}(3,4)$
17	9	S_6
18	9	S_7
19	9	A_8
20	11	M_{11}
21	11	HS
22	11	M_{23}
23	11	$\mathrm{PSL}(3,4).2^2$
24	11	$\mathrm{Aut}(S_6)$

Proof. By [15, 16], B is splendidly Morita equivalent to one of the given blocks. Using the GAP command `TransformingPermutations`, one can check that each two of those blocks have essentially different decomposition matrices. Hence, they cannot be Morita equivalent (splendid or not). □

According to Scopes [33, Example 2 on p. 455], every block B of a symmetric group with defect group $D \cong Z_3 \times Z_3$ is Morita equivalent to the principal block of S_6, S_7, S_8, to the "second" block of S_8, or to the third block of S_{11}. The first and second block of S_8 are both isomorphic to the principal block of A_8 via restriction of characters (see [3, Théorème 0.1]). The block of S_{11} is a RoCK block and must be Morita equivalent to its Brauer correspondent. Hence, B always belongs to one of 24 blocks in the above theorems.

Nevertheless, we found twelve further Morita equivalence classes among the non-principal blocks while checking the character library in GAP. For instance, a non-principal block of the double cover $2.A_6$. Recall that according to Donovan's Conjecture the total number of Morita equivalence classes of blocks with defect group D should be finite.

Now we turn to $p = 5$. In the table below the examples are always faithful blocks of the given group. The Morita equivalence class of such a block is indeed uniquely determined as we will see in the proof. In order to distinguish Morita equivalence classes, we also list the multiplicity $c(B_D)$ of 1 as an elementary divisor of the Cartan matrix of B_D and the Loewy length $LL(ZB_D)$ of the center of B_D (considered as an F-algebra).

Theorem 5. *Let B be a block of a finite group G with defect group $D \cong Z_5 \times Z_5$. Then the Brauer correspondent B_D of B in $\mathrm{N}_G(D)$ is Morita equivalent to exactly one of the following blocks:*

no.	$I(B)$	*group*	$k(B_D)$	$l(B_D)$	$c(B_D)$	$LL(ZB_D)$	*isotypy*	*comments*
1	1	$25:2$	25	1	0	9	✓	*D, nilpotent*
2	Z_2	$50:3$	20	2	0	7	✓	$D_{10}\times Z_5$
3	Z_2	$50:4$	14	2	1	5	✓	*Frobenius group*
4	Z_3	$75:2$	11	3	2	5	✓	*Frobenius group*
5	Z_4	$100:9$	25	4	0	6	✓	$(Z_5\rtimes Z_4)\times Z_5$
6	Z_4	$100:10$	13	4	2	4	✓	
7	Z_4	$100:11$	10	4	3	3	✓	*Frobenius group*
8	Z_4	$100:12$	10	4	3	5	✓	*Frobenius group*
9	Z_2^2	$100:13$	16	4	1	5	✓	D_{10}^2
10	Z_2^2	$200:24$	13	1	0	5	✓	*non-principal*
11	C_6	$150:6$	10	6	5	5		*Frobenius group*
12	S_3	$150:5$	13	3	1	5	✓	
13	Z_8	$200:40$	11	8	7	3		*Frobenius group*
14	$Z_4\times Z_2$	$200:41$	20	8	3	4	✓	$(Z_5\rtimes Z_4)\times D_{10}$
15	$Z_4\times Z_2$	$400:118$	14	2	0	4	✓	*non-principal*
16	$Z_4\times Z_2$	$200:42$	14	8	5	3		
17	$Z_4\times Z_2$	$400:125$	8	2	1	3	✓	*non-principal*
18	Q_8	$200:44$	8	5	4	3		*Frobenius group*
19	D_8	$200:43$	14	5	2	5	✓	$D_{10}\wr Z_2$
20	D_8	$400:131$	11	2	1	5	✓	*non-principal*
21	Z_{12}	$300:24$	14	12	11	3		*Frobenius group*
22	D_{12}	$300:25$	14	6	3	5		
23	D_{12}	$600:59$	11	3	2	5	✓	*non-principal*
24	$Z_3\rtimes Z_4$	$300:23$	8	6	5	2		*Frobenius group*
25	Z_4^2	$400:205$	25	16	9	3		$(Z_5\rtimes Z_4)^2$
26	Z_4^2	$800:957$	13	4	1	3	✓	*non-principal*
27	Z_4^2	$1600:5606$	10	1	0	3	✓	*non-principal*
28	D_8*Z_4	$400:207$	16	10	6	3		
29	D_8*Z_4	$800:968$	10	4	2	3	✓	*non-principal*
30	M_{16}	$400:206$	13	10	8	3		
31	Z_{24}	$600:149$	25	24	23	2		$\mathrm{AGL}(1,25)$
32	$\mathrm{SL}_2(3)$	$600:150$	8	7	6	2		
33	$Z_4\times S_3$	$600:151$	16	12	9	3		
34	$Z_4\times S_3$	$1200:491$	10	6	5	3		*non-principal*
35	$Z_3\rtimes Z_8$	$600:148$	13	12	11	2		*Frobenius group*
36	$Z_4\wr Z_2$	$800:1191$	20	14	9	3		$(Z_5\rtimes Z_4)\wr Z_2$
37	$Z_4\wr Z_2$	$1600:9791$	11	5	3	3	✓	*non-principal*
38	$\mathrm{SL}_2(3)*Z_4$	$1200:947$	16	14	12	3		
39	$\Gamma\mathrm{L}_1(25)$	$1200:946$	20	18	16	3		$\mathrm{A\Gamma L}(1,25)$
40	$\mathrm{SL}_2(3)\rtimes Z_4$	$-$	20	16	12	3		`PrimitiveGroup`$(25,19)$

Proof. Most of the arguments work as in Theorem 3, but we have to be careful if the Schur multiplier of $E:=I(B)$ is larger than Z_2. For $E\cong Z_4^2$ the Schur multiplier is Z_4 by the Künneth formula. A full cover of $L:=D\rtimes E$ is given by $\widehat{L}:=$ `SmallGroup`$(1600,5606)$. This group has four blocks: the principal block, two faithful blocks and a non-faithful block. One can show by computer that $\widehat{L}$ has an automorphism acting as inversion on $\mathrm{Z}(\widehat{L})\cong Z_4$. It follows that the two faithful blocks are isomorphic. Hence, $\widehat{L}$ has only three types of blocks and they have pairwise distinct invariants. In this way we obtain the lines 25, 26 and 27 in the table.

For $E\cong D_8*Z_4$ the Schur multiplier is Z_2^2. A full cover of L is given by $\widehat{L}:=$ `SmallGroup`$(1600,5725)$. Fortunately, $\widehat{L}$ has an automorphism of order 3 which permutes $\mathrm{Z}(\widehat{L})$. Hence, the three non-principal blocks of $\widehat{L}$ are all isomorphic and the Morita equivalence class of B is uniquely determined in this case. This yields lines 28 and 29 in the table. The existence of an isotypy between B and B_D is an outcome of our algorithm.

Finally, we need to verify that all forty blocks are pairwise not Morita equivalent. Comparing the numerical invariants leads to the blocks no. 4 and 23. These can be distinguished using [11] just as in Theorem 3. □

The cases 1, 10 and 27 on our list confirm a result of Kessar–Linckelmann [12] (for a recent generalization see [9]).

An analysis of the Brauer trees mentioned above reveals that the Morita equivalence classes of blocks with defect group Z_5 are represented by the principal blocks of the groups Z_5, D_{10}, $Z_5 \rtimes Z_4$, A_5, S_5 or $\mathrm{Sz}(8)$. Now taking direct products of two of these groups already yields 15 Morita equivalence classes of principal blocks which do not belong to the classes in Theorem 5. For symmetric groups there are precisely 26 Morita equivalent classes of blocks with defect group $Z_5 \times Z_5$ (see [30]). We have found over 100 more classes by checking the character table library in GAP.

4 The Cartan method revisited

As we have seen in the last section, Usami and Puig's method fails in some situations. We provide an alternative by improving the Cartan method described in [26, Section 4.2]. This reduces the possible Cartan matrices of blocks to a handful of choices which can be discussed individually (most cases contradict Alperin's Weight Conjecture). As another advantage, the method applies equally well to non-abelian defect groups. To do so, the inertial quotient $I(B)$ must be replaced by the fusion system $\mathcal{F}$ of B. Nevertheless, the reader will notice many similarities to Usami–Puig's approach (in fact both methods can produce perfect isometries, see [32, Theorem 6.1]).

The key idea is the orthogonality between the decomposition matrix Q_1 of B and the generalized decomposition matrices Q_u for $u \in D \setminus \{1\}$ (see [26, Theorem 1.14]). We wish to compute Q_u with Plesken's algorithm applied to the equation $Q_u^{\mathrm{t}}\overline{Q_u} = C_u$ where C_u is the Cartan matrix of b_u (as before, we obtain C_u from the dominated block $\overline{b_u}$). To this end, we first need to "integralize" Q_u by expressing its columns as linear combinations of an integral basis in a suitable cyclotomic field (in our situation we use the basis 1, $\zeta = e^{2\pi i/3}$ of $\mathbb{Q}_3$). We put these new integral columns in a "fake" generalized decomposition matrix $\widetilde{Q}_u$. Although $\widetilde{Q}_u$ has more columns than Q_u, both matrices generated the same orthogonal space. In practice we will remove linearly dependent columns from $\widetilde{Q}_u$ to obtain matrices with $l(b_u)|\operatorname{Aut}(\langle u\rangle) : \mathcal{N}_u|$ columns where $\mathcal{N}_u := \mathrm{N}_G(\langle u\rangle, b_u)/\mathrm{C}_G(u)$ (this step is not strictly necessary). The scalar products between the columns of $\widetilde{Q}_u$ can be computed by studying the action of $\mathcal{N}_u$ on $\mathrm{IBr}(b_u)$. This gives rise to the "fake" Cartan matrix $\widetilde{C}_u := \widetilde{Q}_u^{\mathrm{t}}\widetilde{Q}_u$ (this was developed in general in [29, Theorem 2.1], but in our case hand calculations will do). We obtain such a matrix for every $\mathcal{F}$-conjugacy class of cyclic subgroups of D. Unfortunately, $\widetilde{C}_u$ depends crucially on the chosen basic set for b_u. We introduce the following results to find "good" basic sets.

Lemma 6. *Let $C = (d + \delta_{ij})_{i,j=1}^{n} \in \mathbb{Z}^{n\times n}$ where d and n are positive integers. Let*

$$\operatorname{Aut}(C) := \{A \in \mathrm{GL}(n, \mathbb{Z}) : A^{\mathrm{t}}CA = C\}.$$

Then there exists a natural isomorphism

$$\operatorname{Aut}(C) \cong \begin{cases} S_n \times Z_2 & \text{if } d > 1 \text{ or } n = 1, \\ S_{n+1} \times Z_2 & \text{if } d = 1 < n \end{cases}$$

sending $A \in \mathrm{Aut}(C)$ to $\pm P$ where P is a permutation matrix.

Proof. We may assume that $n > 1$. We first solve the matrix equation $(c_{ij}) = C = Q^{\mathrm{t}}Q$ where $Q \in \mathbb{Z}^{(n+d)\times n}$ has non-zero rows. To this end, we define the positive definite matrix

$$W := (w_{ij}) = \frac{1}{2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \in \mathbb{Q}^{n\times n}.$$

For the rows $q_1, \ldots, q_{n+d}$ of our putative solution $Q = (q_{ij})$ we obtain

$$\begin{aligned} n + d \le \sum_{i=1}^{n+d} q_i W q_i^{\mathrm{t}} &= \sum_{i=1}^{n+d} \sum_{1\le s,t\le n} w_{st} q_{is} q_{it} = \sum_{1\le s,t\le n} w_{st} c_{st} \\ &= \sum_{i=1}^{n} c_{ii} - \sum_{i=1}^{n-1} c_{i,i+1} = n(d+1) - (n-1)d = n + d. \end{aligned}$$

It follows that

$$\frac{1}{2}\Big(q_{i1}^2 + q_{in}^2 + \sum_{j=1}^{n-1}(q_{ij} - q_{i,j+1})^2\Big) = \sum_{j=1}^{n} q_{ij}^2 - \sum_{j=1}^{n-1} q_{ij} q_{i,j+1} = q_i W q_i^{\mathrm{t}} = 1$$

for $i = 1, \ldots, n + d$. Hence, every row of Q has the form $\pm(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$. Now it is easy to see that

$$Q := \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ -1 & \cdots & -1 \\ \vdots & & \vdots \\ -1 & \cdots & -1 \end{pmatrix} \in \mathbb{Z}^{(n+d)\times n}$$

is the only solution of the equation $C = Q^{\mathrm{t}}Q$ (up to permutations and signs of rows) of size $(n+d)\times n$. Hence, for $A \in \mathrm{Aut}(C)$ there exists a signed permutation matrix P such that $QA = PQ$, since $(QA)^{\mathrm{t}}(QA) = A^{\mathrm{t}}CA = C$. Note that A is just the upper part of PQ. At closer look at line $d+1$ reveals that P has in fact a uniform sign, i. e. P or $-P$ is a permutation matrix. The map $f : \mathrm{Aut}(C) \to S_{n+d} \times Z_2$, $A \mapsto P$ is clearly a monomorphism.

For $d = 1$ the matrix P has size $(n + 1) \times (n + 1)$ and conversely every such permutation matrix P gives rise to some $A \in \mathrm{Aut}(C)$ such that $PQ = QA$, since the upper part of PQ has determinant ± 1. Hence, f is surjective in this case. On the other hand, if $d > 1$, then P must fix the last d rows of Q and therefore A or $-A$ itself must be a permutation matrix. Thus, in this case $\mathrm{Aut}(C)$ consists of the permutation matrices and their negatives. □

The next proposition generalizes [29, Lemma 3.3].

Proposition 7. *Let B be a p-block of a finite group G with abelian defect group D such that $E := I(B)$ is abelian and $D \rtimes E$ is a Frobenius group. Suppose that $p > 2$ or $|E| < |D| - 1$. Suppose further that B is perfectly isometric to its Brauer correspondent B_D in $\mathrm{N}_G(D)$. Let $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(B) = B$.*

Then there exists a basic set Φ of B such that $\mathrm{IBr}(B)$ and Φ are isomorphic α-sets and the Cartan matrix of B with respect to Φ is

$$C=\Big(\frac{|D|-1}{|E|}+\delta_{ij}\Big)_{i,j=1}^{|E|}.$$

Proof. It is well-known that the abelian Frobenius complement E is in fact cyclic. Therefore, E has trivial Schur multiplier and B_D is Morita equivalent to the group algebra of the Frobenius group $L:=D\rtimes E$. The irreducible characters of L are either inflations from E or induced from D. The inflations from E can be identified with the irreducible Brauer characters of L. On the other hand, the number of distinct irreducible characters induced from D is $d:=\frac{|D|-1}{|E|}$. Consequently, the decomposition matrix of B_D is

$$Q_D:=\begin{pmatrix}1&&0\\&\ddots&\\0&&1\\1&\cdots&1\\\vdots&&\vdots\\1&\cdots&1\end{pmatrix}$$

and the Cartan matrix is $Q_D^{\mathrm{t}}Q_D=C$ as given in the statement.

Let $Q_B:=(d_{\chi\varphi})$ and $C_B:=Q_B^{\mathrm{t}}Q_B=(c_{\varphi\mu})$ be the decomposition matrix and the Cartan matrix of B respectively. Since B is perfectly isometric to B_D, there exist $S\in\mathrm{GL}(l(B),\mathbb{Z})$ and a signed permutation matrix $T\in\mathrm{GL}(k(B),\mathbb{Z})$ such that $Q_DS=TQ_B$ (see [32, Theorem 4.2]). Note that Q_D differs from Q in the proof of Lemma 6 only by the signs of the last rows. We replace T by T' accordingly such that

$$QS=T'Q_B.$$

After rearranging $\mathrm{Irr}(B)$, we may assume that T' is just an identity matrix with signs.

The action of α on $\mathrm{IBr}(B)$ permutes the columns of Q_B. Let P be the corresponding permutation matrix. Since

$$c_{\varphi^\alpha,\mu^\alpha}=\sum_{\chi\in\mathrm{Irr}(B)}d_{\chi,\varphi^\alpha}d_{\chi,\mu^\alpha}=\sum_{\chi\in\mathrm{Irr}(B)}d_{\chi^{\alpha^{-1}},\varphi}d_{\chi^{\alpha^{-1}},\mu}=c_{\varphi\mu}$$

for $\varphi,\mu\in\mathrm{IBr}(B)$, it follows that P commutes with C_B. We compute

$$(S^{-\mathrm{t}}P^{\mathrm{t}}S^{\mathrm{t}})C(SPS^{-1})=(S^{-\mathrm{t}}P^{\mathrm{t}}S^{\mathrm{t}})S^{-\mathrm{t}}C_BS^{-1}(SPS^{-1})=S^{-\mathrm{t}}P^{\mathrm{t}}C_BPS^{-1}=S^{-\mathrm{t}}C_BS^{-1}=C,$$

i. e. $A:=SPS^{-1}\in\mathrm{Aut}(C)$. By Lemma 6, there exist a permutation matrix P_A and a sign $\epsilon=\pm1$ such that

$$QA=\epsilon P_AQ.$$

Suppose first that P_A has no fixed points. Then $d=1$ and neither A nor $-A$ is a permutation matrix. It follows that $-\epsilon=\mathrm{tr}(A)=\mathrm{tr}(SPS^{-1})=\mathrm{tr}(P)\ge0$ and $\epsilon=-1$. We compute

$$T'Q_BP=QSP=QAS=-P_AQS=-P_AT'Q_B$$

and $Q_BP=-T'P_AT'Q_B$. Now Q_B and Q_BP are non-negative matrices and $-T'P_AT'$ is a signed permutation matrix. This can only fit together if $-T'P_AT'=P_A$. In particular,

$$\det P_A=\det(-T'P_AT')=(-1)^{|E|+d}\det P_A$$

and $|D|=|E|+1=|E|+d$ is even. This contradicts the hypothesis $p>2$ (whenever $|E|=|D|-1$).

Now since P_A has a fixed point, there exists yet another permutation matrix U such that UP_AU^{-1} fixes the last coordinate. We regard UP_AU^{-1} as a permutation matrix P' of size $n \times n$. Let $A_U \in \mathrm{Aut}(C)$ be the preimage of U under the isomorphism in Lemma 6, that is, $QA_U = UQ$. Then

$$QA_UAA_U^{-1} = UQAA_U^{-1} = \epsilon UP_AQA_U^{-1} = \epsilon UP_AU^{-1}Q.$$

By the shape of Q, it follows that $A_UAA_U^{-1} = \epsilon P'$. We may replace S by A_US if we adjust T and T' accordingly. Then we obtain $SPS^{-1} = \epsilon P'$. By way of contradiction, we assume that $\epsilon = -1$. Let $(a_1, \ldots, a_l)$ be the last row of Q_B. Since $QS = T'Q_B$, we obtain

$$(a_1, \ldots, a_l)P = \pm(1, \ldots, 1)SP = \mp(1, \ldots, 1)P'S = \mp(1, \ldots, 1)S = -(a_1, \ldots, a_l).$$

This is impossible since $a_1, \ldots, a_l \geq 0$ and at least one $a_i > 0$. Hence, $\epsilon = 1$. A comparison of the eigenvalues shows that P and P' have the same cycle type. Consequently, P and P' are conjugate inside $S_n \leq \mathrm{Aut}(C)$ as is well-known. Hence, we may change S, T and T' again such that $SPS^{-1} = P$.

Finally, we define $\Phi = \{\widehat{\varphi}_i : i = 1, \ldots, l\}$ with $\widehat{\varphi}_i := \sum_{j=1}^{l} s_{ji}\varphi_j$ where $\mathrm{IBr}(B) = \{\varphi_1, \ldots, \varphi_l\}$ and $S = (s_{ij})$. Then

$$(\widehat{\varphi}_1^\alpha, \ldots, \widehat{\varphi}_l^\alpha) = (\varphi_1^\alpha, \ldots, \varphi_l^\alpha)S = (\varphi_1, \ldots, \varphi_l)PS = (\varphi_1, \ldots, \varphi_l)SP = (\widehat{\varphi}_1, \ldots, \widehat{\varphi}_l)P,$$

i. e. Φ and $\mathrm{IBr}(B)$ are isomorphic α-sets. Moreover, the decomposition matrix of B with respect to Φ is $Q_BS^{-1} = Q$ and the Cartan matrix is $Q^{\mathrm{t}}Q = C$. $\square$

The proof of Proposition 7 does not go through for $p = 2$ and $|E| = |D| - 1$ as one can see from the possibility

$$Q_B = \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ 1 & . & 1 \\ . & 1 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & . & 1 \\ . & 1 & . \\ . & -1 & -1 \end{pmatrix}$$

with α being the transposition $(1, 2)$ on $\mathrm{IBr}(B)$. In those exceptions D is elementary abelian and E is a Singer cycle. It has been shown recently by McKernon [19] that in this situation B is Morita equivalent to the Brauer correspondent B_D or to the principal block of $\mathrm{SL}(2, |D|)$. From the shape of the Cartan matrix of $\mathrm{SL}(2, |D|)$ (see [1]), it can be deduced that Proposition 7 still holds in those cases. Hence, the hypothesis $p > 2$ or $|E| < |D| - 1$ is actually superfluous.

Now we get back to the explanation of the Cartan method. If D is abelian, then all characters in $\mathrm{Irr}(B)$ have height 0 (by [13]) and therefore every row of Q_u (and of $\widetilde{Q}_u$) is non-zero. In general, the heights of the characters influence the p-adic valuation of the so-called *contribution matrix*

$$M^u := (m^u_{\chi\psi})_{\chi,\psi \in \mathrm{Irr}(B)} = |D|Q_uC_u^{-1}Q_u^{\mathrm{t}} \in \mathbb{C}^{k(B) \times k(B)}$$

(see [26, Proposition 1.36]). This matrix is also of interest, because it only depends on the order of $\mathrm{Irr}(B)$ and possible signs, but not on the chosen basic set of b_u. In particular, there are at most $2^{k(B)}k(B)!$ choices for M^u, while there are potentially infinitely many choices for Q_u (one for every basic set). Note that

$$\widetilde{M}^u := |D|\widetilde{Q}_u\widetilde{C}_u^{-1}\widetilde{Q}_u^{\mathrm{t}} = \sum_{\gamma \in \mathrm{Aut}(\langle u \rangle)/\mathcal{N}_u} M^{\gamma(u)} \in \mathbb{Z}^{k(B) \times k(B)},$$

since there exists an invertible complex matrix U such that $(Q_{\gamma(u)} : \gamma \in \mathrm{Aut}(\langle u \rangle)/\mathcal{N}_u) = \widetilde{Q}_uU$.

Finally, Broué–Puig's $*$-construction, introduced in [6], gives congruence relations between the M^u where u runs through D. Specifically, if λ is an $\mathcal{F}$-invariant generalized character of D, then

$$\sum_{u\in\mathcal{S}}\lambda(u)M^u=|D|(\lambda*\chi,\psi)_{\chi,\psi\in\mathrm{Irr}(B)}$$

where $\mathcal{S}$ is a set of representatives for the $\mathcal{F}$-conjugacy classes of elements of D. In particular, $\sum_{u\in\mathcal{S}}M^u=|D|1_{k(B)}$ and

$$\sum_{u\in\mathcal{S}}\lambda(u)M^u\equiv 0\pmod{|D|}. \tag{4.1}$$

We summarize the steps of the Cartan method under the assumption that a defect group D and a fusion system $\mathcal{F}$ on D are given:

(1) Determine the $\mathcal{F}$-conjugacy classes of fully $\mathcal{F}$-centralized cyclic subgroups of D. Let $\mathcal{R}$ be a set of representatives of generators of these subgroups.

(2) For $u\in\mathcal{R}\setminus\{1\}$ determine the Cartan matrix C_u (up to E-compatible basic sets) of a Brauer correspondent b_u of B in $\mathrm{C}_G(u)$ by considering the dominated block $\overline{b_u}$ with defect group $\mathrm{C}_D(u)/\langle u\rangle$ and fusion system $\mathrm{C}_{\mathcal{F}}(u)/\langle u\rangle$ (see [27, Lemma 3]).

(3) For every possible action of $\mathcal{N}_u$ on $\mathrm{IBr}(b_u)$ compute the fake Cartan matrix $\widetilde{C}_u$.

(4) Solve the matrix equation $\widetilde{Q}_u^{\mathrm{t}}\widetilde{Q}_u=\widetilde{C}_u$ with Plesken's algorithm.

(5) Reduce the number of possibilities for $\widetilde{Q}_u$ by comparing contribution matrices (make use of heights and the $*$-construction).

(6) Form the matrix $\widetilde{Q}:=(\widetilde{Q}_u:u\in\mathcal{R}\setminus\{1\})$ of size $k(B)\times(k(B)-l(B))$.

(7) Compute an orthogonal complement $Q\in\mathbb{Z}^{k(B)\times l(B)}$ of $\widetilde{Q}$.

(8) $C:=Q^{\mathrm{t}}Q$ is the Cartan matrix of B up to basic sets.

5 Cartan matrices of local blocks

In this section we compute Cartan matrices of many 3-blocks of defect at most 4. They all occur as dominated Brauer correspondents of blocks with larger defect in the subsequent sections.

We first apply the Usami–Puig algorithm to the following situations.

Lemma 8. *Let B be a block of a finite group G with defect group $D\cong Z_3^3$ and inertial quotient $I(B)\cong D_8$ such that $D\rtimes I(B)\cong$ `SmallGroup`$(6^3,158)$. Then B is perfectly isometric to its Brauer correspondent in $\mathrm{N}_G(D)$. In particular, $l(B)\in\{2,5\}$ and if $l(B)=5$, then the Cartan matrix of B is given by*

$$\begin{pmatrix}7&5&2&1&6\\5&7&1&2&6\\2&1&7&5&6\\1&2&5&7&6\\6&6&6&6&15\end{pmatrix}$$

up to basic sets.

Proof. The given group $L := D \rtimes I(B)$ can be represented by $D = \langle x, y, z\rangle$ and $E := I(B) = \langle a, b, c\rangle$ such that

$$ {}^a x = x^{-1}, \quad {}^b y = y^{-1}, \quad {}^c x = y, \quad {}^c z = z^{-1}, \quad a^2 = b^2 = c^2 = [a, y] = [b, x] = [a, z] = [b, z] = 1. $$

Since $E \cong D_8$ has Schur multiplier Z_2, there are two possible cocycles γ. For $\gamma \neq 1$ we can consider the (unique) non-principal block of $\widehat{L} = D \rtimes D_{16} \cong \texttt{SmallGroup}(432, 582)$. The set $\mathcal{Q}$ in Step (b) of the Usami–Puig algorithm consists only of those subgroups $Q < D$ which are normal in L, since otherwise $|\mathrm{N}_E(Q)| \leq 4$. In all (three) cases we obtain the existence and uniqueness of the isometry Δ by Plesken's algorithm. $\square$

Lemma 9. *Let B be a block of a finite group G with defect group $D \cong Z_3^3$ and inertial quotient $I(B) \cong Z_4 \times Z_2$ such that $D \rtimes I(B) \cong \texttt{SmallGroup}(6^3, 156)$. Then B is perfectly isometric to its Brauer correspondent in $\mathrm{N}_G(D)$. In particular, $l(B) \in \{2, 8\}$ and if $l(B) = 8$, then the Cartan matrix of B is given by*

$$\begin{pmatrix} 6 & 2 & 3 & 4 & 2 & 2 & 4 & 4 \\ 2 & 6 & 4 & 3 & 4 & 4 & 2 & 2 \\ 3 & 4 & 6 & 2 & 4 & 4 & 2 & 2 \\ 4 & 3 & 2 & 6 & 2 & 2 & 4 & 4 \\ 2 & 4 & 4 & 2 & 6 & 4 & 3 & 2 \\ 2 & 4 & 4 & 2 & 4 & 6 & 2 & 3 \\ 4 & 2 & 2 & 4 & 3 & 2 & 6 & 4 \\ 4 & 2 & 2 & 4 & 2 & 3 & 4 & 6 \end{pmatrix}$$

up to basic sets.

Proof. As in the previous lemma, $E := I(B) = \langle a\rangle \times \langle b\rangle \cong Z_4 \times Z_2$ acts reducibly on $D = \langle x, y, z\rangle$ such that $L := D \rtimes I(B) = \langle x, y, a\rangle \times \langle z, b\rangle$. Again E has Schur multiplier Z_2 and there are two possible cocycles γ. For $\gamma \neq 1$ we can consider the non-principal block of $\texttt{SmallGroup}(432, 568)$.

The set $\mathcal{Q}$ in Step (b) of the Usami–Puig algorithm consists of 1, $\langle z\rangle$ and $\langle x, y\rangle$. Our algorithm works for $Q = 1$ without intervention, but needs some additional argument for the remaining two cases. We only deal with $Q = \langle z\rangle$, since the final case is similar, in fact easier. The arguments go along the lines of Proposition 2. We may assume that $G = \mathrm{N}_G(Q, b_Q)$. Let β_Q be a block of a suitable stem extension $\widehat{L}$ of L such that β_Q is isomorphic to $F_\gamma \overline{\mathrm{C}_L(Q)}$. Note that β_Q and $\overline{b_Q}$ have defect group $\langle x, y\rangle$ and inertial quotient $\mathrm{C}_E(Q) = \langle a\rangle \cong Z_4$. By Theorem 3 we know that $l(\beta_Q) = l(\overline{b_Q}) = 4$. Since $E/\mathrm{C}_E(Q) = \langle b\rangle$ is cyclic, Proposition 7 provides us with a basic set Φ of $\overline{b_Q}$ such that Φ and $\mathrm{IBr}(\overline{b_Q})$ are isomorphic E-sets and the Cartan matrix of $\overline{b_Q}$ with respect to Φ is $C = (2+\delta_{ij})_{i,j=1}^4$. This happens to be the Cartan matrix of β_Q. As in the proof of Proposition 2 we can extend Δ_0 by any bijection $\mathrm{PIM}(\beta_Q) \to \{\widehat{\varphi} : \varphi \in \Phi\}$. It suffices to show that $\mathrm{IBr}(\beta_Q)$ and $\mathrm{IBr}(\overline{b_Q})$ are isomorphic E-sets.

Suppose first that $\gamma = 1$. Then E acts trivially on $\mathrm{Irr}(\beta_Q)$ and $\overline{L} \cong \overline{\mathrm{C}_L(Q)} \times \langle b\rangle$. Therefore, β_Q is covered by two blocks of $\overline{L}$ and they both have inertial quotient Z_4. By [22, Proposition 3.14], $\overline{b_Q}$ is covered by two blocks of $\overline{G}$ with inertial quotient Z_4. According to Clifford theory, E must act trivially on $\mathrm{Irr}(\overline{b_Q})$.

Now suppose that $\gamma \neq 1$. Then β_Q is covered by only one block $\overline{B_Q}$ of $\widehat{L}/Q$ and $I(\overline{B_Q}) \cong Z_2$. Hence, $l(\overline{B_Q}) = 2$ by Theorem 3. Clifford theory implies that E acts as a double transposition on $\mathrm{IBr}(\beta_Q)$. Again by [22, Proposition 3.14], $\overline{b_Q}$ is covered by a unique block of $\overline{G}$ and this block has also two irreducible Brauer characters. Consequently, E also acts as a double transposition on $\mathrm{IBr}(\overline{b_Q})$. Hence, in any event, $\mathrm{IBr}(\beta_Q)$ and $\mathrm{IBr}(\overline{b_Q})$ are isomorphic E-sets. $\square$

In the following lemmas the Usami–Puig method fails basically because one of the two "bad" groups Z_8 and Q_8 from Theorem 3 is involved in $I(B)$. We make use of the Cartan method instead.

Lemma 10. *Let B be a block of a finite group G with defect group $D \cong Z_3^3$ and inertial quotient $I(B) \in \{Z_8, Q_8\}$ such that $D \rtimes I(B) \cong \texttt{SmallGroup}(6^3, s)$ where $s \in \{155, 161\}$. Then $l(B) \in \{2,5,8\}$ and in the latter two cases the Cartan matrix of B is*

$$\begin{pmatrix} 5&4&3&3&6\\ 4&5&3&3&6\\ 3&3&5&4&6\\ 3&3&4&5&6\\ 6&6&6&6&15 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 5&4&3&3&3&3&3&3\\ 4&5&3&3&3&3&3&3\\ 3&3&5&4&3&3&3&3\\ 3&3&4&5&3&3&3&3\\ 3&3&3&3&5&4&3&3\\ 3&3&3&3&4&5&3&3\\ 3&3&3&3&3&3&5&4\\ 3&3&3&3&3&3&4&5 \end{pmatrix}$$

up to basic sets.

Proof. Let $D = \langle x, y, z\rangle$ and $E := I(B)$. In both cases E acts regularly on $\langle x, y\rangle$ and inverts z. We will see that the local analysis does not depend on the isomorphism type of E and we will end up with exactly the same possibilities for the generalized decomposition matrices. Let $L := D \rtimes E$. Since the fusion system of B is the fusion system of L and every subgroup of D is fully $\mathcal{F}$-centralized, we may choose $\mathcal{R} = \{1, x, xz, z\}$ in the algorithm of the Cartan method. Since $\mathrm{C}_E(x) = \mathrm{C}_E(xz) = 1$ and $\mathrm{C}_E(z) \cong Z_4$ we obtain $l(b_x) = l(b_{xz}) = 1$ and $l(b_z) = 4$ by applying Theorem 3 to the dominated blocks $\overline{b_x}$ and so on. Since x is conjugate to x^{-1} in L, we have $Q_x = \widetilde{Q}_x$ and $C_x = \widetilde{C_x} = (27)$. On the other hand, xz is not conjugate to $(xz)^{-1}$. We may write $Q_{xz} = (a + b\zeta)$ and form $\widetilde{Q}_{xz} = (a, b)$ with $\zeta = e^{2\pi i/3}$ and $a, b \in \mathbb{Z}^{k(B)\times 1}$. Since $C_{xz} = (27)$, we compute

$$\widetilde{C}_{xz} = \begin{pmatrix} 1 & 1\\ \zeta & \overline{\zeta} \end{pmatrix}^{-\mathrm{t}} \begin{pmatrix} 27 & 0\\ 0 & 27 \end{pmatrix} \begin{pmatrix} 1 & 1\\ \overline{\zeta} & \zeta \end{pmatrix}^{-1} = 9\begin{pmatrix} 2 & 1\\ 1 & 2 \end{pmatrix}.$$

By Theorem 3, $\overline{b_z}$ is perfectly isometric to its Brauer first main theorem correspondent. Moreover, $\overline{D} \rtimes I(b_z) \cong Z_3^2 \rtimes Z_4$ is a Frobenius group. Hence by Proposition 7, there exists a basic set Φ of $\overline{b_z}$ such that

$$\mathcal{N}_z = \mathrm{N}_G(\langle z\rangle, b_z)/\mathrm{C}_G(z) \cong Z_2$$

acts on Φ and $\overline{b_z}$ has Cartan matrix $(2 + \delta_{ij})_{i,j=1}^4$ with respect to Φ. By [20, Theorem 9.10], we may regard Φ as a basic set of b_z and then $C_z = 3(2 + \delta_{ij})_{i,j=1}^4$. Applying [26, Theorem 4.2] to b_z yields $k(B) \le 18$. Additionally, $k(B)$ is always divisible by 3 (this can be seen from Q_x or [26, Proposition 1.31] in general). On the other hand,

$$k(B) - l(B) = l(b_x) + l(b_{xz}) + l(b_{(xz)^{-1}}) + l(b_z) = 7.$$

Since we may assume that $l(B) > 2$, we are left with the cases $(k(B), l(B)) \in \{(18, 11), (15, 8), (12, 5)\}$.

If ρ is the regular character of $\langle x, y\rangle$, then

$$\lambda := (9 \cdot 1_{\langle x,y\rangle} - \rho) \times 1_{\langle z\rangle}$$

is an E-invariant generalized character of D such that $\lambda(1) = \lambda(z) = 0$ and $\lambda(x) = \lambda(xz) = 9$. We obtain $M^x + \widetilde{M}^{xz} \equiv 0 \pmod 3$ from (4.1). Similarly, there exists an E-invariant λ with $\lambda(1) = \lambda(x) = 0$

and $\lambda(xz)=\lambda(z)=3$. This implies $\widetilde{M}^{xz}+M^z\equiv 0 \pmod 9$. Additionally, $M^1+M^x+\widetilde{M}^{xz}+M^z = 27\cdot 1_{k(B)}$. In particular,

$$2m^{xz}_{\chi\chi}+m^z_{\chi\chi}\in\{9,18\} \tag{5.1}$$

for all $\chi\in\mathrm{Irr}(B)$. We discuss the possible actions of $\mathcal{N}_z$ on Φ.

Case 1: $\mathcal{N}_z$ acts trivially on Φ.
Here we have $Q_z=\widetilde{Q}_z$ and $C_z=\widetilde{C}_z$. We apply Plesken's algorithm directly to the block matrix $\widetilde{C}_{xz}\oplus C_z$ with a prescribed set of rows fulfilling (5.1). Solutions exist only if $k(B)=15$. Taking also the congruence $\widetilde{M}^{xz}+M^z\equiv 0 \pmod 9$ into account, there is a unique solution up to basic sets:

$$(\widetilde{Q}_{xy},Q_z)=\begin{pmatrix}1&-2&1&1&1&1\\-2&1&1&1&1&1\\-1&-1&2&2&2&2\\1&1&1&.&.&.\\1&1&1&.&.&.\\1&1&1&.&.&.\\1&1&.&1&.&.\\1&1&.&1&.&.\\1&1&.&1&.&.\\1&1&.&.&1&.\\1&1&.&.&1&.\\1&1&.&.&1&.\\1&1&.&.&.&1\\1&1&.&.&.&1\\1&1&.&.&.&1\end{pmatrix}$$

Now it is easy to add the column Q_x under the condition $M^x+\widetilde{M}^{xz}\equiv 0 \pmod 3$. In the end, the Cartan matrix C of B is uniquely determined up to basic sets.

Case 2: $\mathcal{N}_z$ interchanges two characters of Φ.
We may assume that the first two characters of Φ are interchanged by $\mathcal{N}_z$. We express the first columns of Q_z with the basis $1,\zeta$ and compute

$$\widetilde{C}_z=\begin{pmatrix}1&1&.&.\\\zeta&\overline{\zeta}&.&.\\.&.&1&.\\.&.&.&1\end{pmatrix}^{-\mathrm{t}} C_z\begin{pmatrix}1&1&.&.\\\overline{\zeta}&\zeta&.&.\\.&.&1&.\\.&.&.&1\end{pmatrix}^{-1}=\begin{pmatrix}8&1&6&6\\1&2&0&0\\6&0&9&6\\6&0&6&9\end{pmatrix}.$$

It is convenient to reduce this matrix with the LLL algorithm to

$$\begin{pmatrix}2&1&1&1\\1&5&2&2\\1&2&5&2\\1&2&2&8\end{pmatrix}$$

(this amounts a change of basic sets). Plesken's algorithm applied to $\widetilde{C}_{xz}\oplus\widetilde{C}_z$ under the restriction

(5.1) yields $k(B)=12$. As in Case 1, there is in fact a unique solution:

$$(\widetilde{Q}_{xy},\widetilde{Q}_z)=\begin{pmatrix}1&1&1&1&1&1\\2&2&.&1&1&1\\-2&1&.&.&.&1\\1&-2&.&.&.&1\\-1&-1&1&.&.&.\\-1&-1&.&.&.&2\\-1&-1&.&1&.&.\\-1&-1&.&1&.&.\\-1&-1&.&1&.&.\\-1&-1&.&.&1&.\\-1&-1&.&.&1&.\\-1&-1&.&.&1&.\end{pmatrix}$$

Combined with the possibilities for Q_x one gets the Cartan matrix of B up to basic sets.

Case 3: $\mathcal{N}_z$ has two orbits of length 2 on Φ.
Let $(1,2)(3,4)$ be the cycle structure of $\mathcal{N}_z$ on Φ. Then

$$\widetilde{C}_z=\begin{pmatrix}1&1&.&.\\\zeta&\overline{\zeta}&.&.\\.&.&1&1\\.&.&\zeta&\overline{\zeta}\end{pmatrix}^{-\mathrm{t}}C_z\begin{pmatrix}1&1&.&.\\\overline{\zeta}&\zeta&.&.\\.&.&1&1\\.&.&\overline{\zeta}&\zeta\end{pmatrix}^{-1}=\begin{pmatrix}8&1&6&0\\1&2&0&0\\6&0&8&1\\0&0&1&2\end{pmatrix}\sim_{LLL}\begin{pmatrix}2&.&1&1\\.&2&1&.\\1&1&4&2\\1&.&2&8\end{pmatrix}.$$

It turns out that Plesken's algorithm for $\widetilde{C}_{xz}\oplus\widetilde{C}_z$ only has solutions if $k(B)=9$. This case was already excluded. □

With the notation of the proof above, we note that Case 1 occurs if $I(B)\cong Z_8$ and Case 2 occurs if $I(B)\cong Q_8$. Case 3 contradicts Alperin's Weight Conjecture (cf. remark after Theorem 3).

We also need an extension of the defect group in Lemma 10.

Lemma 11. *Let B be a block of a finite group G with defect group $D\cong Z_3^4$ and inertial quotient $I(B)\cong Z_8$ such that $D\rtimes I(B)\cong$ `SmallGroup`$(6^3,155)\times Z_3$. Then the Cartan matrix of B is $3C$ where C is one of the possible Cartan matrices in Lemma 10.*

Proof. The proof follows along the lines of [27, Proposition 16]. Let $E:=I(B)$. We note that $D=D_1\times D_2$ with $D_1:=[D,E]=\langle x,y,z\rangle\cong Z_3^3$ and $D_2:=\mathrm{C}_E(D)=\langle w\rangle\cong Z_3$. With the notation of Lemma 10, a set of representatives for the E-orbits on D is given by $\mathcal{R}=\{w^i,xw^i,zw^i,xzw^i:i=0,1,2\}$. The character group $\mathrm{Irr}(D_2)$ acts semiregularly on $\mathrm{Irr}(B)$ via the $*$-construction. By [27, Lemma 10], the generalized decomposition matrix Q_{uw^i} (where $u\in\{1,x,z,xz\}$) has the form

$$Q_{uw^i}=\begin{pmatrix}A_{uw^i}\\\zeta^iA_{uw^i}\\\overline{\zeta}^iA_{uw^i}\end{pmatrix}.$$

Now the ordinary decomposition matrix Q_1 is orthogonal to Q_r for all $r\in\mathcal{R}\setminus\{1\}$ if and only if A_1 is orthogonal to A_x, A_z and A_{xz}. Just as in Lemma 10 we have $I(b_x)=I(b_{xz})=1$ and $I(b_z)\cong Z_4$. Therefore the corresponding Cartan matrices C_x, C_z and C_{xz} are given by [23]. By the structure of the matrix above, we obtain $3A_x^{\mathrm{t}}\overline{A_x}=C_x$ and similarly for z and xz. Consequently, the matrices A_x, A_z and A_{xz} fulfill the very same relations as the matrices Q_x, Q_z and Q_{xz} in the proof of Lemma 10. In particular, we obtain exactly the same possibilities for A_1. From that we compute $C=3A_1^{\mathrm{t}}A_1$. □

Lemma 12. *Let B be a block of a finite group with defect group $D \cong Z_3^3$ and $I(B) \cong Q_8$ such that $D \rtimes I(B) \cong M_9 \times Z_3$. Then $l(B) \in \{2, 5, 8\}$ and in the latter two cases the Cartan matrix of B is*

$$3 \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 5 \end{pmatrix} \qquad \text{or} \qquad 3(1+\delta_{ij})_{i,j=1}^{8}$$

up to basic sets.

Proof. Let $D := \langle x, y, z\rangle$ such that $E := I(B) \cong Q_8$ acts regularly on $\langle x, y\rangle$ and $\mathrm{C}_D(E) = \langle z\rangle$. The proof is similar as the previous one. With the notation used there, A_1 is just the orthogonal complement of A_x. Observe that $l(b_x) = 1$ and x is conjugate to x^{-1} under E. So there are essentially three choices for A_x:

$$(2,2,1)^{\mathrm{t}}, \qquad (2,1,1,1,1)^{\mathrm{t}}, \qquad (1,\ldots,1)^{\mathrm{t}}.$$

The claim follows as usual. $\square$

For our next local block we need to establish E-compatible basic sets of virtually non-existent blocks.

Lemma 13. *Let B be a block of finite group G with defect group $D \cong Z_3^2$ such that $D \rtimes I(B) \cong M_9$. Suppose that B is fixed by some automorphism $\alpha \in \mathrm{Aut}(G)$ of order 2. Then there exists a basic set Φ such that Φ and $\mathrm{IBr}(B)$ are isomorphic α-sets and the Cartan matrix of B with respect to Φ is one of the following*

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 5 \end{pmatrix}, \qquad (1+\delta_{ij})_{i,j=1}^{8}.$$

Proof. Since $I(B)$ acts regularly on D, there is only one non-trivial subsection (x, b_x) and $l(b_x) = 1$. As in the previous lemma, the generalized decomposition matrix Q_x is one of the following: $(2,2,1)^{\mathrm{t}}$, $(2,1,\ldots,1)^{\mathrm{t}}$ or $(1,\ldots,1)^{\mathrm{t}}$ up to signs. From that we obtain the decomposition matrix Q of B up to basic sets. More precisely, there exists some $S \in \mathrm{GL}(l(B),\mathbb{Z})$ and an identity matrix with signs T such that TQS is one of the following

$$\begin{pmatrix} 1 & . \\ . & 1 \\ 2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 1 & \cdots & 1 \end{pmatrix} \in \mathbb{Z}^{9\times 8}.$$

If α acts trivially on $\mathrm{IBr}(B)$, then we choose Φ according to S. The Cartan matrix with respect to Φ is then given by $\widehat{C} := S^{\mathrm{t}}Q^{\mathrm{t}}QS$ as in the statement. Thus, we may assume that α acts non-trivially on $\mathrm{IBr}(B)$. Suppose first that $l(B) = 2$. Since α interchanges the two Brauer characters of B, the Cartan matrix of B has the form $C = Q^{\mathrm{t}}Q = \left(\begin{smallmatrix} s & t \\ t & s \end{smallmatrix}\right)$. Since C has the same elementary divisors as $\widehat{C} = \left(\begin{smallmatrix} 5 & 4 \\ 4 & 5 \end{smallmatrix}\right)$, we conclude that $9 = \det C = s^2 - t^2 = (s+t)(s-t)$. This easily implies $C = \widehat{C}$. Hence, we may choose $\Phi = \mathrm{IBr}(B)$.

Suppose next that $l(B) = 5$. Since α has order 2, we may arrange $\mathrm{IBr}(B) = \{\varphi_1, \ldots, \varphi_5\}$ such that α fixes φ_5. Let P be the permutation matrix on the columns of Q induced by α. Then

$$(S^{\mathrm{t}} P^{\mathrm{t}} S^{-\mathrm{t}}) \widehat{C} (S^{-1} P S) = \widehat{C},$$

i. e. $S^{-1}PS \in \mathrm{Aut}(\widehat{C})$. One can show by computer (or as in Lemma 6) that $\mathrm{Aut}(\widehat{C}) \cong S_5 \times Z_2$ where Z_2 is generated by the negative identity matrix and S_5 contains the permutation matrices on the first four coordinates. In particular, $P \in \mathrm{Aut}(\widehat{C})$. Since P and $S^{-1}PS$ have the same rational canonical form, the computer tells us that P and $S^{-1}PS$ are conjugate inside $\mathrm{Aut}(\widehat{C})$. Hence, we may assume that $PS = SP$. Now the claim follows as in the proof of Proposition 7.

Now let $l(B) = 8$. Then $\mathrm{Aut}(\widehat{C})$ was already computed in Lemma 6 and we can repeat the arguments in Proposition 7 word by word. □

Lemma 14. *Let B be a block of a finite group with defect group $D \cong Z_3^3$ and $I(B) \cong Q_8 \times Z_2$ such that $D \rtimes I(B) \cong M_9 \times S_3$. Let C be the Cartan matrix of B. Then there exists a matrix $W \in \mathbb{R}^{l(B) \times l(B)}$ such that $xWx^{\mathrm{t}} \geq 1$ for all $x \in \mathbb{Z}^{l(B)} \setminus \{0\}$ and $\mathrm{tr}(WC) \leq 27$.*

Proof. Let $D = \langle x, y, z \rangle$ and $E := I(B) = \langle a, b, c \rangle$ such that

$$L := D \rtimes E = \langle x, y, a, b \rangle \times \langle z, c \rangle \cong M_9 \times S_3.$$

The B-subsections are represented by $\mathcal{R} = \{1, x, z, xz\}$ and all of these elements are conjugate to their inverses under L. As usual, we obtain $l(b_x) = 2$, $l(b_{xz}) = 1$ and $l(b_z) \in \{2, 5, 8\}$. Moreover, $\widetilde{Q}_{xz} = Q_{xz}$ and $\widetilde{C}_{xz} = C_{xz} = (27)$. If $\mathcal{N}_x = \mathrm{N}_G(\langle x \rangle, b_x)/\mathrm{C}_G(x) \cong \langle a^2 \rangle$ acts non-trivially on $\mathrm{IBr}(b_x)$, then the *exact* Cartan matrix of b_x is $C_x = 9\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ as can be seen from the elementary divisors just as in the proof of Lemma 13. If, on the other hand, $\mathcal{N}_x$ acts trivially on $\mathrm{IBr}(b_x)$ we choose a basic set such that $C_x = 9\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ So there are two possibilities

$$\widetilde{C}_x = C_x = 9 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \text{or} \qquad \widetilde{C}_x = \begin{pmatrix} 1 & 1 \\ \zeta & \overline{\zeta} \end{pmatrix}^{-\mathrm{t}} C_x \begin{pmatrix} 1 & 1 \\ \overline{\zeta} & \zeta \end{pmatrix}^{-1} = 3 \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix},$$

which we denote by (A) and (B). The block $\overline{b_z}$ fulfills the hypothesis of Lemma 13. We choose a corresponding basic set Φ and observe that $\mathcal{N}_z = \mathrm{N}_G(\langle z \rangle, b_z)/\mathrm{C}_G(z) \cong \langle c \rangle$ has cycle type (1), (2), (2^2),

(2^3), (2^4) provided $l(b_z)$ is large enough. The relevant fake Cartan matrices $\widetilde{C}_z$ are

$$3\begin{pmatrix}5&4\\4&5\end{pmatrix},\begin{pmatrix}14&1\\1&2\end{pmatrix},3\begin{pmatrix}2&1&1&1&2\\1&2&1&1&2\\1&1&2&1&2\\1&1&1&2&2\\2&2&2&2&5\end{pmatrix},\begin{pmatrix}5&1&3&3&6\\1&2&.&.&.\\3&.&6&3&6\\3&.&3&6&6\\6&.&6&6&15\end{pmatrix},\begin{pmatrix}5&1&3&.&6\\1&2&.&.&.\\3&.&5&1&6\\.&.&1&2&.\\6&.&6&.&15\end{pmatrix}$$

$$3(1+\delta_{ij})_{i,j=1}^{8},\begin{pmatrix}5&1&3&3&3&3&3&3\\1&2&.&.&.&.&.&.\\3&.&6&3&3&3&3&3\\3&.&3&6&3&3&3&3\\3&.&3&3&6&3&3&3\\3&.&3&3&3&6&3&3\\3&.&3&3&3&3&6&3\\3&.&3&3&3&3&3&6\end{pmatrix},\begin{pmatrix}5&1&3&.&3&3&3&3\\1&2&.&.&.&.&.&.\\3&.&5&1&3&3&3&3\\.&.&1&2&.&.&.&.\\3&.&3&.&6&3&3&3\\3&.&3&.&3&6&3&3\\3&.&3&.&3&3&6&3\\3&.&3&.&3&3&3&6\end{pmatrix},$$

$$\begin{pmatrix}5&1&3&.&3&.&3&3\\1&2&.&.&.&.&.&.\\3&.&5&1&3&.&3&3\\.&.&1&2&.&.&.&.\\3&.&3&.&5&1&3&3\\.&.&.&.&1&2&.&.\\3&.&3&.&3&.&6&3\\3&.&3&.&3&.&3&6\end{pmatrix},\begin{pmatrix}5&1&3&.&3&.&3&.\\1&2&.&.&.&.&.&.\\3&.&5&1&3&.&3&.\\.&.&1&2&.&.&.&.\\3&.&3&.&5&1&3&.\\.&.&.&.&1&2&.&.\\3&.&3&.&3&.&5&1\\.&.&.&.&.&.&1&2\end{pmatrix}.$$

We number these cases from (1) to (10). In total there are 20 cases to consider. As usual,

$$k(B)-l(B)=l(b_x)+l(b_z)+l(b_{xz})\in\{5,8,11\}.$$

Note that $\widetilde{M^x}=M^x$ and $\widetilde{M^z}=M^z$. As in the proof of Lemma 10 we obtain $M^x+M^{xz}\equiv 0 \pmod 3$ and $M^z+M^{xz}\equiv 0 \pmod 9$ from (4.1). It is remarkable that Plesken's algorithm applied to $C_{xz}\oplus\widetilde{C}_z$ under the condition $M^z+M^{xz}\equiv 0 \pmod 9$ always has a unique solution up to basic sets. In particular, $k(B)$ and $l(B)$ are uniquely determined in each of the cases (1)–(10). If $k(B)$ is given, there are only a few solutions $\widetilde{Q}_x$ for $\widetilde{Q}_x^{\mathrm{t}}\widetilde{Q}_x=\widetilde{C}_x$. Eventually we combine $(Q_{xz},\widetilde{Q}_z)$ with $\widetilde{Q}_x$. We collect the possible pairs $(k(B),l(B))$ in the following table:

Case	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(A)	$(9,4)$	$(6,1)$	$(18,10)$	$-$	$(12,4)$	$(27,16)$	$-$	$(21,10)$	$(18,7)$	$(15,4)$
(B)	$-$	$-$	$-$	$(15,7)$	$(12,4)$	$-$	$-$	$-$	$(18,7)$	$(15,4)$

In each case we obtain a handful of possible Cartan matrices C of B. If

$$m:=\min\{xC^{-1}x^{\mathrm{t}}:x\in\mathbb{Z}^{l(B)}\setminus\{0\}\}\geq\frac{l(B)}{27},$$

then the claim follows with $W:=\frac{1}{m}C^{-1}$. This works in all cases except Case (8A). For this exception we construct W from an identity matrix by adding some entries $\pm\frac{1}{2}$ matching the positions of the "large" off-diagonal entries of C. Then we check if W is positive definite (cf. [26, proof of Theorem 13.7]). This can all be done in an automatic fashion. Finally, we remark that the existence of W does not depend on the basic set for B (see [31, Introduction]). □

Now we turn to non-abelian defect groups. For the definitions of controlled and constrained fusion systems (and blocks) we refer to [26, p. 11].

Lemma 15. *Let B be a controlled block with extraspecial defect group $D \cong 3^{1+2}_+$ of order 27 and exponent 3. Suppose that $I(B) \cong Z_2$ and $D \rtimes I(B) = \texttt{SmallGroup}(54, 5)$. Then $l(B) \le 2$.*

Proof. Let $D = \langle x, y, z\rangle$ and $E := I(B) = \langle a\rangle$ such that

$$[x, y] = z, \qquad x^3 = y^3 = z^3 = [x, z] = [y, z] = 1, \qquad x^a = x^{-1}, \qquad y^a = y, \qquad z^a = z^{-1}.$$

Since the fusion system $\mathcal{F}$ of B is controlled, every subgroup of D is fully $\mathcal{F}$-centralized. Hence, for $u \in D$ the Brauer correspondent b_u of B in $\mathrm{C}_G(u)$ has defect group $\mathrm{C}_D(u)$ and $I(b_u) \cong \mathrm{C}_E(u)$. It can be checked that the E-orbits on D are represented by the elements $1, z, x, y, y^{-1}, xy, xy^{-1}$. Since $\mathrm{C}_E(z) = \mathrm{C}_E(x) = \mathrm{C}_E(xy) = \mathrm{C}_E(xy^{-1}) = 1$ and $\mathrm{C}_E(y) = E$, we have $l(b_z) = l(b_x) = l(b_{xy}) = l(b_{xy^{-1}}) = 1$ and $l(b_y) = l(b_{y^{-1}}) = 2$. It follows that

$$k(B) - l(B) = 4 \cdot 1 + 2 \cdot 2 = 8.$$

By [26, Theorem 4.2] applied to b_x, we obtain $k_0(B) \le 9$ where $k_h(B)$ denotes the number characters of height h in B. Hence, we may assume that $l(B) \ge 3$ and $k_1(B) \ge 2$. Now [26, Theorem 4.7] applied to b_z yields $k(B) = k_0(B) + k_1(B) = 9 + 2 = 11$ and $l(B) = 3$. Consequently, the generalized decomposition numbers corresponding to (z, b_z) have the form $(\pm1, \ldots, \pm1, \pm3, \pm3)^{\mathrm{t}}$ where the first nine entries correspond to the characters of height 0. Similarly, the generalized decomposition numbers with respect to (x, b_x) are $(\pm1, \ldots, \pm1, 0, 0)^{\mathrm{t}}$. But now these two columns cannot be orthogonal (regardless of the signs). This contradicts [26, Theorem 1.14]. Hence, $l(B) \le 2$. $\square$

In the following the famous group

$$\mathrm{Qd}(3) = \mathrm{ASL}(2, 3) = Z_3^2 \rtimes \mathrm{SL}(2, 3) \cong \texttt{SmallGroup}(6^3, 153)$$

plays a role.

Lemma 16. *Let B be a constrained block with extraspecial defect group $D \cong 3^{1+2}_+$ and fusion system $\mathcal{F} = \mathcal{F}(\mathrm{Qd}(3))$. Then $l(B) \le 2$ or the Cartan matrix of B is*

$$C = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 2 & 7 \end{pmatrix}$$

up to basic sets.

Proof. Let $D = \langle x, y, z\rangle$ as in Lemma 15 and $E = \langle a, b\rangle \cong Q_8$ such that E acts on $\langle x, z\rangle$ (but not on D). Then $I(B) = \langle a^2\rangle = \langle b^2\rangle$ and $x^{a^2} = x^{-1}$, $y^{a^2} = y$ and $z^{a^2} = z^{-1}$. The $\mathcal{F}$-conjugacy classes of fully centralized cyclic subgroups of D are represented by $\mathcal{R} = \{1, z, y, xy\}$. We compute $l(b_z) = l(b_{xy}) = 1$ and $l(b_y) = 2$. Since z is $\mathcal{F}$-conjugate to z^{-1}, we obtain $Q_z = \widetilde{Q}_z$ and $C_z = \widetilde{C}_z = (27)$. On the other hand, xy and $(xy)^{-1}$ are not $\mathcal{F}$-conjugate. From $C_{xy} = (9)$ we conclude

$$\widetilde{C}_{xy} = \begin{pmatrix} 1 & 1 \\ \zeta & \overline{\zeta} \end{pmatrix}^{-\mathrm{t}} \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \overline{\zeta} & \zeta \end{pmatrix}^{-1} = 3 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Similarly, y and y^{-1} are not $\mathcal{F}$-conjugate. It follows from Brauer's theory of blocks of defect 1 (see [20, Theorem 11.4]) that $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ is the *exact* Cartan matrix of $\overline{b_y}$ (no just up to basic sets). Therefore,

$C_y = 3\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$ is the Cartan matrix of b_y and we may assume that $\overline{Q_y} = Q_{y^{-1}}$. Thus, we do not need to find E-compatible basic sets as in Proposition 7. We compute

$$\widetilde{C}_y = \begin{pmatrix}1&1&.&.\\ \zeta&\overline{\zeta}&.&.\\ .&.&1&1\\ .&.&\zeta&\overline{\zeta}\end{pmatrix}^{-\mathrm{t}} \begin{pmatrix}6&.&3&.\\ .&6&.&3\\ 3&.&6&.\\ .&3&.&6\end{pmatrix} \begin{pmatrix}1&1&.&.\\ \overline{\zeta}&\zeta&.&.\\ .&.&1&1\\ .&.&\overline{\zeta}&\zeta\end{pmatrix}^{-1} = \begin{pmatrix}4&2&2&1\\ 2&4&1&2\\ 2&1&4&2\\ 1&2&2&4\end{pmatrix}.$$

Moreover,

$$k(B) - l(B) = l(b_z) + l(b_y) + k(b_{y^{-1}}) + l(b_{xy}) + l(b_{(xy)^{-1}}) = 7.$$

We may assume that $l(B) > 2$ and therefore, $k(B) \geq 10$. By [26, Theorem 4.2] applied to b_{xy}, it follows that $k_0(B) \leq 9$. Hence, $k_0(B) \in \{3, 6, 9\}$ by [26, Proposition 1.31]. Now [26, Proposition 4.7] applied to b_z yields $k_0(B) = 9$ and $k_1(B) \leq 2$. In total, $k(B) \leq 11$. Recall that the height zero characters correspond to non-zero rows in $\widetilde{Q}_y$ and $\widetilde{Q}_{xy}$. By Plesken's algorithm, the non-zero parts $\widetilde{Q}_y^0$ and $\widetilde{Q}_{xy}^0$ of these matrices are essentially unique:

$$\widetilde{Q}_y^0 = \begin{pmatrix}1&1&1&1\\ 1&1&.&.\\ 1&.&1&.\\ 1&.&.&.\\ .&1&.&1\\ .&1&.&.\\ .&.&1&1\\ .&.&1&.\\ .&.&.&1\end{pmatrix}, \qquad \widetilde{Q}_{xy}^0 = \begin{pmatrix}1&.\\ 1&.\\ 1&.\\ .&1\\ .&1\\ .&1\\ 1&1\\ 1&1\\ 1&1\end{pmatrix}$$

The $*$-construction with suitable $\mathcal{F}$-invariant characters implies

$$3M^z \equiv \widetilde{M}^y \equiv -\widetilde{M}^{xy} \pmod 9$$

by (4.1). Under these restrictions $\widetilde{Q}_y$ and $\widetilde{Q}_{xy}$ can only be combined in a few ways.

Suppose that $k(B) = 11$. Then $Q_z = (\pm1, \ldots, \pm1, \pm3, \pm3)^{\mathrm{t}}$ where the first nine characters have height 0. However, one can show with GAP that Q_z cannot be orthogonal to the matrix $(\widetilde{Q}_y, \widetilde{Q}_{xy})$ formed above. Consequently, $k(B) = 10$ and $l(B) = 3$. Then $Q_z = (\pm2, \pm2, \pm2, \pm1, \ldots, \pm1, \pm3)^{\mathrm{t}}$ up to permutations. All possible combinations lead to the desired Cartan matrix. □

We remark that the case $l(B) = 3$ in Lemma 16 occurs and is predicted in general by Alperin's Weight Conjecture.

6 Abelian defect groups

We are now in a position to prove the first main result of this paper.

Theorem 17. *Let B be a 3-block of a finite group G with abelian defect group D of rank at most 5. Then $k(B) \leq |D|$.*

Proof. Let $E := I(B)$ as usual. We decompose D into indecomposable E-invariant subgroups

$$D = D_1 \times \ldots \times D_n$$

where $n \le 5$ since D has rank at most 5. Since E is a $3'$-automorphism group of D, we know that each D_i is homocyclic, i. e. a direct product of isomorphic cyclic groups (see [8, Theorem 5.2.2]). If D_i is not elementary abelian or $|D_i| = p$, then there always exists $x_i \in D_i$ such that $\mathrm{C}_E(x_i) = \mathrm{C}_E(D_i)$ by [27, Proposition 19]. If $|D_i| \le 27$, then there exists $x_i \in D_i$ such that $|\mathrm{C}_E(x_i) : \mathrm{C}_E(D_i)| \le 2$. Hence, if $n \ge 3$, the element $x := x_1 \ldots x_n$ satisfies $|\mathrm{C}_E(x)| \le 4$. In this case the claim follows from [26, Lemma 14.5].

Now suppose that $n = 2$. Then we may assume that $D_1 \cong Z_3^4$. Let x_2 be a generator of the cyclic group D_2. We may assume that there is no $x_1 \in D_1$ such that $|\mathrm{C}_E(x_1) : \mathrm{C}_E(D_1)| \le 4$, because otherwise $|\mathrm{C}_E(x_1x_2)| \le 4$. The action of E on D_1 determines an irreducible $3'$-subgroup $\overline{E} := E/\mathrm{C}_E(D_1)$ of $\mathrm{GL}(4, 3)$. In other words, $L_1 := D_1 \rtimes \overline{E}$ is a primitive permutation group on D_1 of affine type. These groups are fully classified and available in GAP. It turns out that there are three possibilities:

(i) $\overline{E} \cong SD_{32} \wr Z_2$ (a Sylow 2-subgroup of $\mathrm{GL}(4, 3)$) and $L_1 \cong$ `PrimitiveGroup`$(3^4, 95)$,

(ii) $\overline{E} \cong$ `SmallGroup`$(2^8, 6662)$ and $L_1 \cong$ `PrimitiveGroup`$(3^4, 83)$,

(iii) $\overline{E} \cong$ `SmallGroup`$(640, 21454)$ and $L_1 \cong$ `PrimitiveGroup`$(3^4, 99)$.

In the first two cases there exists $x_1 \in D_1$ such that $\mathrm{C}_{\overline{E}}(x_1) \cong D_8$. With $x := x_1x_2$ we obtain $\mathrm{C}_E(x) \cong D_8$. Let b_x be the corresponding Brauer correspondent of B in $\mathrm{C}_G(x)$ and let $\overline{b_x}$ be the dominated block in $\mathrm{C}_G(x)/\langle x\rangle$ with defect group $\overline{D} := D/\langle x\rangle$. Another GAP computation shows that $\overline{D} \rtimes I(\overline{b_x}) \cong$ `SmallGroup`$(6^3, 158) \times D_2/\langle x_2^3\rangle$ where the second factor has order 3 if $D_2 \ne 1$. Our Usami–Puig algorithm applied in Lemma 8 works equally well if $D_2 \ne 1$ (only Plesken takes a little longer). Consequently, $l(b_x) = l(\overline{b_x}) \in \{2, 5\}$. If $l(b_x) \le 2$, then the claim follows from [26, Theorem 4.9]. Otherwise we know the Cartan matrix $\overline{C_x}$ of $\overline{b_x}$ up to basic sets from Lemma 8 (for $D_2 \ne 1$ the given matrix must be multiplied by 3). The Cartan matrix of b_x is $3\overline{C_x}$ (up to basic sets). Now the claim follows from [26, Theorem 4.2].

In the third case above we find $x \in D$ such that $\overline{D} \rtimes I(\overline{b_x}) \cong$ `SmallGroup`$(6^3, 155) \times D_2/\langle x_2^3\rangle$. This time we get the Cartan matrix of b_x from Lemmas 10 and 11 (again assuming $l(b_x) > 2$). Here the claim follows from [26, Theorem 4.4] (the minimum of the quadratic form can be computed with the GAP command `ShortestVectors`).

Finally, it remains to handle $n = 1$, i. e. E acts irreducibly on D. By [27, Proposition 11], we may assume that $|\mathrm{C}_E(x)| > 7$ for all $x \in D$. From the GAP library we see that the only primitive group to consider is $L = D \rtimes E \cong$ `PrimitiveGroup`$(3^5, 15)$. Here we find $x \in D$ such that $\overline{D} \rtimes I(\overline{b_x}) \cong$ `SmallGroup`$(6^3, 158) \times Z_3$. This is the same instance already discussed above. □

The inertial index 256 occurring in Case (ii) of the above proof is in fact the smallest inertial index where Brauer's $k(B)$-Conjecture is not known to hold in general (see [26, Proposition 14.13]).

7 Non-abelian defect groups

In order to investigate non-abelian defect groups, we first generalize [27, Theorem 8].

Theorem 18. *Let B be a block of a finite group with defect group D such that $D/\langle z\rangle$ is metacyclic for some $z \in \mathrm{Z}(D)$. Then $k(B) \le |D|$.*

Proof. If $p = 2$ or $\overline{D} := D/\langle z\rangle$ is abelian, then the claim follows from [26, Theorem 13.9] or [27, Theorem 5] respectively. Thus, we may assume that $p > 2$ and $\overline{D}$ is non-abelian. Let b_z be a Brauer correspondent of B in $\mathrm{C}_G(z)$. As usual, b_z dominates a unique block $\overline{b_z}$ of $\overline{C_z} := \mathrm{C}_G(z)/\langle z\rangle$ with defect group $\overline{D}$. In order to apply [26, Theorem 4.2], we need to compute the Cartan matrix of $\overline{b_z}$ up to basic sets. To this end, we may assume that $\overline{b_z}$ is non-nilpotent. Then by [26, Theorem 8.8],

$$\overline{D} = \langle x, y : x^{p^m} = y^{p^n} = 1,\ yxy^{-1} = x^{1+p^l}\rangle$$

where $0 < l < m$ and $m - l \le n$. By a result of Stancu [34], $\overline{b_z}$ is a controlled block. Moreover, $E := I(\overline{b_z})$ is cyclic of order dividing $p-1$ and E acts semiregularly on $\langle x\rangle$ and trivially on $\langle y\rangle$ (see [26, proof of Theorem 8.8]). In particular, the (hyper)focal subgroup $[\overline{D}, E] = \langle x\rangle$ is cyclic. By the main result of [35], $\overline{b_z}$ is perfectly isometric to its Brauer correspondent β_z in $\mathrm{N}_{\overline{C_z}}(\overline{D})$. In particular, $\overline{b_z}$ and β_z have the same Cartan matrices up to basic sets. By [17], we may assume that β_z is the unique block of $L := \overline{D} \rtimes E \cong C_{p^m} \rtimes (C_{p^n} \times E)$. By result of Fong (see [20, Theorem 10.13]), the projective indecomposable characters of $L_1 := \langle x\rangle \rtimes E$ are $\Phi'_\lambda := \lambda^{L_1}$ where $\lambda \in \mathrm{Irr}(E)$. Similarly, the projective indecomposable characters of L are $\Phi_\lambda := \lambda^L = (\Phi'_\lambda)^L$. Since $\langle y\rangle$ centralizes E, we have $\Phi_\lambda(g) = p^n(\Phi'_\lambda)(g)$ if $g \in L_1$ and 0 otherwise. Consequently,

$$[\Phi_\lambda, \Phi_\mu] = \frac{1}{|L|}\sum_{g\in L_1} p^{2n}\Phi'_\lambda(g)\overline{\Phi'_\mu(g)} = p^n[\Phi'_\lambda, \Phi'_\mu]$$

for $\lambda, \mu \in \mathrm{Irr}(E)$. The Cartan matrix of L_1 is $(d + \delta_{ij})_{i,j=1}^e$ where $e := |E|$ and $d := (p^m - 1)/e$ (see proof of Proposition 7). Hence, the Cartan matrix of L is $p^n(d + \delta_{ij})$ and the Cartan matrix of b_z is $|\langle z\rangle|p^n(d + \delta_{ij})$. Now [26, Theorem 4.2] yields

$$k(B) \le |\langle z\rangle|p^n\Big(\frac{p^m - 1}{e} + e\Big) \le |\langle z\rangle|p^{n+m} = |\langle z\rangle||\overline{D}| = |D|. \qquad \square$$

We can now prove our second main theorem.

Theorem 19. *Brauer's $k(B)$-Conjecture holds for the 3-blocks of defect at most 4.*

Proof. Brauer's Conjecture has been verified for all p-blocks of defect at most 3 in [28]. Hence, let B be a block with defect group D of order 81. By Theorems 17 and 18 we may assume that D is non-abelian and $D/\langle z\rangle$ has order 27 and exponent 3 for every $z \in \mathrm{Z}(D) \setminus \{1\}$.

Case 1: $|\mathrm{Z}(D)| = 3$.
Since there exists no extraspecial group of order 3^4, we must have $D/\mathrm{Z}(D) \cong 3^{1+2}_+$. Using GAP we are left with four possible groups: $D \cong \mathtt{SmallGroup}(81, s)$ with $s = 7, \ldots, 10$. Let $z \in \mathrm{Z}(D) \setminus \{1\}$, and let b_z and $\overline{b_z}$ as usual. The possible fusion systems of $\overline{b_z}$ were classified in [25].

We compute further that the $3'$-part of $|\mathrm{C}_{\mathrm{Aut}(D)}(\mathrm{Z}(D))|$ is at most 2. In particular, $|I(\overline{b_z})| \le 2$. If $I(\overline{b_z}) = 1$, then $\overline{b_z}$ is nilpotent by the main theorem of [25] (this happens if $s = 10$). In this case the claim follows from [26, Proposition 4.7]. Hence, we may assume that $|I(\overline{b_z})| = 2$ in the following. A further calculation shows that $\overline{D} \rtimes I(\overline{b_z}) \cong Z_3^2 \rtimes Z_2 \cong \mathtt{SmallGroup}(54, 5)$.

By [25], the fusion system $\mathcal{F}_z$ of $\overline{b_z}$ is constrained. More precisely, $\mathcal{F}_z$ is the fusion system of the group $\overline{D} \rtimes I(\overline{b_z})$ or of the group $\mathrm{Qd}(3)$. In the first case, we obtain $l(b_z) \le 2$ from Lemma 15. Then Brauer's $k(B)$-Conjecture follows from [26, Theorem 9.4]. In the remaining case, the claim follows from Lemma 16 and [26, Theorem 4.2].

Case 2: $|\mathrm{Z}(D)| = 9$.
Here $D \cong Z_3 \times 3^{1+2}_+$. Let $z \in D' \setminus \{1\}$ and b_z, $\overline{b_z}$ as usual. Note that $\overline{b_z}$ has defect group $\overline{D} = D/D' = D/\Phi(D) \cong Z_3^3$. The $3'$-group $E := I(B) \le \mathrm{Aut}(D)$ acts faithfully on $\overline{D}$ and normalizes $\mathrm{Z}(D)/D'$. Hence, E is a 2-group and $I(b_z) \cong \mathrm{C}_E(z) \le Q_8 \times Z_2$. If $|I(b_z)| \le 4$, then the claim follows from [26, Lemma 14.5]. Now suppose that $|I(b_z)| \in \{8, 16\}$. Up to isomorphism there are four possibilities for $L := \overline{D} \rtimes I(b_z)$:

(i) $I(b_z) \cong Z_4 \times Z_2$ and $L \cong (Z_3^2 \rtimes Z_4) \times S_3 \cong \mathtt{SmallGroup}(6^3, 156)$: Here the claim follows from Lemma 9 and [26, Theorem 4.2].

(ii) $I(b_z) \cong Q_8$ and $L \cong M_9 \times Z_3$: Apply [26, Theorem 4.2] with Lemma 12.

(iii) $I(b_z) \cong Q_8$ and $L \cong \mathtt{SmallGroup}(6^3, 161)$: Apply [26, Theorem 4.2] with Lemma 10.

(iv) $I(b_z) \cong Q_8 \times Z_2$ and $L \cong M_9 \times S_3$: In this case we use [31, Theorem A] in combination with Lemma 14. □

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References

[1] J. L. Alperin, *Projective modules for* $\mathrm{SL}(2, 2^n)$, J. Pure Appl. Algebra **15** (1979), 219–234.

[2] C. G. Ardito, *Morita equivalence classes of blocks with elementary abelian defect groups of order 32*, J. Algebra **573** (2021), 297–335.

[3] M. Broué, *Isométries de caractères et équivalences de Morita ou dérivées*, Inst. Hautes Études Sci. Publ. Math. **71** (1990), 45–63.

[4] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque **181-182** (1990), 61–92.

[5] M. Broué, *Equivalences of blocks of group algebras*, in: Finite-dimensional algebras and related topics (Ottawa, ON, 1992), 1–26, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 424, Kluwer Acad. Publ., Dordrecht, 1994.

[6] M. Broué and L. Puig, *Characters and local structure in G-algebras*, J. Algebra **63** (1980), 306–317.

[7] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.0*; 2020, (`http://www.gap-system.org`).

[8] D. Gorenstein, *Finite groups*, Chelsea Publishing Co., New York, 1980.

[9] X. Hu, *Blocks with abelian defect groups of rank 2 and one simple module*, J. Algebra **559** (2020), 496–509.

[10] R. Kessar, S. Koshitani and M. Linckelmann, *Conjectures of Alperin and Broué for* 2*-blocks with elementary abelian defect groups of order* 8, J. Reine Angew. Math. **671** (2012), 85–130.

[11] R. Kessar and M. Linckelmann, *On blocks with Frobenius inertial quotient*, J. Algebra **249** (2002), 127–146.

[12] R. Kessar and M. Linckelmann, *On stable equivalences and blocks with one simple module*, J. Algebra **323** (2010), 1607–1621.

[13] R. Kessar and G. Malle, *Quasi-isolated blocks and Brauer's height zero conjecture*, Ann. of Math. (2) **178** (2013), 321–384.

[14] M. Kiyota, *On* 3*-blocks with an elementary abelian defect group of order* 9, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 33–58.

[15] S. Koshitani, *Conjectures of Donovan and Puig for principal 3-blocks with abelian defect groups*, Comm. Algebra **31** (2003), 2229–2243.

[16] S. Koshitani, *Corrigendum: "Conjectures of Donovan and Puig for principal 3-blocks with abelian defect groups"*, Comm. Algebra **32** (2004), 391–393.

[17] B. Külshammer, *Crossed products and blocks with normal defect groups*, Comm. Algebra **13** (1985), 147–168.

[18] P. Landrock and B. Sambale, *On centers of blocks with one simple module*, J. Algebra **472** (2017), 339–368.

[19] E. McKernon, *2-blocks whose defect group is homocyclic and whose inertial quotient contains a Singer cycle*, J. Algebra **563** (2020), 30–48.

[20] G. Navarro, *Characters and blocks of finite groups*, London Mathematical Society Lecture Note Series, Vol. 250, Cambridge University Press, Cambridge, 1998.

[21] W. Plesken, *Solving $XX^{\mathrm{tr}} = A$ over the integers*, Linear Algebra Appl. **226/228** (1995), 331–344.

[22] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and Klein four inertial quotients*, J. Algebra **160** (1993), 192–225.

[23] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and cyclic inertial quotients of order* 4, J. Algebra **172** (1995), 205–213.

[24] R. Rouquier, *Block theory via stable and Rickard equivalences*, in: Modular representation theory of finite groups (Charlottesville, VA, 1998), 101–146, de Gruyter, Berlin, 2001.

[25] A. Ruiz and A. Viruel, *The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p*, Math. Z. **248** (2004), 45–65.

[26] B. Sambale, *Blocks of finite groups and their invariants*, Springer Lecture Notes in Math., Vol. 2127, Springer-Verlag, Cham, 2014.

[27] B. Sambale, *Cartan matrices and Brauer's $k(B)$-Conjecture IV*, J. Math. Soc. Japan **69** (2017), 735–754.

[28] B. Sambale, *On blocks with abelian defect groups of small rank*, Results Math. **71** (2017), 411–422.

[29] B. Sambale, *Refinements of the orthogonality relations for blocks*, Algebr. Represent. Theory **20** (2017), 1109–1131.

[30] B. Sambale, *Morita equivalent blocks of symmetric groups*, SIGMA Symmetry Integrability Geom. Methods Appl. **14** (2018), Paper No. 100, 8.

[31] B. Sambale, *Bounding the number of characters in a block of a finite group*, Adv. Math. **358** (2019), 106861, 20 pp.

[32] B. Sambale, *Survey on perfect isometries*, Rocky Mountain J. Math. **50** (2020), 1517–1539.

[33] J. Scopes, *Cartan matrices and Morita equivalence for blocks of the symmetric groups*, J. Algebra **142** (1991), 441–455.

[34] R. Stancu, *Control of fusion in fusion systems*, J. Algebra Appl. **5** (2006), 817–837.

[35] F. Tasaka and A. Watanabe, *Isotypies for p-blocks with non-abelian metacyclic defect groups, p odd*, J. Algebra **535** (2019), 251–285.

[36] Y. Usami, *On p-blocks with abelian defect groups and inertial index* 2 *or* 3. *I*, J. Algebra **119** (1988), 123–146.

[37] Y. Usami, *Perfect isometries for blocks with abelian defect groups and dihedral inertial quotients of order 6*, J. Algebra **172** (1995), 113–125.