Further evidence for conjectures in block theory

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September 16, 2018

Abstract

We prove new inequalities concerning Brauer's $k(B)$-Conjecture and Olsson's Conjecture by generalizing old results from [Olsson, 1981]. After that we obtain the invariants for 2-blocks of finite groups with certain bicyclic defect groups. Here, a bicyclic group is a product of two cyclic subgroups. This provides an application for the classification of the corresponding fusion systems in [Sambale, 2012]. To some extent this generalizes previously known cases with defect groups of type $D_{2^n} \times C_{2^m}$, $Q_{2^n} \times C_{2^m}$ and $D_{2^n} * C_{2^m}$. As a consequence we prove Alperin's Weight Conjecture and other conjectures for several new infinite families of nonnilpotent blocks. We also prove Brauer's $k(B)$-Conjecture and Olsson's Conjecture for the 2-blocks of defect at most 5. This completes results from [Sambale, 2011]. The $k(B)$-Conjecture is also verified for defect groups with a cyclic subgroup of index at most 4. Finally, we consider Olsson's Conjecture for certain 3-blocks.

Keywords: 2-blocks, bicyclic defect groups, Brauer's $k(B)$-Conjecture, Alperin's Weight Conjecture
AMS classification: 20C15, 20C20

1 Introduction

Let B be a p-block of a finite group G. One aim of this paper is to establish new inequalities on the number of irreducible characters of B in terms of subsections. We outline the idea behind these things.

Olsson proved in [38] the following:

$$l(B) \le 2 \Longrightarrow k(B) \le p^d$$

where d is the defect of B. In particular this gives an example for Brauer's $k(B)$-Conjecture. However, in praxis this implication is not so useful, because usually the knowledge of $l(B)$ already implies the exact value of $k(B)$. Since the proofs in [38] only rely on computations with the contributions of the trivial subsection $(1, B)$, it seems likely that one can extend this result to major subsections. Then we would be able to apply induction on d (see Theorem 4.9). Hence, let (z, b_z) be a major subsection such that $l(b_z) \le 2$. In case $l(b_z) = 1$ we have $\sum k_i(B)p^{2i} \le p^d$ by Theorem 3.4 in [40] (a stronger bound can be found in [20]).

In the first part of this paper we show

$$l(b_z) \le 2 \Longrightarrow k(B) \le p^d.$$

In contrast to Olsson's paper we use methods from [40] and [45]. For $p = 2$, Olsson proved the stronger statement $l(B) \le 3 \Longrightarrow k(B) \le p^d$. Using his ideas we generalize this to major subsections as well. The underlying properties of the contribution matrices were first discovered by Brauer in [5]. However, we will refer to Feit's book [16] for a more accessible account. Using Galois theory we overcome the difficulty that the contributions are not necessarily integers in this general setting.

More generally we consider arbitrary subsections for the prime 2 in order to give bounds on the number of characters of height 0. Here it is known by [8] (more recent accounts can be found in [41, 32]) that the corresponding contributions for characters of height 0 do not vanish. Using exactly the same method we show that $k_0(B) \le 2^q$ if there is a subsection (u, b_u) such that b_u has defect q and $l(b_u) \le 3$.

In the third section of this article we present new infinite families of defect groups for which the block invariants can be calculated. These defect groups are examples of bicyclic 2-groups (i. e. $D = \langle x\rangle\langle y\rangle$ for some $x, y \in D$). The proofs make use of the classification of the corresponding fusion systems in [51]. However, we cannot handle all bicyclic 2-groups. We also remark that these defect groups are in a sense noncommutative versions of the groups $D_{2^n} \times C_{2^m}$, $Q_{2^n} \times C_{2^m}$ and $D_{2^n} * C_{2^m}$ covered in [47, 50, 49]. As a consequence we verify numerous conjectures including Alperin's Weight Conjecture for these blocks.

After that we collect some more or less related examples for block invariants. In particular we discuss some defect groups of order 32. One of the main results here is the verification of Brauer's $k(B)$-Conjecture for the 2-blocks of defect at most 5. This completes Theorem 3 in [46]. The new ingredient here is in fact an old result of Brauer which uses the inverse of the Cartan matrix of a major subsection.

In the last section we obtain new cases for Olsson's Conjecture. In particular we handle the 2-blocks of defect at most 5 and some 3-blocks with defect group of 3-rank 2 which were left over in [20].

2 New Inequalities

Let B be a p-block of a finite group G with defect group D. We define the *height* $h(\chi)$ of a character $\chi \in \mathrm{Irr}(B)$ by $\chi(1)_p = p^{h(\chi)}|G : D|_p$. Moreover, $\mathrm{Irr}_i(B) := \{\chi \in \mathrm{Irr}(B) : h(\chi) = i\}$, $k(B) := |\mathrm{Irr}(B)|$ and $k_i(B) := |\mathrm{Irr}_i(B)|$ for $i \geq 0$. As usual we denote the set of irreducible Brauer characters of B by $\mathrm{IBr}(B)$ and its cardinality by $l(B) := |\mathrm{IBr}(B)|$.

In the following we choose an element $z \in \mathrm{Z}(D)$. Then there exists a Brauer correspondent b_z of B in $\mathrm{C}_G(z)$. The pair (z, b_z) is called *major subsection*.

Theorem 2.1. *Let B be a p-block of a finite group with defect d, and let (z, b_z) be a major subsection such that $l(b_z) \leq 2$. Then one of the following holds:*

(i)

$$\sum_{i=0}^{\infty} k_i(B)p^{2i} \leq p^d.$$

(ii)

$$k(B) \leq \begin{cases} \frac{p+3}{2} p^{d-1} & \text{if } p > 2, \\ \frac{2}{3} 2^d & \text{if } p = 2. \end{cases}$$

In particular Brauer's $k(B)$-Conjecture holds for B.

Proof. In case $l(b_z) = 1$ Eq. (i) holds. Hence, let $l(b_z) = 2$, and let $C_z = (c_{ij})$ be the Cartan matrix of b_z up to basic sets. We consider the number

$$q(b_z) := \min\{xp^d C_z^{-1} x^{\mathrm{T}} : 0 \neq x \in \mathbb{Z}^{l(b_z)}\} \in \mathbb{N}.$$

If $q(b_z) = 1$, Eq. (i) follows from Theorem 3.4.1 in [40]. Therefore, we may assume $q(b_z) \geq 2$. Then Brauer's $k(B)$-Conjecture already holds by Theorem V.9.17 in [16], but we want to obtain the stronger bound (ii). Since p^d is always an elementary divisor of C_z, we see that C_z is not a diagonal matrix. This allows us to apply Theorem 2.4 in [20]. All entries of C_z are divisible by the smallest elementary divisor $\gamma := p^{-d} \det C_z$. Hence, we may consider the integral matrix $\widetilde{C}_z = (\widetilde{c}_{ij}) := \gamma^{-1} C_z$. After changing the basic set we may assume that $0 < 2\widetilde{c}_{12} \leq \widetilde{c}_{11} \leq \widetilde{c}_{22}$. Then

$$\widetilde{c}_{11}\widetilde{c}_{22} - \frac{\widetilde{c}_{11}^2}{4} \leq \widetilde{c}_{11}\widetilde{c}_{22} - \widetilde{c}_{12}^2 = \det \widetilde{C}_z = \frac{p^d}{\gamma}$$

and

$$\widetilde{c}_{11} + \widetilde{c}_{22} \leq \frac{5}{4}\widetilde{c}_{11} + \frac{\det \widetilde{C}_z}{\widetilde{c}_{11}} =: f(\widetilde{c}_{11}).$$

A discussion of the convex function $f(\widetilde{c}_{11})$ as in Theorem 1 in [45] shows that $\widetilde{c}_{11} + \widetilde{c}_{22} \leq f(2)$. Now Theorem 2.4 in [20] leads to

$$k(B) \leq \gamma(\widetilde{c}_{11} + \widetilde{c}_{22} - \widetilde{c}_{12}) \leq \gamma(f(2) - 1) \leq \frac{p^d + 3\gamma}{2}.$$

Since $\gamma \le p^{d-1}$, we get (ii) for p odd. In order to deduce the $k(B)$-Conjecture we need to consider the case $p = 2$. If $\widetilde{c}_{11} = 2$, we must have $\widetilde{c}_{12} = 1$. Hence, under these circumstances $p > 2$, since otherwise $\det \widetilde{C}_z$ is not a p-power. Now assume $\widetilde{c}_{11} \ge 3$ and $p = 2$. Since

$$p^d C_z^{-1} = \frac{p^d}{\gamma}\widetilde{C}_z^{-1} = \begin{pmatrix} \widetilde{c}_{22} & -\widetilde{c}_{12} \\ -\widetilde{c}_{12} & \widetilde{c}_{11} \end{pmatrix},$$

we have $q(b_z) \ge 3$. Now Theorem V.9.17 in [16] implies (ii). We will derive another estimation for $p = 2$ in Theorem 2.2 below. $\square$

It is conjectured that the matrix C_z for $l(b_z) \ge 2$ in the proof of Theorem 2.1 cannot have diagonal shape (this holds for p-solvable groups by Lemma 1 in [45]). Hence for $l(b_z) = 2$ Theorem 2.1(ii) might always apply. Then $k(B) < p^d$ unless $p = 3$.

In order to improve Theorem 2.1 for $p = 2$ we need more notation. Suppose as before that (z, b_z) is a major subsection. We denote the corresponding part of the generalized decomposition matrix by $D_z := (d^z_{\chi\varphi} : \chi \in \mathrm{Irr}(B),\ \varphi \in \mathrm{IBr}(b_z))$. Then the Cartan matrix of b_z is given by $C_z := D_z^{\mathrm{T}}\overline{D_z}$. Moreover, the contribution matrix of b_z is defined as

$$M_z := (m^z_{\chi\psi})_{\chi,\psi\in\mathrm{Irr}(B)} = |D| D_z C_z^{-1} \overline{D_z}^{\mathrm{T}}.$$

In case $|\langle z\rangle| \le 2$, it can be seen easily that M_z is an integral matrix. Then most proofs of [38] remain true without any changes. This was more or less done in [39] (compare also with Corollary 3.5 in [40]). In the general case we have to put a bit more effort into the proof.

Theorem 2.2. *Let B be a 2-block of a finite group with defect d, and let (z, b_z) be a major B-subsection such that $l(b_z) \le 3$. Then*

$$k(B) \le k_0(B) + \frac{2}{3}\sum_{i=1}^{\infty} 2^i k_i(B) \le 2^d.$$

In particular Brauer's $k(B)$-Conjecture is satisfied for B.

Proof. Observe that by construction $m^z_{\chi\chi}$ is a positive real number for every $\chi \in \mathrm{Irr}(B)$, since C_z is positive definite. Since all elementary divisors of C_z are divisors of 2^d, the matrix $2^d C_z^{-1}$ is integral. In particular the numbers $m^z_{\chi\psi}$ are also algebraic integers. Let $\chi \in \mathrm{Irr}(B)$ be a character of height 0. Let $|\langle z\rangle| = 2^n$. In case $n \le 1$ the proof is much easier. For this reason we assume $n \ge 2$. We write

$$m^z_{\chi\chi} = \sum_{j=0}^{2^{n-1}-1} a_j(\chi)\zeta^j$$

with $\zeta := e^{2\pi i/2^n}$ and $a_j(\chi) \in \mathbb{Z}$ for $j = 0, \ldots, 2^{n-1} - 1$. As usual the Galois group $\mathcal{G}$ of the 2^n-th cyclotomic field acts on $\mathrm{Irr}(B)$, on the rows of D_z, and thus also on M_z in an obvious manner. Let Γ be the orbit of χ under $\mathcal{G}$. Set $m := |\Gamma|$. Then we have

$$m a_0(\chi) = \sum_{\psi\in\Gamma} m^z_{\psi\psi} > 0.$$

Assume first that $a_0(\chi) = 1$. Since $M_z^2 = M_z\overline{M_z}^{\mathrm{T}} = 2^d M_z$ (see Theorem V.9.4 in [16]), it follows that

$$m2^d = \sum_{\substack{\psi\in\Gamma,\\ \tau\in\mathrm{Irr}(B)}} |m^z_{\psi\tau}|^2.$$

Applying Galois theory gives

$$\prod_{\substack{\psi\in\Gamma,\\ \tau\in\mathrm{Irr}_i(B)}} |m^z_{\psi\tau}|^2 \in \mathbb{Q}$$

for all $i \geq 0$. By Theorem V.9.4 in [16] we also know $\nu(m_{\psi\tau}^z) = h(\tau)$ where ν is the 2-adic valuation and $\psi \in \Gamma$. Hence, also the numbers $m_{\psi\tau}^z 2^{-h(\tau)}$ are algebraic integers. This implies

$$\mathbb{Z} \ni \prod_{\substack{\psi \in \Gamma, \\ \tau \in \operatorname{Irr}_i(B)}} 2^{-2i} |m_{\psi\tau}^z|^2 \geq 1.$$

Now using the inequality of arithmetic and geometric means we obtain

$$\sum_{\substack{\psi \in \Gamma, \\ \tau \in \operatorname{Irr}_i(B)}} |m_{\psi\tau}^z|^2 \geq m 2^{2i} k_i(B)$$

for all $i \geq 0$. Summing over i gives

$$m 2^d = \sum_{\substack{\psi \in \Gamma, \\ \tau \in \operatorname{Irr}(B)}} |m_{\psi\tau}^z|^2 \geq m \sum_{i=0}^{\infty} 2^{2i} k_i(B)$$

which is even more than we wanted to prove.

Hence, we can assume that $a_0(\chi) \geq 2$ for all $\chi \in \operatorname{Irr}(B)$ such that $h(\chi) = 0$. It is well known that the ring of integers of $\mathbb{Q}(\zeta) \cap \mathbb{R}$ has basis $1, \zeta^j + \zeta^{-j} = \zeta^j - \zeta^{2^{n-1}-j}$ for $j = 1, \ldots, 2^{n-2}-1$. In particular the numbers $a_j(\chi)$ for $j \geq 1$ come in pairs modulo 2. Since $\nu(m_{\chi\chi}^z) = 0$, we even have $a_0(\chi) \geq 3$. For an arbitrary character $\psi \in \operatorname{Irr}(B)$ of positive height we already know that $m_{\psi\psi}^z 2^{-h(\psi)}$ is a positive algebraic integer. Hence, $2^{h(\psi)} \mid a_j(\psi)$ for all $j \geq 0$. By Theorem V.9.4 in [16] we have $\nu(m_{\psi\psi}^z) > h(\psi)$. Thus, we even have $2^{h(\psi)+1} \mid a_0(\psi)$. As above we also have $a_0(\psi) > 0$. This implies $\sum_{\psi \in \operatorname{Irr}_i(B)} m_{\psi\psi}^z \geq 2^{i+1} k_i(B)$ for $i \geq 1$ via Galois theory. Using $\operatorname{tr} M_z = 2^d l(b_z)$ it follows that

$$3 \cdot 2^d \geq \sum_{\psi \in \operatorname{Irr}(B)} m_{\psi\psi}^z \geq 3k_0(B) + \sum_{i=1}^{\infty} 2^{i+1} k_i(B).$$

This proves the claim. □

We remark that in Theorem 6(ii) in [38] it should read $l(B) \leq p^2 - 1$ (compare with Theorem 6*(ii)).

It is easy to see that the proof of Theorem 2.2 can be generalized to the following.

Proposition 2.3. *Let B be a 2-block of a finite group with defect d, and let (z, b_z) be a major B-subsection. Then for every odd number α one of the following holds:*

(1) $\sum_{i=0}^{\infty} 2^{2i} k_i(B) \leq 2^d \alpha$,

(2) $(\alpha + 2) k_0(B) + \sum_{i=1}^{\infty} 2^{i+1} k_i(B) \leq 2^d l(b_z)$.

Proof. As in Theorem 2.2 let $\chi \in \operatorname{Irr}_0(B)$ and define $a_0(\chi)$ similarly. In case $a_0(\chi) \leq \alpha$ the first inequality applies. Otherwise the second inequality applies. □

Observe that Proposition 2.3 also covers (a generalization of) Theorem 8 in [38] for $p = 2$.

Going over to arbitrary subsections (i. e. the element does not necessarily belong to $\mathrm{Z}(D)$) we can prove the following result concerning Olsson's Conjecture. This improves Theorem 3.1 in [41] for $p = 2$.

Theorem 2.4. *Let B be a 2-block of a finite group, and let (u, b_u) be a B-subsection such that b_u has defect q. Set $\alpha := \lfloor \sqrt{l(b_u)} \rfloor$ if $\lfloor \sqrt{l(b_u)} \rfloor$ is odd and $\alpha := \frac{l(b_u)}{\lfloor \sqrt{l(b_u)} \rfloor + 1}$ otherwise. Then $k_0(B) \leq \alpha 2^q$. In particular $k_0(B) \leq 2^q$ if $l(b_u) \leq 3$.*

Proof. The contributions for (u, b_u) are defined by

$$M_u := (m^u_{\chi\psi})_{\chi,\psi\in\mathrm{Irr}(B)} = p^q D_u C_u^{-1} \overline{D_u}^{\mathrm{T}}.$$

By Corollary 1.15 in [32] we still have $m^u_{\chi\psi} \neq 0$ as long as $h(\chi) = h(\psi) = 0$. However, in all other cases it is possible that $m^u_{\chi\psi} = 0$. So we can copy the proof of Theorem 2.2 by leaving out the characters of positive height. This gives $k_0(B) \le \alpha 2^q$ or $k_0(B) \le 2^q l(b_u)/(\alpha+2)$ for every odd number α. If $\lfloor\sqrt{l(b_u)}\rfloor$ is odd, we choose $\alpha := \lfloor\sqrt{l(b_u)}\rfloor$. Otherwise we take $\alpha := \lfloor\sqrt{l(b_u)}\rfloor - 1$. The result follows. □

Finally, we generalize the "dual" inequalities in [38]. For this let $M'_z := (m'_{\chi\psi}) = 2^d 1_{k(B)} - M_z$.

Proposition 2.5. *Let B be a 2-block of a finite group with defect d, and let (z, b_z) be a major B-subsection. Then for every odd number α one of the following holds:*

(1) $\sum_{i=0}^{\infty} 2^{2i} k_i(B) \le 2^d\alpha$,

(2) $(\alpha+2)k_0(B) + \sum_{i=1}^{\infty} 2^{i+1} k_i(B) \le 2^d(k(B) - l(b_z))$.

In particular Brauer's $k(B)$-Conjecture holds if $k(B) - l(b_z) \le 3$.

Proof. By Lemma V.9.3 in [16] the numbers $m'_{\chi\chi}$ for $\chi \in \mathrm{Irr}(B)$ are still real, positive algebraic integers. As in Theorem 2.2 we may assume $|\langle z\rangle| = 2^n \ge 4$. Let us write

$$m'_{\chi\chi} = \sum_{j=0}^{2^{n-1}-1} a_j(\chi)\zeta^j$$

with $\chi \in \mathrm{Irr}_0(B)$, $\zeta := e^{2\pi i/2^n}$ and $a_j(\chi) \in \mathbb{Z}$ for $j = 0, \ldots, 2^{n-1}-1$. The Galois group still acts on M'_z. Also the equation $(M'_z)^2 = M'_z\overline{M'_z}^{\mathrm{T}} = 2^d M'_z$ remains true. For $\tau \in \mathrm{Irr}(B)$ we have $\nu(m'_{\chi\tau}) = \nu(2^d - m^z_{\chi\tau}) = \nu(m^z_{\chi\tau}) = h(\tau)$. Hence, in case $a_0(\chi) \le \alpha$ we can carry over the arguments in Theorem 2.2.

Now assume that $a_0(\chi) > \alpha$ for all characters $\chi \in \mathrm{Irr}_0(B)$. Here the proof works also quite similar as in Theorem 2.2. In fact for a character $\psi \in \mathrm{Irr}(B)$ of positive height we have $\nu(m'_{\psi\psi}) = \nu(2^d - m^z_{\psi\psi}) \ge \min\{\nu(2^d), \nu(m^z_{\psi\psi})\} > h(\psi)$ by Theorem V.9.4 in [16]. Moreover, $\operatorname{tr} M'_z = 2^d(k(B) - l(B))$. The claim follows. □

It should be pointed out that usually $k(B) - l(B) = k(B) - l(b_1) \le k(B) - l(b_z)$ for a major subsection (z, b_z) (this holds for example if z lies in the center of the fusion system of B, see [26]). However, this is not true in general as we see in Proposition 2.1(vii) in [27]. Another problem is that $k(B) - l(b_z)$ for $z \neq 1$ is not locally determined (in contrast to $k(B) - l(B)$). By combining with Proposition 2.3 we can replace (2) in the last proposition by

$$(\alpha+2)k_0(B) + \sum_{i=1}^{\infty} 2^{i+1}k_i(B) \le 2^d \min\{l(b_z), k(B) - l(b_z)\}.$$

3 Bicyclic defect groups

As mentioned in the introduction, we consider in this section blocks with defect groups coming from Theorem 4.19 in [51]. A key feature of the groups in the next three theorems is that all their irreducible characters have degree 1 or 2. We also remark that Olsson's Conjecture was verified for all blocks with bicyclic defect groups in [51].

Theorem 3.1. *Let B be a nonnilpotent 2-block of a finite group with defect group*

$$D \cong \langle v, x, a \mid v^{2^n} = x^2 = a^{2^m} = 1,\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx \rangle \cong D_{2^{n+1}} \rtimes C_{2^m}$$

for some $n, m \ge 2$. Then $k(B) = 2^{m-1}(2^n + 3)$, $k_0(B) = 2^{m+1}$, $k_1(B) = 2^{m-1}(2^n - 1)$ and $l(B) = 2$. In particular Brauer's $k(B)$-Conjecture and Alperin's Weight Conjecture are satisfied.

Proof. Let $\mathcal{F}$ be the fusion system of B, and let $z := v^{2^{n-1}}$. Then by Theorem 4.19 in [51], $Q := \langle z, x, a^2 \rangle$ is the only $\mathcal{F}$-essential subgroup up to conjugation. In order to calculate $k(B)$ we use Brauer's formula (Theorem 5.9.4 in [33]). We will see that it is not necessary to obtain a complete set of representatives for the $\mathcal{F}$-conjugacy classes. Since $\langle v, ax \rangle$ is an abelian maximal subgroup of D, all characters in $\mathrm{Irr}(D)$ have degree 1 or 2. In particular $k(D) := |\mathrm{Irr}(D)| = |D/D'| + (|D| - |D/D'|)/4 = 2^{m-1}(2^n+3)$. Now we have to count how many conjugacy classes of D are fused under $\mathrm{Aut}_{\mathcal{F}}(Q)$. According to Theorem 4.19 in [51] there are two possibilities $\mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Z}(\mathcal{F}) \in \{\langle a^2 \rangle, \langle a^2 z \rangle\}$. In the first case the elements of the form xa^{2j} are conjugate to corresponding elements za^{2j} under $\mathrm{Aut}_{\mathcal{F}}(Q)$. In the second case a similar statement is true for a^{2j}. Observe that the elements xa^{2^j} and xza^{2j} are already conjugate in D. Since $\langle a^2, z \rangle \subseteq \mathrm{Z}(D)$, no more fusion can occur. Hence, the number of $\mathcal{F}$-conjugacy classes is $2^{m-1}(2^n + 3) - 2^{m-1} = 2^m(2^{n-1} + 1)$.

Now we have to determine at least some of the numbers $l(b_u)$ where $u \in D$. The group $\overline{D}_1 := D/\langle a^2 \rangle$ (resp. $\overline{D}_2 := D/\langle a^2 z \rangle$) has commutator subgroup $D'\langle a^2 \rangle/\langle a^2 \rangle$ (resp. $D'\langle a^2 z \rangle/\langle a^2 z \rangle$) of index 4. Hence $\overline{D}_1$ (resp. $\overline{D}_2$) has maximal class. The block b_{a^2} (resp. $b_{a^2 z}$) dominates a block $\overline{b_{a^2}}$ (resp. $\overline{b_{a^2 z}}$) with defect group $\overline{D}_1$. Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be the fusion system of $\overline{b_{a^2}}$ (resp. $\overline{b_{a^2 z}}$). Then in case $\mathrm{Z}(\mathcal{F}) = \langle a^2 \rangle$ (resp. $\mathrm{Z}(\mathcal{F}) = \langle a^2 z \rangle$) $\overline{Q}$ is the only $\mathcal{F}_1$-essential (resp. $\mathcal{F}_2$-essential) subgroup of $\overline{D}_1$ (resp. $\overline{D}_2$) up to conjugation. Thus, [7, 36] imply $l(b_{a^2}) = l(\overline{b_{a^2}}) = 2$ (resp. $l(b_{a^2 z}) = l(\overline{b_{a^2 z}}) = 2$). The same holds for all odd powers of a^2 (resp. $a^2 z$). Next we consider the elements $u := a^{2^j}$ for $2 \le j \le m - 1$. It can be seen that the isomorphism type of $D/\langle u \rangle$ is the same as for D except that we have to replace m by j. Also the essential subgroup Q carries over to the block $\overline{b_u}$. Hence, induction on m gives $l(b_u) = 2$ as well. For all other nontrivial subsections (u, b_u) we only know $l(b_u) \ge 1$. Finally, $l(B) \ge 2$, since B is centrally controlled (Theorem 1.1 in [26]). Applying Brauer's formula gives

$$k(B) \ge 2^m + 2^m(2^{n-1} + 1) - 2^{m-1} = 2^{m-1}(2^n + 3) = k(D).$$

We already know from Theorem 5.3 in [51] that Olsson's Conjecture holds for B, i. e. $k_0(B) \le |D : D'| = 2^{m+1}$. Now we apply Theorem 3.4 in [40] to the subsection (z, b_z) which gives

$$|D| = 2^{m+1} + 2^{m+1}(2^n - 1) \le k_0(B) + 4(k(B) - k_0(B)) \le \sum_{i=0}^{\infty} 2^{2i} k_i(B) \le |D|.$$

This implies $k(B) = k(D) = 2^{m-1}(2^n + 3)$, $k_0(B) = 2^{m+1}$, $k_1(B) = 2^{m-1}(2^n - 1)$ and $l(B) = 2$. Brauer's $k(B)$-Conjecture follows immediately. In order to prove Alperin's Weight Conjecture (see Proposition 5.4 in [22]) it suffices to show that Q and D are the only $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroups of D. Thus, assume by way of contradiction that Q_1 is another $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup. Since Q_1 is $\mathcal{F}$-centric it cannot lie inside Q. Moreover, $\mathrm{Out}_{\mathcal{F}}(Q_1)$ must provide an morphism of odd order, because $Q_1 < D$. However, by Alperin's Fusion Theorem $\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(Q)$ and $\mathrm{Aut}_{\mathcal{F}}(D)$. This gives the desired contradiction. □

We add some remarks. First, the direct products of similar type $D_{2^{n+1}} \times C_{2^m}$ were already handled in [47]. Also if $n = 1$ we obtain the minimal nonabelian group $C_2^2 \rtimes C_{2^m}$ for which the block invariants are also known by [44]. Moreover, it is an easy exercise to check that various other conjectures (for example [14, 12, 31]) are also true in the situation of Theorem 3.1. We will not go into the details here.

The next theorem concerns defect groups which have a similar structure as the central products $Q_{2^{n+1}} * C_{2^m}$ discussed in [49]. Also, this result is needed for the induction step in the theorem after that.

Theorem 3.2. *Let B be a nonnilpotent 2-block of a finite group with defect group*

$$D \cong \langle v, x, a \mid v^{2^n} = 1,\ a^{2^m} = x^2 = v^{2^{n-1}},\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx \rangle \cong Q_{2^{n+1}}.C_{2^m} \cong D_{2^{n+1}}.C_{2^m}$$

for some $n, m \ge 2$ and $m \ne n$. Then $k(B) = 2^{m+1}(2^{n-2} + 1)$, $k_0(B) = 2^{m+1}$, $k_1(B) = 2^{m-1}(2^n - 1)$, $k_n(B) = 2^{m-1}$ and $l(B) = 2$. In particular Brauer's $k(B)$-Conjecture and Alperin's Weight Conjecture are satisfied.

Proof. First observe that the proof of Theorem 4.20 in [51] shows that in fact

$$D \cong \langle v, x, a \mid v^{2^n} = x^2 = 1,\ a^{2^m} = v^{2^{n-1}},\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx \rangle \cong D_{2^{n+1}}.C_{2^m}.$$

Let $\mathcal{F}$ be the fusion system of B, and let $y := v^{2^{n-2}}$ and $z := x^2$. Then by Theorem 4.19 in [51], $Q := \langle x, y, a^2 \rangle \cong Q_8 * C_{2^m}$ is the only $\mathcal{F}$-essential subgroup up to conjugation (since $n \neq m$, D is not a wreath product). Again we use Brauer's formula (Theorem 5.9.4 in [33]) to get a lower bound for $k(B)$. The same argumentation as in Theorem 3.1 shows that D has $2^{m-1}(2^n + 3)$ conjugacy classes and we need to know which of them are fused in Q. It is easy to see that xa^{2j} is conjugate to ya^{2j} under $\mathrm{Aut}_{\mathcal{F}}(Q)$ for $j \in \mathbb{Z}$. Observe that xa^{2j} is already conjugate to xya^{2^j} and $x^{-1}a^{2^j} = xa^{2j+2^m}$ in D. Since $\mathrm{Z}(\mathcal{F}) = \langle a^2 \rangle$, this is the only fusion which occurs. Hence, the number of $\mathcal{F}$-conjugacy classes is again $2^m(2^{n-1} + 1)$.

Again $D/\langle a^2 \rangle$ has maximal class and $l(b_{a^2}) = 2$ by [7, 36]. The same is true for the odd powers of a^2. Now let $u := a^{2^j}$ for some $2 \leq j \leq m$. Then it turns out that $D/\langle u \rangle$ is isomorphic to the group $D_{2^n} \rtimes C_{2^j}$ as in Theorem 3.1. So we obtain $l(b_u) = 2$ as well. For the other nontrivial subsections (u, b_u) we have at least $l(b_u) \geq 1$. Finally $l(B) \geq 2$, since B is centrally controlled (see Theorem 1.1 in [26]). Therefore,

$$k(B) \geq 2^{m+1} + 2^m(2^{n-1} + 1) - 2^m = 2^{m+1}(2^{n-2} + 1). \tag{1}$$

Also, $k_0(B) \leq 2^{m+1}$ by Theorem 5.3 in [51]. However, in this situation we cannot apply [40]. So we use Theorem 2.4 in [20] for the major subsection (a^2, b_{a^2}). Let us determine the isomorphism type of $\overline{D} := D/\langle a^2 \rangle$ precisely. Since $(ax)^2 = axax = vx^2a^2 \equiv v \pmod{\langle a^2 \rangle}$, ax generates a cyclic maximal subgroup $\overline{D}$. Since ${}^a(ax) = avx = axv^{-1} \equiv (ax)^{-1} \pmod{\langle a^2 \rangle}$, $\overline{D} \cong D_{2^{n+1}}$. Hence, the Cartan matrix of b_{a^2} is given by

$$2^m \begin{pmatrix} 2^{n-1} + 1 & 2 \\ 2 & 4 \end{pmatrix}$$

up to basic sets (see [15]). This gives $k(B) \leq 2^m(2^{n-1} + 3)$ which is not quite what we wanted. However, the restriction on $k_0(B)$ will show that this maximal value for $k(B)$ cannot be reached. For this we use the same method as in [49], i. e. we analyze the generalized decomposition numbers $d^u_{\chi\varphi_i}$ for $u := a^2$ and $\mathrm{IBr}(b_u) = \{\varphi_1, \varphi_2\}$. Since the argument is quite similar except that n has a slightly different meaning, we only present some key observations here. As in [49] we write

$$d^u_{\chi\varphi_i} = \sum_{j=0}^{2^{m-1}-1} a^i_j(\chi)\zeta^j$$

where $\zeta := e^{2\pi i/2^m}$. It follows that

$$(a^1_i, a^1_j) = (2^n + 2)\delta_{ij}, \qquad (a^1_i, a^2_j) = 4\delta_{ij}, \qquad (a^2_i, a^2_j) = 8\delta_{ij}.$$

Moreover, $h(\chi) = 0$ if and only if $\sum_{j=0}^{2^{m-1}-1} a^2_j(\chi) \equiv 1 \pmod 2$. This gives three essentially different possibilities for a^1_j and a^2_j as in [49]. Let the numbers α, β, γ and δ be defined as there. Then

$$\begin{aligned} \gamma &= 2^{m-1} - \alpha - \beta, \\ k(B) &\leq (2^n + 6)\alpha + (2^n + 4)\beta + (2^n + 2)\gamma - \delta/2 \\ &= 2^{m+n-1} + 6\alpha + 4\beta + 2\gamma - \delta/2 \\ &= 2^{m+n-1} + 2^m + 4\alpha + 2\beta - \delta/2, \\ 8\alpha + 4\beta - \delta &\leq k_0(B) \leq 2^{m+1}. \end{aligned}$$

This shows $k(B) \leq 2^{m+n-1} + 2^{m+1} = 2^{m+1}(2^{n-2} + 1)$. Together with (1) we have $k(B) = 2^{m+1}(2^{n-2} + 1)$ and $l(B) = 2$. The inequalities above also show $k_0(B) = 2^{m+1}$. Now we can carry over the further discussion in [49] word by word. In particular we get $\delta = 0$,

$$k_1(B) = (2^n - 2)\alpha + (2^n - 1)\beta + 2^n\gamma = 2^{n+m-1} - 2\alpha - \beta = 2^{n+m-1} - 2^{m-1} = 2^{m-1}(2^n - 1)$$

and finally $k_n(B) = 2^{m-1}$. The conjectures follow as usual. □

Now we can also handle defect groups of type $Q_{2^{n+1}} \rtimes C_{2^m}$. It is interesting to see that we get the same number of characters, although the groups are nonisomorphic as shown in [51].

Theorem 3.3. *Let B be a nonnilpotent 2-block of a finite group with defect group*

$$D \cong \langle v, x, a \mid v^{2^n} = a^{2^m} = 1,\ x^2 = v^{2^{n-1}},\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx \rangle \cong Q_{2^{n+1}} \rtimes C_{2^m}$$

for some $n, m \geq 2$. Then $k(B) = 2^{m+1}(2^{n-2} + 1)$, $k_0(B) = 2^{m+1}$, $k_1(B) = 2^{m-1}(2^n - 1)$, $k_n(B) = 2^{m-1}$ and $l(B) = 2$. In particular Brauer's $k(B)$-Conjecture and Alperin's Weight Conjecture are satisfied.

Proof. Let $\mathcal{F}$ be the fusion system of B, and let $y := v^{2^{n-2}}$ and $z := x^2$. Then by Theorem 4.19 in [51], $Q := \langle x, y, a^2 \rangle \cong Q_{2^{n+1}} \times C_{2^{m-1}}$ is the only $\mathcal{F}$-essential subgroup up to conjugation. Again we use Brauer's formula (Theorem 5.9.4 in [33]) to get a lower bound for $k(B)$.

The same argument as in Theorem 3.1 shows that D has $2^{m-1}(2^n + 3)$ conjugacy classes and we need to know which of them are fused in Q. It is easy to see that xa^{2j} is conjugate to ya^{2j} under $\mathrm{Aut}_{\mathcal{F}}(Q)$ for $j \in \mathbb{Z}$. Since $\mathrm{Z}(\mathcal{F}) = \langle z, a^2 \rangle$, this is the only fusion which occurs. Hence, the number of $\mathcal{F}$-conjugacy classes is again $2^m(2^{n-1} + 1)$. In case $n = 2$ the group $D/\langle z \rangle \cong C_2^2 \rtimes C_{2^m}$ is minimal nonabelian, and we get $l(b_z) = 2$ from [44]. Otherwise $D/\langle z \rangle$ is isomorphic to one of the groups in Theorem 3.1. Hence, again $l(b_z) = 2$. As usual the groups $D/\langle a^2 \rangle$ and $D/\langle a^2 z \rangle$ have maximal class and it follows that $l(b_{a^2}) = l(b_{a^2 z}) = 2$. The same holds for all odd powers of a^2 and $a^2 z$. For $2 \leq j \leq m - 1$ the group $D/\langle u \rangle$ with $u := a^{2^j}$ has the same isomorphism type as D where m has to be replaced by j. So induction on m shows $l(b_u) = 2$. It remains to deal with $u := a^{2^j} z$. Here $D/\langle u \rangle \cong Q_{2^{n+1}}.C_{2^j}$ is exactly the group from Theorem 3.2. Thus, for $j \neq n$ we have again $l(b_u) = 2$. In case $j = n$, $D/\langle u \rangle \cong C_{2^n} \wr C_2$. Then (7.G) in [25] gives $l(b_u) = 2$ as well. Now Brauer's formula reveals

$$k(B) \geq 2^{m+1} + 2^m(2^{n-1} + 1) - 2^m = 2^{m+1}(2^{n-2} + 1).$$

For the opposite inequality we apply Theorem 2.4 in [20] to the major subsection (u, b_u) where $u := a^2 z$. A similar calculation as in Theorem 3.2 shows that $D/\langle u \rangle \cong Q_{2^{n+2}}$. Hence, the Cartan matrix of b_u is given by

$$2^m \begin{pmatrix} 2^{n-1} + 1 & 2 \\ 2 & 4 \end{pmatrix}$$

up to basic sets (see [15]). This is the same matrix as in Theorem 3.2, but the following discussion is slightly different, because a^2 has only order 2^{m-1} here. So we copy the proof of the main theorem in [50]. In fact we just have to replace m by $m + 1$ and n by $n - 2$ in order to use this proof word by word. The claim follows. □

We describe the structure of these group extensions in a more generic way.

Proposition 3.4. *Let D be an extension of the cyclic group $\langle a \rangle \cong C_{2^n}$ by a group M which has maximal class or is the fourgroup. Suppose that the corresponding coupling $\omega : \langle a \rangle \to \mathrm{Out}(M)$ satisfies the following: If $\omega \neq 0$, then the coset $\omega(a)$ of $\mathrm{Inn}(M)$ contains an involution which acts nontrivially on $M/\varphi(M)$. Moreover, assume that $D \ncong C_{2^m} \wr C_2$ for all $m \geq 3$. Then the invariants for every block of a finite group with defect group D are known.*

Proof. Assume first that $M \cong C_2^2$. Then in case $\omega = 0$ we get the groups $C_{2^n} \times C_2^2$ and $C_{2^{n+1}} \times C_2$ for which the block invariants can be calculated by [53, 23]. So let $\omega \neq 0$. If D is nonsplit, it must contain a cyclic maximal subgroup. In particular D is metacyclic and the block invariants are known. If the extension splits and we obtain the minimal nonabelian group $C_2^2 \rtimes C_{2^n}$. Here the block invariants are known by [44].

Hence, let M be a 2-group of maximal class. Then $|\mathrm{Z}(M)| = 2$. Thus, for $\omega = 0$ we obtain precisely two extensions for every group M. All these cases were handled in [47, 50, 49]. Let us now consider the case $\omega \neq 0$. Since the three maximal subgroups of a semidihedral group are pairwise nonisomorphic, M must be a dihedral or quaternion group. Write $M = \langle v, x \mid v^{2^m} = 1,\ x^2 \in \langle v^{2^{m-1}} \rangle,\ {}^x v = v^{-1} \rangle$. Let $\alpha \in \mathrm{Aut}(M)$ be an involution which acts nontrivially on $M/\varphi(M)$. Then there is an odd integer i such that ${}^\alpha x = v^i x$. Since $\alpha^2 = 1$, it follows that ${}^\alpha v = v^{-1}$. Hence, the coset $\alpha\,\mathrm{Inn}(M) \in \mathrm{Out}(M)$ is determined uniquely. Hence, ω is unique. So we get four group extensions for every pair (n, m). Two of them are isomorphic and all cases are covered in Theorem 3.1, 3.2 and 3.3 (and [25] for $C_4 \wr C_2$). □

4 More examples

Since almost all block invariants for 2-blocks of defect 4 are known (see [27]), it is natural to look at 2-blocks of defect 5. Here for the abelian defect group $C_4 \times C_2^3$ the invariants are not known so far. We handle more general abelian defect groups in the next theorem. This result relies on the classification of the finite simple groups. We denote the inertial index of B by $e(B)$.

Theorem 4.1. *Let B be a block of a finite group G with defect group $C_{2^n} \times C_2^3$ for some $n \geq 2$. Then we have $k(B) = k_0(B) = |D| = 2^{n+3}$ and one of the following holds:*

(i) $e(B) = l(B) = 1$.

(ii) $e(B) = l(B) = 3$.

(iii) $e(B) = l(B) = 7$.

(iv) $e(B) = 21$, $l(B) = 5$.

Proof. Let $D = C_{2^n} \times C_2^3$. Since $\mathrm{Aut}(D)$ acts faithfully on $\Omega(D)/\varphi(D) \cong C_2^3$, we have $e(B) \in \{1, 3, 7, 21\}$. In case $e(B) = 1$, the block is nilpotent and the result is clear. Now we consider the remaining cases.

Case 1: $e(B) = 3$.
Then there are 2^{n+2} subsections (u, b_u) up to conjugation and 2^{n+1} of them satisfy $l(b_u) = 1$. For the other 2^{n+1} subsections Theorem 1 in [55] implies $l(b_u) = 3$. This gives $k(B) = 2^{n+3} = |D|$. The Height Zero Conjecture follows from Theorem 1.1 in [24].

Case 2: $e(B) = 7$.
Here we have 2^{n+1} subsections (u, b_u) up to conjugation where 2^n of them satisfy $l(b_u) = 1$. For the other 2^n subsections we use Theorem 1 in [55] in connection with Theorem 1.1 in [23] (instead of [23] we could also use [24] which we need anyway). This gives $l(b_u) = 7$ for these subsections. It follows that $k(B) = |D|$ and $k(B) = k_0(B)$ by Theorem 1.1 in [24].

Case 3: $e(B) = 21$.
Here we have again 2^{n+1} subsections (u, b_u) up to conjugation. But this time 2^n subsections satisfy $l(b_u) = 3$ and the other 2^n subsections satisfy $l(b_u) = 5$ by [55, 23]. The result follows as before. □

Next we study another group of order 32 with an easy structure. For this let $MNA(r, s)$ be the minimal nonabelian group given by

$$\langle x, y \mid x^{2^r} = y^{2^s} = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle$$

for some $r \geq s \geq 1$ (see [43]). For the notion of a *constrained* fusion system we refer to Definition 2.3 in [35].

Proposition 4.2. *Let B be a nonnilpotent block of a finite group with defect group $D \cong MNA(2, 1) \times C_2$. Then $k(B) = 20$, $k_0(B) = 16$, $k_1(B) = 4$ and $l(B) = 2$. In particular Olsson's Conjecture and Alperin's Weight Conjecture hold for B.*

Proof. Let $\mathcal{F}$ be the fusion system of B. Since $|D : \mathrm{Z}(D)| = 4$, every $\mathcal{F}$-essential subgroup is maximal, and there are three candidates for these groups. Let $\mathrm{Z}(D) < M < D$ such that $M \cong C_4 \times C_2^2$. Then $\mathrm{Aut}_{\mathcal{F}}(M)$ must act nontrivially on $\Omega(M)/\varphi(M)$. However, it can be seen that $\mathrm{N}_D(M)$ acts trivially on $\Omega(M)/\varphi(M)$. In particular M is not $\mathcal{F}$-radical. Hence, there is only one $\mathcal{F}$-essential subgroup $Q \cong C_2^4$ (up to conjugation). Since $Q \trianglelefteq D$, $\mathcal{F}$ is constrained and thus uniquely determined by $\mathrm{Out}_{\mathcal{F}}(Q)$ (see Theorem 4.6 in [30]). By Lemma 3.11 in [51] we have some possibilities for $\mathrm{Out}_{\mathcal{F}}(Q)$. However, a GAP calculation shows that only $\mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$ is realizable. Then $\mathcal{F}$ is the fusion system on the group $\texttt{SmallGroup}(96, 194) \cong (A_4 \rtimes C_4) \times C_2$. In particular there are exactly 16 $\mathcal{F}$-conjugacy classes on D. Moreover, $\mathrm{Z}(\mathcal{F}) \cong C_2^2$, and for $1 \neq z \in \mathrm{Z}(\mathcal{F})$ we have $D/\langle z \rangle \in \{MNA(2, 1), D_8 \times C_2\}$. Hence, we get $l(b_z) = 2$ as usual. For all other nontrivial subsections (u, b_u) we have $l(b_u) \geq 1$. Since B is centrally controlled, Theorem 1.1 in [26] implies $l(B) \geq 2$. Brauer's formula for $k(B)$ gives $k(B) \geq 20$. If $x \in D$ has order 4, then $\mathrm{C}_D(x)/\langle x \rangle$ has order 4. Hence, Olsson's Conjecture follows from Theorem 2.5 in [20], i. e.

$k_0(B) \le |D : D'| = 16$. For an element $z \in \mathrm{Z}(D) \setminus \mathrm{Z}(\mathcal{F})$ the block b_z is nilpotent. Thus, Theorem 3.4 in [40] implies

$$|D| = 32 \le k_0(B) + 4(k(B) - k_0(B)) \le \sum_{i=0}^{\infty} 2^{2i} k_i(B) \le |D|.$$

The claim follows as usual. $\square$

In the classification of the simple groups of 2-rank 2 the sole exception $\mathrm{PSU}(3,4)$ shows up (see [1]). This group has a Suzuki Sylow 2-subgroup P of order 64 (see Definition 1.4 in [10]). The group P also occurs in the classification of the centerfree fusion systems on 2-groups of 2-rank 2 (see [10]). It can also be described as the smallest 2-group with exactly three involutions and an automorphism of order 5. This answers as question raised in Exercise 82.3 in [3]. In fact P admits an automorphism of order 15. Moreover, $\mathrm{Z}(P) = \varphi(P) = P' = \Omega(P) \cong C_2^2$; so P is *special* (see p. 183 in [18]).

Using this as a motivation it seems worth to obtain the invariants of blocks with defect group P (this will be done in an upcoming diploma thesis). Doing so we need to handle the extraspecial group $P/\langle z\rangle \cong D_8 * Q_8$ for $1 \ne z \in \mathrm{Z}(P)$ for the induction step.

Proposition 4.3. *Let B be a block of a finite group G with defect group $D_8 * Q_8$ and inertial index 5. Then $l(B) = 5$, $k(B) = 13$, $k_0(B) = 8$ and $k_2(B) = 5$. Moreover, the Cartan matrix of B is given by*

$$2\begin{pmatrix} 2 & 1 & 1 & 1 & 1\\ 1 & 2 & 1 & 1 & 1\\ 1 & 1 & 2 & 1 & 1\\ 1 & 1 & 1 & 2 & 1\\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

up to basic sets.

Proof. Let $D = D_8 * Q_8$, and let $\mathcal{F}$ be the fusion system of B. By Theorem 5.3 in [52], $\mathcal{F}$ is controlled by $\mathrm{Aut}_{\mathcal{F}}(D)$. Let $\mathrm{Z}(D) = \langle z\rangle$. As usual we denote the subsections by (u, b_u). Then b_z covers a block $\overline{b_z}$ with elementary abelian defect group of order 16. It follows from Proposition 2.1 in [27] that $5 = e(B) = e(b_z) = e(\overline{b_z}) = l(\overline{b_z}) = l(b_z)$. Moreover, B is centrally controlled; in particular Theorem 1.1 in [26] implies $l(B) \ge 5$.

There are three nonmajor subsections (u_1, b_1), (u_2, b_2) and (u_3, b_3). Since $|D'| = 2$, every conjugacy class in D has at most two elements. In particular $|\mathrm{C}_D(u_i)| = 16$ for $i = 1, 2, 3$. By Proposition 5.1 in [20] we have $l(b_i) = 1$ for $i = 1, 2, 3$. Now let us look at the major subsection (z, b_z). By the proof of Proposition 1 in [46] the Cartan matrix of b_z is given by

$$2\begin{pmatrix} 4 & 3 & 3 & 3 & 3\\ 3 & 4 & 3 & 3 & 3\\ 3 & 3 & 4 & 3 & 3\\ 3 & 3 & 3 & 4 & 3\\ 3 & 3 & 3 & 3 & 4 \end{pmatrix}$$

up to basic sets. If we change the basic set, we get the following matrix with smaller entries:

$$C_z := 2\begin{pmatrix} 2 & 1 & 1 & 1 & 1\\ 1 & 2 & 1 & 1 & 1\\ 1 & 1 & 2 & 1 & 1\\ 1 & 1 & 1 & 2 & 1\\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}.$$

Now we consider the matrix $D_z := (d_{ij}^z)$. Since z has order 2, D_z is an integral matrix such that $D_z^{\mathrm{T}} D_z = C$. Since all columns of D_z are orthogonal to the columns of ordinary decomposition numbers, we see that the first four columns consist of exactly four entries ± 1 each. By way of contradiction assume that the first two columns of D_z have the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & . & \cdots & .\\ 1 & 1 & 1 & -1 & . & \cdots & . \end{pmatrix}^{\mathrm{T}}.$$

Then there is at least one column of ordinary decomposition numbers with is not orthogonal to the difference of these two columns of D_z. This contradiction shows that D_z has the form

$$D_z = \begin{pmatrix} 1 & 1 & 1 & 1 & . & . & . & . & . & . & . & \cdots & . \\ 1 & 1 & . & . & 1 & 1 & . & . & . & . & . & \cdots & . \\ 1 & 1 & . & . & . & . & 1 & 1 & . & . & . & \cdots & . \\ 1 & 1 & . & . & . & . & . & . & 1 & 1 & . & \cdots & . \\ * & * & * & * & * & * & * & * & * & * & * & \cdots & * \end{pmatrix}^{\mathrm{T}}$$

up to signs and permutations. It holds that $k(B) = l(B) + l(b_z) + l(b_1) + l(b_2) + l(b_3) \geq 13$. Hence, for the last column of D_z we have essentially the following possibilities:

$$\begin{aligned}
&(\,1 \;\; 1 \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1\,)^{\mathrm{T}},\\
&(\,1 \;\; . \;\; 1 \;\; . \;\; 1 \;\; . \;\; 1 \;\; . \;\; 1 \;\; . \;\; 1 \;\; 1 \;\; 1\,)^{\mathrm{T}},\\
&(\,1 \;\; 1 \;\; 1 \;\; -1 \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; 1 \;\; 1 \;\; 1 \;\; 1\,)^{\mathrm{T}},\\
&(\,2 \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; 1 \;\; 1 \;\; 1 \;\; 1\,)^{\mathrm{T}},\\
&(\,1 \;\; 1 \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; 2 \;\; 1 \;\; 1\,)^{\mathrm{T}}.
\end{aligned}$$

This already implies $k(B) \in \{13, 14, 16\}$. In order to investigate to heights of the irreducible characters we consider the matrix $M^z = (m_{ij}^z) = 32 D_z C_z^{-1} D_z^{\mathrm{T}}$ of contributions. We have

$$32C_z^{-1} = \begin{pmatrix} 13 & -3 & -3 & -3 & -1 \\ -3 & 13 & -3 & -3 & -1 \\ -3 & -3 & 13 & -3 & -1 \\ -3 & -3 & -3 & 13 & -1 \\ -1 & -1 & -1 & -1 & 5 \end{pmatrix}.$$

By (5G) and (5H) in [5] we have

$$h(\chi) = 0 \Longleftrightarrow m_{\chi\chi}^z \equiv 1 \pmod 2 \Longleftrightarrow \sum_{\varphi \in \mathrm{IBr}(b_z)} d_{\chi\varphi}^z \equiv 1 \pmod 2.$$

This gives $k_0(B) \in \{8, 12, 16\}$ according to the last column of D^z. By Proposition 1 in [8] we also have $h(\chi) = 0 \Leftrightarrow d_{\chi\varphi_i}^{u_i} \equiv 1 \pmod 2$ for $i = 1, 2, 3$ where $\mathrm{IBr}(b_{u_i}) = \{\varphi_i\}$. Since the norm of these nonmajor columns is 16, we have the following possibilities for the nonvanishing entries according to $k_0(B)$: sixteen ± 1, twelve ± 1 and one ± 2, eight ± 1 and two ± 2, seven ± 1 and one ± 3.

Taking this together we can enumerate all the possibilities for the decomposition numbers of nontrivial subsections with GAP. Then the ordinary decomposition matrix (up to multiplication with an invertible matrix) can be determined as the orthogonal space. Finally the square of the ordinary decomposition matrix is the Cartan matrix C of B. Now we determine the elementary divisors of C by considering the lower defect groups.

By (7G) in [6] the multiplicity $m(d)$ of the elementary divisor $d \in \mathbb{N}$ of C is given by

$$m(d) = \sum_{R \in \mathcal{R}} m_B^{(1)}(R)$$

where $\mathcal{R}$ is a set of representatives for the G-conjugacy classes of subgroups of G of order d. After combining this with the formula (2S) of [9] we get

$$m(d) = \sum_{(R, b_R) \in \mathcal{R}'} m_B^{(1)}(R, b_R)$$

where $\mathcal{R}'$ is a set of representatives for the G-conjugacy classes of B-subpairs (R, b_R) such that R has order d. We have to emphasize that in contrast to other papers we regard b_R as a block of $\mathrm{C}_G(R)$ instead of $R\,\mathrm{C}_G(R)$. Let b_D be a Brauer correspondent of B in $\mathrm{C}_G(D)$. Then, after changing the representatives if necessary we may assume $(R, b_R) \le (D, b_D)$ for $(R, b_R) \in \mathcal{R}'$. Then it is well known that b_R is uniquely determined by R. Since the fusion of these subpairs is controlled by $\mathrm{N}_G(D, b_D)$, we get

$$m(d) = \sum_{R \in \mathcal{R}''} m_B^{(1)}(R, b_R)$$

where $\mathcal{R}''$ is a set of representatives for the $\mathrm{Aut}_{\mathcal{F}}(D)$-conjugacy classes of subgroups of D of order d.

It is well known that we have $m(32) = 1$. Now we discuss smaller values for d. We begin with the case $d = 2$. For this let $m_B^{(1)}(Q, b_Q) > 0$ for some Q with $|Q| = 2$. Then (Q, b_Q) is in fact a subsection and 2 is also an elementary divisor of the Cartan matrix of b_Q. In particular $l(b_Q) > 1$. This shows that $Q = \mathrm{Z}(D)$. One can show that 2 occurs as elementary divisor of C_z exactly four times. If we apply the same arguments to the block b_z instead of B, we see that $m(2) = m_B^{(1)}(Q, b_Q) = 4$.

Now let $2 < d < 32$ and $Q \le D$ such that $|Q| = d$. Then by (2Q) in [9] we have $m_{B_Q}^{(1)}(Q) > 0$ where $B_Q := b_Q^{\mathrm{N}_G(Q, b_Q)}$. Since Q is fully $\mathcal{F}$-normalized, Theorem 2.4 in [29] implies that $\mathrm{C}_D(Q)$ is a defect group of b_Q and $\mathrm{N}_D(Q)$ is a defect group of B_Q. By Proposition 2.1 in [2] also the block b_Q is controlled. If we follow the proof of this proposition more closely it turns out that $(\mathrm{C}_D(Q), b_{Q\,\mathrm{C}_D(Q)})$ is a Sylow b_Q-subpair. So the inertial quotient of b_Q is

$$\begin{aligned}\mathrm{N}_{\mathrm{C}_G(Q)}(\mathrm{C}_D(Q), b_{Q\,\mathrm{C}_D(Q)}) / \mathrm{C}_D(Q)\,\mathrm{C}_{\mathrm{C}_G(Q)}(\mathrm{C}_D(Q)) \le \\ \mathrm{N}_G(Q\,\mathrm{C}_D(Q), b_{Q\,\mathrm{C}_D(Q)}) \cap \mathrm{C}_G(Q) / \mathrm{C}_D(Q)\,\mathrm{C}_G(Q\,\mathrm{C}_D(Q)).\end{aligned}$$

All odd order automorphisms of $\mathrm{Aut}_{\mathcal{F}}(Q\,\mathrm{C}_D(Q)) = \mathrm{N}_G(Q\,\mathrm{C}_D(Q), b_{Q\,\mathrm{C}_D(Q)}) / \mathrm{C}_G(Q\,\mathrm{C}_D(Q))$ come from restrictions of $\mathrm{Aut}_{\mathcal{F}}(D)$. However the automorphism of order 5 in $\mathrm{Aut}_{\mathcal{F}}(D)$ cannot centralize Q, since $2 < d$. Hence, the inertial index of b_Q is 1 and $l(b_Q) = 1$. Finally, Theorem 5.11 in [37] and the remark following it show

$$1 = l(b_Q) \ge m_{B_Q}^{(1)}(Q) + m_{B_Q}^{(1)}(\mathrm{N}_D(Q)) = m_{B_Q}^{(1)}(Q) + 1$$

and $m_{B_Q}^{(1)}(Q) = 0$. Taking these arguments together, we proved that the elementary divisors of C are 32, 2, 2, 2, 2, $1, \ldots, 1$ (including the possibility of no 1 at all).

Using this, our GAP program reveals that the only possibility for the generalized decomposition numbers is:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & . & . & . & . & . & . & . & . & . \\
1 & 1 & . & . & 1 & 1 & . & . & . & . & . & . & . \\
1 & 1 & . & . & . & . & 1 & 1 & . & . & . & . & . \\
1 & 1 & . & . & . & . & . & . & 1 & 1 & . & . & . \\
1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & 1 & 1 \\
-1 & . & . & 1 & . & 1 & . & 1 & . & 1 & 3 & -1 & -1 \\
-1 & . & . & 1 & . & 1 & . & 1 & . & 1 & -1 & 3 & -1 \\
-1 & . & . & 1 & . & 1 & . & 1 & . & 1 & -1 & -1 & 3
\end{pmatrix}^{\mathrm{T}}$$

(up to permutations and choosing signs as described earlier). In particular $k(B) = 13$, $k_0(B) = 8$ and $l(B) = 5$. Moreover, C is uniquely determined up to basic sets. Hence, $C = C_z$ up to basic sets, because in case $z \in \mathrm{Z}(G)$, B and b_z would coincide. It remains to determine $k_i(B)$ for $i > 0$. For this let $\psi \in \mathrm{Irr}(B)$ be the fourth character in the numbering above. In particular ψ has height 0. Then for a character $\chi \in \mathrm{Irr}(B)$ with $h(\chi) > 0$ we can see that $m_{\chi\psi}^z$ is divisible by 4 but not by 8. Thus, (5H) in [5] implies $k_2(B) = 5$. □

For the defect group in the last proposition the inertial index could also be 3. However, in this case the computational effort is too big.

In [46] we verified Brauer's $k(B)$-Conjecture for defect groups of order at most 32, but not isomorphic to the extraspecial group $D_8 * D_8$. We are finally able to handle this remaining group as well.

Theorem 4.4. *Brauer's $k(B)$-Conjecture holds for defect groups with a central cyclic subgroup of index at most 16. In particular, the $k(B)$-Conjecture holds for the 2-blocks of defect at most 5.*

Proof. Let B be a p-block with defect group D of the stated form. By Theorem 1 and Theorem 3 in [46] we may assume that there is a major B-subsection (z, b_z) such that $D/\langle z\rangle \cong C_2^4$ (in particular $p = 2$) and B has inertial index 9. We apply Theorem V.9.17 in [16]. For this it suffices to determine the Cartan matrix of b_z (only up to basic sets). Thus, we may consider a 2-block B with elementary abelian defect group D of order 16 and inertial index 9. As in Lemma 2.2 in [27] we obtain a list of possible Cartan matrices of B. However, since we are considering 9×9 matrices it is very hard to see if two of these candidates only differ by basic sets. In order to reduce the set of possible Cartan matrices further we apply various ad hoc matrix manipulations as permutations of rows and columns and elementary row/column operations. After this procedure we end up with a list of only ten possible Cartan matrices of B which might be all equal up to basic sets. For the purpose of illustration, we display one of this matrices:

$$\begin{pmatrix} 4 & -1 & 1 & . & 1 & 1 & 2 & . & . \\ 1 & 4 & . & 1 & -1 & 1 & . & 1 & 1 \\ 1 & . & 4 & 1 & -1 & 1 & 2 & -1 & -1 \\ . & 1 & 1 & 4 & . & . & . & 2 & . \\ 1 & -1 & -1 & . & 4 & . & 1 & 1 & 1 \\ 1 & 1 & 1 & . & . & 4 & 1 & 1 & 1 \\ 2 & . & 2 & . & 1 & 1 & 4 & . & -2 \\ . & 1 & -1 & 2 & 1 & 1 & . & 4 & . \\ . & 1 & -1 & . & 1 & 1 & -2 & . & 4 \end{pmatrix}.$$

It can be seen that all diagonal entries are 4 (for every one of these ten matrices). In order to apply Theorem V.9.17 in [16] let C be one of these ten matrices. Then we have a positive definite integral quadratic form q corresponding to the matrix $16C^{-1}$. We need to find the minimal nonzero value of q among all integral vectors. More precisely, we have to check if a value strictly smaller than 9 is assumed by q. By Theorem 1 in [28] it suffices to consider only vectors with entries in $\{0, \pm 1\}$ (observe that the notation of a quadratic form given by a matrix is the same in [16] and [28]). Hence, there are only 3^9 values to consider. An easy computer computation shows that in fact the minimum of q is at least 9. So Brauer's $k(B)$-Conjecture follows from Theorem V.9.17 in [16]. □

We like to point out that we do not know a single Cartan matrix such that Brauer's $k(B)$-Conjecture would not follow from Theorem 2.4 in [20] or from Theorem V.9.17 in [16]. Since these two results are somehow related, it seems interesting to investigate the following problem: Let $C = (c_{ij}) \in \mathbb{Z}^{l \times l}$ be the Cartan matrix of a p-block with defect d. Assume that for all integral, positive definite quadratic forms $q(x_1, \ldots, x_{l(b_u)}) = \sum_{1 \le i \le j \le l} q_{ij} x_i x_j$ we have

$$\sum_{1 \le i \le j \le l} q_{ij} c_{ij} > p^d.$$

Then prove that $xp^d C^{-1} x^{\mathrm{T}} \ge l$ for all $0 \ne x \in \mathbb{Z}^l$. If this can be done, the $k(B)$-Conjecture would follow in full generality. A diagonal matrix shows that this argument fails for arbitrary positive definite, symmetric matrices C.

In the next proposition we take a closer look at the defect group $D_8 * D_8$.

Proposition 4.5. *Let B be a block of a finite group G with defect group $D \cong D_8 * D_8$. Suppose that the inertial quotient $\mathrm{Out}_{\mathcal{F}}(D)$ has order 3 and acts freely on $D/\varphi(D)$. Then $k(B) = 11$, $k_0(B) = 8$ and $l(B) = 3$. Moreover, the Cartan matrix of B is given by*

$$2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

up to basic sets. For the numbers $k_i(B)$ $(i \ge 1)$ we have the following cases $(k_1(B), k_2(B)) \in \{(0, 3), (2, 1)\}$.

Proof. Let $\mathcal{F}$ be the fusion system of B. By Theorem 5.3 in [52], $\mathcal{F}$ is controlled by $\mathrm{Aut}_{\mathcal{F}}(D)$. By hypothesis $\mathrm{Out}_{\mathcal{F}}(D) \cong C_3$ acts freely on $D/\varphi(D)$. Hence, there are two major and five nonmajor subsections. The Cartan matrix of the nontrivial major subsection (z, b_z) is given by

$$2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

up to basic sets. In particular $k(B) \le 16$. The nonmajor subsections (u, b_u) all satisfy $l(b_u) = 1$. Since B is centrally controlled, we have $k(B) \ge 11$. The first two columns of the b_z decomposition numbers have the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & . & . & . & . & . & . & . & . & \cdots & . \\ 1 & 1 & . & . & 1 & 1 & . & . & . & . & . & . & \cdots & . \end{pmatrix}^{\mathrm{T}}$$

up to signs and permutations (compare with proof of Proposition 4.3). For the third column we have essentially 17 possibilities which we do not list explicitly here. Similarly as in Proposition 4.3 we get $k_0(B) \in \{8, 12, 16\}$ and also the positions of the characters of height 0 depending on the third column of D_z. Since every element of order 4 in D is conjugate to its inverse, all generalized decomposition numbers are integers. For every of the 17 cases we proceed by enumerating the five columns of nonmajor subsections with the help of a computer. Entirely similar as in the proof of Proposition 4.3 we see that the elementary divisors of the Cartan matrix of B are 32, 2, 2, $1, \ldots, 1$. Now the computations reveal $k(B) = 11$, $k_0(B) = 8$, $l(B) = 3$ and the Cartan matrix of B up to basic sets. However, the value of $k_1(B)$ does not follow immediately from these calculations. Instead we obtain the two cases $(k_1(B), k_2(B)) \in \{(0, 3), (2, 1)\}$. □

It is easy to construct examples for Proposition 4.5 such that $k_1(B) = 0$. In contrast, $k_1(B) = 2$ would contradict the Ordinary Weight Conjecture (see [42]).

The next proposition concerns the Sylow 2-subgroup of $\mathrm{PSU}(3, 4)$ as mentioned above. This will result will be used in an upcoming diploma thesis.

Proposition 4.6. *Let B be a block of a finite group G with defect group $D \in \mathrm{Syl}_2(\mathrm{PSU}(3, 4))$ and inertial index 15. Then the elementary divisors of the Cartan matrix of B lie in $\{1, 4, 64\}$ where 4 occurs with multiplicity at most 4.*

Proof. Since D is a Suzuki 2-group, Theorem 4.4 in [10] tells us that the fusion system $\mathcal{F}$ of B is controlled. So similarly as in the proof of Proposition 4.3 the multiplicity of d as an elementary divisor of the Cartan matrix C of B is given by

$$m(d) = \sum_{R \in \mathcal{R}''} m_B^{(1)}(R, b_R)$$

where $\mathcal{R}''$ is a set of representatives for the $\mathrm{Aut}_{\mathcal{F}}(D)$-conjugacy classes of subgroups of D of order d. Assume first $d = 2$ and $m_B^{(1)}(Q, b_Q) > 0$ for $|Q| = 2$. Then (Q, b_Q) is a subsection and $Q \subseteq \mathrm{Z}(D)$. One can show that b_Q has defect group D and inertial index 5. Moreover, b_Q covers a block $\overline{b_Q}$ of $\mathrm{C}_G(Q)/Q$ with defect group $D/Q \cong D_8 * Q_8$. Hence, Proposition 4.3 implies that all elementary divisors of the Cartan matrix of b_Q are divisible by 4. This contradiction shows that $m(2) = 0$. Now suppose that $2 < d < 64$. Again we assume $m_B^{(1)}(Q, b_Q) > 0$ such that $|Q| = d$. We argue as in the proof of Proposition 4.3. The inertial quotient of b_Q is given by

$$\begin{aligned} &\mathrm{N}_{\mathrm{C}_G(Q)}(\mathrm{C}_D(Q), b_{Q\,\mathrm{C}_D(Q)}) / \mathrm{C}_D(Q)\,\mathrm{C}_{\mathrm{C}_G(Q)}(\mathrm{C}_D(Q)) \le \\ &\qquad \mathrm{N}_G(Q\,\mathrm{C}_D(Q), b_{Q\,\mathrm{C}_D(Q)}) \cap \mathrm{C}_G(Q) / \mathrm{C}_D(Q)\,\mathrm{C}_G(Q\,\mathrm{C}_D(Q)). \end{aligned}$$

Every odd order automorphism in $\mathrm{N}_G(Q\,\mathrm{C}_D(Q), b_{Q\,\mathrm{C}_D(Q)}) / \mathrm{C}_G(Q\,\mathrm{C}_D(Q)) = \mathrm{Aut}_{\mathcal{F}}(Q\,\mathrm{C}_D(Q))$ comes from a restriction of $\mathrm{Aut}_{\mathcal{F}}(D)$. Moreover, $\mathrm{Out}_{\mathcal{F}}(D)$ acts freely on $D/\varphi(D)$. So in case $d > 4$ we see that these odd order automorphisms cannot lie in $\mathrm{C}_G(Q)$. Hence, in this case $l(b_Q) = 1$ and $m(d) = 0$ (compare with Proposition 4.3). It remains to deal with the case $Q = \mathrm{Z}(D) = \varphi(D)$. Then we have $b_Q = b_D^{\mathrm{C}_G(\mathrm{Z}(D))}$. Moreover, b_Q has defect group D and inertial index 5. Looking at the covered block of $\mathrm{C}_G(Q)/Q$, we see that $l(b_Q) = 5$. Hence,

$$5 = l(b_Q) \ge m_{B_Q}^{(1)}(Q) + m_{B_Q}^{(1)}(\mathrm{N}_D(Q)) = m_{B_Q}^{(1)}(Q) + 1$$

by Theorem 5.11 in [37] and the remark following it. This gives $m(4) = m_{B_Q}^{(1)}(Q) \le 4$ and the proof is complete. □

Our next result handles rather unknown groups of order 32. The key observation here is that the fusion system is constrained and thus quite easy to understand.

Proposition 4.7. *Let B be a nonnilpotent block of a finite group G with defect group $D \cong \texttt{SmallGroup}(32, q)$ for $q \in \{28, 29\}$. Then $k(B) = 14$, $k_0(B) = 8$, $k_1(B) = 6$ and $l(B) = 2$.*

Proof. Let $\mathcal{F}$ be the fusion system of B. Using GAP one can show that $\mathrm{Aut}(D)$ is 2-group. In particular $\mathrm{Out}_{\mathcal{F}}(D) = 1$. Moreover, one can show using general results in [51] that D contains only one $\mathcal{F}$-essential subgroup Q. Here $C_2^2 \times C_4 \cong Q \trianglelefteq D$. In particular $\mathcal{F}$ is constrained. Another GAP calculation shows that $\mathcal{F}$ is the fusion system of the group $\texttt{SmallGroup}(96, 187)$ or $\texttt{SmallGroup}(96, 185)$ for $q \in \{28, 29\}$ respectively. We have ten B-subsections up to conjugation. The center of D is a four-group and $\varphi(Q) \subseteq \mathrm{Z}(D)$. Hence, an odd order automorphism of Q cannot act on $\mathrm{Z}(D)$. It follows that we have four major subsections $(1, B)$, (z, b_z), (v, b_v) and (w, b_w) up to conjugation. Here we may assume that $l(b_v) = l(b_w) = 1$. On the other hand b_z dominates a nonnilpotent block with defect group $D/\langle z\rangle \cong D_8 \times C_2$. Thus, by Proposition 3 in [46] we have $l(b_z) = 2$. Also we find an element $u \in Q$ such that b_u is nonnilpotent with defect group Q. Here Proposition 2 in [46] implies $l(b_u) = 3$. The remaining nonmajor subsections split in one subsection (u_1, b_1) of defect 16 and four subsections (u_i, b_i) $(i = 2, 3, 4, 5)$ of defect 8. Here $l(b_i) = 1$ for $i = 1, \ldots, 5$. In particular Olsson's Conjecture $k_0(B) \le 8 = |D : D'|$ follows at once. Since B is centrally controlled, we also obtain $l(B) \ge 2$ and $k(B) \ge 14$. So the generalized decomposition numbers d^v_{ij} consist of eight entries ± 1 and six entries ± 2. Hence, $k(B) = 14$, $k_0(B) = 8$, $k_1(B) = 6$ and $l(B) = 2$. □

Also in the next proposition the corresponding fusion is easy to understand, since it is controlled. Another advantage here is that $k(B)$ is relatively small so that computational effort is small as well.

Proposition 4.8. *Let D be a central cyclic extension of $\texttt{SmallGroup}(32, q)$ for $q \in \{33, 34\}$. Then Brauer's $k(B)$-Conjecture holds for all blocks with defect group D.*

Proof. As usual, it suffices to consider a block B with defect group $D \cong \texttt{SmallGroup}(32, q)$ for $q \in \{33, 34\}$. GAP shows that B is a controlled block with inertial index 3. Hence, the fusion system of B is the same as the fusion system of the group $D \rtimes C_3$. It follows that there are only six B-subsections up to conjugation; two of them are major. For $1 \ne z \in \mathrm{Z}(D)$ we have $l(b_z) = 1$. Let us denote the four nonmajor subsections by (u_i, b_i) for $i = 1, \ldots, 4$. We may assume that b_1 has defect group C_2^3. It is easy to see that $\mathrm{Aut}_{\mathcal{F}}(D)$ restricts to the inertial group of b_1. In particular $l(b_1) = e(b_1) = 3$. Moreover, the Cartan matrix of b_1 is given by

$$2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

up to basic sets (see Theorem 3 in [45]). Moreover, b_2 has defect 3 and b_3 and b_4 have defect 4. Here, $l(b_2) = l(b_3) = l(b_4) = 1$. In particular Olsson's Conjecture $k_0(B) \le 8 = |D : D'|$ follows. Looking at d^z_{ij} we get $k(B) \le 14$. The numbers $d^{u_1}_{ij}$ can certainly be arranged in the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & . & . & . & . & . & \cdots & . \\ 1 & 1 & . & . & 1 & 1 & . & . & . & \cdots & . \\ 1 & 1 & . & . & . & . & 1 & 1 & . & \cdots & . \end{pmatrix}^{\mathrm{T}}$$

Using the contributions it follows that $k_0(B) = 8$. We can easily add the column for (u_2, b_2) as

$$(1, 1, -1, \ldots, -1, 0, \ldots, 0)^{\mathrm{T}} \text{ or } (1, -1, 1, -1, 1, -1, 1, -1, 0, \ldots, 0)^{\mathrm{T}}.$$

We investigate next the elementary divisors of the Cartan matrix of B. For this we consider the multiplicity of $\langle u_1\rangle$ as a lower defect group. The multiplicity of 2 as an elementary divisor of the Cartan matrix of b_1 is certainly 2. Since $\langle u_1\rangle$ is the only lower defect group of order 2 of b_1, we have $m(2) = m^{(1)}_B(\langle u_1\rangle, b_1) = m^{(1)}_{b_1}(\langle u_1\rangle, b_1) = 2$. This shows $l(B) \ge 3$ and $k(B) \ge 10$. Every automorphism of order 3 of D fixes only two elements in D. Thus, it follows as in Proposition 4.3 that $m(d) = 0$ for $2 < d < 32$. We have essentially four possibilities for the numbers d^z_{ij}:

- eight entries ± 1 and six entries ± 2,
- eight entries ± 1, two entries ± 2 and one entry ± 4,

- seven entries ± 1, four entries ± 2 and one entry ± 3,
- six entries ± 1, two entries ± 2 and two entries ± 3.

In particular $k(B)$ determines $k_i(B)$ for $i \geq 1$ uniquely. It remains to add the generalized decomposition numbers corresponding to (u_3, b_3) and (u_4, b_4). Here the situation is distinguished by $q \in \{33, 34\}$. Assume first that $q = 34$. Then u_3^{-1} (resp. u_4^{-1}) is conjugate to u_3 (resp. u_4). Hence, the numbers $d_{ij}^{u_3}$ and $d_{ij}^{u_4}$ are integers. It is easy to see that such a column must consist of the following (nonzero) entries:

- eight entries ± 1 and two entries ± 2,
- seven entries ± 1 and one entry ± 3.

In contrast, for $q = 33$ the elements u_3^{-1} and u_4 are conjugate. So we may assume $u_4 := u_3^{-1}$, and it suffices to consider the column $d_{ij}^{u_3}$ whose entries are Gaussian integers. Let us write $d_{\chi\varphi_3}^{u_3} := a(\chi) + b(\chi)i$ where $\mathrm{IBr}(b_3) = \{\varphi_3\}$, $a, b \in \mathbb{Z}^{k(B)}$ and $i := \sqrt{-1}$. Then $(a, a) = (b, b) = 8$ and $(a, b) = 0$. Since we have only one pair of algebraically conjugate subsections, there is only one pair of 2-conjugate characters (see Lemma IV.6.10 in [16]). This shows that b consists of two entries ± 2. Now $k_0(B) = 8$ implies that a has eight entries ± 1.

As usual we enumerate all these configurations of the generalized decomposition matrix and obtain the Cartan matrix of B as orthogonal space. However, we get two possibilities $l(B) \in \{3, 4\}$. We are not able to exclude the case $l(B) = 4$ despite it contradicts Alperin's Weight Conjecture. Anyway in both cases $l(B) \in \{3, 4\}$ all candidates for the Cartan matrix satisfy Theorem 2.4 in [20]. The claim follows. □

We add a short discussion about the defect group

$$D := \texttt{SmallGroup}(32, 27) \cong \langle a, b, c \mid a^2 = b^2 = c^2 = [a, b] = [a, {}^c a] = [{}^c a, b] = [b, {}^c b] = 1\rangle \cong C_2^4 \rtimes C_2.$$

Let $\mathcal{F}$ be a nonnilpotent fusion system on D. It can be shown that $Q := \langle a, b, {}^c a, {}^c b\rangle \cong C_2^4$ is the only possible $\mathcal{F}$-essential subgroup. In particular $\mathcal{F}$ is controlled or constrained (note that controlled is a strong form of constrained). In the controlled case we have $\mathcal{F} = \mathcal{F}_D(D \rtimes C_3) = \mathcal{F}_D(\texttt{SmallGroup}(96, 70))$. In the noncontrolled case we have various possibilities for $\mathcal{F}$ according to $\mathrm{Out}_{\mathcal{F}}(Q) \in \{S_3, D_{10}, S_3 \times C_3, \texttt{SmallGroup}(18, 4), D_{10} \times C_3\}$ (see Lemma 3.11 in [51]). These possibilities are represented by the following groups:

- $\texttt{SmallGroup}(96, 195)$,
- $\texttt{SmallGroup}(96, 227)$,
- $\texttt{SmallGroup}(160, 234)$,
- $\texttt{SmallGroup}(288, 1025)$,
- $\texttt{SmallGroup}(288, 1026)$,
- $\texttt{SmallGroup}(480, 1188)$.

Here observe that in case $\mathrm{Out}_{\mathcal{F}}(Q) = S_3$ there are essentially two different actions of $\mathrm{Out}_{\mathcal{F}}(Q)$ on Q. The cases $\mathrm{Out}_{\mathcal{F}}(Q) \in \{S_3 \times C_3, \texttt{SmallGroup}(18, 4)\}$ also differ by $\mathrm{Out}_{\mathcal{F}}(D) \in \{C_3, 1\}$ respectively. Additionally, for $\mathrm{Out}_{\mathcal{F}}(Q) = \texttt{SmallGroup}(18, 4)$ there exists a nontrivial 2-cocycle on $\mathrm{Out}_{\mathcal{F}}(Q)$ (on the other hand the Künneth formula implies $\mathrm{H}^2(S_3 \times C_3, F^\times) = 0$ for an algebraically closed field F of characteristic 2). This gives even more examples for blocks with defect group D. For example a nonprincipal 2-block of $\texttt{SmallGroup}(864, 3996)$ has defect group D and only one irreducible Brauer character. In all these examples $l(B)$ assumes the values $1, 2, 3, 5, 6, 9$. We will not consider the block invariants in full generality although it might be possible. We also end the discussion about the remaining groups of order 32. In most cases (especially when 9×9 Cartan matrices show up) the computational effort to compute the corresponding block invariants is too big.

In the following table we enumerate all groups of order 32 by using the small groups library and give information about blocks with corresponding defect groups. In many cases it can be shown with GAP that there are no nontrivial fusion systems. These cases were also determined in [54]; however with the Hall-Senior enumeration [19]. Using a conversion between both enumerations provided by Eamonn O'Brien, we confirm the results in [54]. We denote the modular group of order $2^n \geq 16$ by M_{2^n}, i.e. the unique group of class 2 with a cyclic maximal subgroup.

Small group no.	description	invariants	comments	reference
1	C_{32}	known	nilpotent	
2	$MNA(2,2)$	known	controlled	[44, 13]
3	$C_8 \times C_4$	known	nilpotent	
4	$C_8 \rtimes C_4$	known	nilpotent	[48]
5	$MNA(3,1)$	known		[44]
6	$MNA(2,1) \rtimes C_2$	known	nilpotent	GAP
7	$M_{16} \rtimes C_2$	known	nilpotent	GAP
8	$C_2.MNA(2,1)$	known	nilpotent	GAP
9	$D_8 \rtimes C_4$	known	bicyclic	Theorem 3.1
10	$Q_8 \rtimes C_4$	known	bicyclic	Theorem 3.3
11	$C_4 \wr C_2$	known		[25]
12	$C_4 \rtimes C_8$	known	nilpotent	[48]
13	$C_8 \rtimes C_4$	known	nilpotent	[48]
14	$C_8 \rtimes C_4$	known	nilpotent	[48]
15	$C_8.C_4$	known	nilpotent	[48]
16	$C_{16} \times C_2$	known	nilpotent	
17	M_{32}	known	nilpotent	[48]
18	D_{32}	known	maximal class	[7]
19	SD_{32}	known	maximal class	[36]
20	Q_{32}	known	maximal class	[36]
21	$C_4^2 \times C_2$	known	controlled	[53]
22	$MNA(2,1) \times C_2$	known	constrained	Proposition 4.2
23	$(C_4 \rtimes C_4) \times C_2$	known	nilpotent	GAP
24	$C_4^2 \rtimes C_2$	known	nilpotent	GAP
25	$D_8 \times C_4$	known		[47]
26	$Q_8 \times C_4$	known		[50]
27	$C_2^4 \rtimes C_2$			
28	$(C_4 \times C_2^2) \rtimes C_2$	known	constrained	Proposition 4.7
29	$(Q_8 \times C_2) \rtimes C_2$	known	constrained	Proposition 4.7
30	$(C_4 \times C_2^2) \rtimes C_2$	known	nilpotent	GAP
31	$(C_4 \times C_4) \rtimes C_2$	known	nilpotent	GAP
32	$C_2^2.C_2^3$	known	nilpotent	GAP
33	$(C_4 \times C_4) \rtimes C_2$		controlled	
34	$(C_4 \times C_4) \rtimes C_2$		controlled	
35	$C_4 \rtimes Q_8$	known	nilpotent	GAP
36	$C_8 \times C_2^2$	known	controlled	[53]
37	$M_{16} \times C_2$	known	nilpotent	GAP
38	$D_8 * C_8$	known		[49]
39	$D_{16} \times C_2$	known		[47]
40	$SD_{16} \times C_2$	known		[50]
41	$Q_{16} \times C_2$	known		[50]
42	$D_{16} * C_4$	known		[49]
43	$(D_8 \times C_2) \rtimes C_2$			
44	$(Q_8 \times C_2) \rtimes C_2$			
45	$C_4 \times C_2^3$	known	controlled	Theorem 4.1
46	$D_8 \times C_2^2$			
47	$Q_8 \times C_2^2$		controlled	
48	$(D_8 * C_4) \times C_2$		controlled	
49	$D_8 * D_8$		controlled	
50	$D_8 * Q_8$		controlled	
51	C_2^5		controlled	

We apply these results to Theorem 2.2.

Theorem 4.9. *Let D be a cyclic central extension of one of the following groups* *(i)* *a metacyclic group,*

(ii) *a minimal nonabelian group,*

(iii) *a group of order at most* 16*,*

(iv) $\prod_{i=1}^{n} C_{2^{m_i}}$ *where* $|\{m_i : i = 1, \ldots, n\}| \ge n-1$*,*

(v) $M \times C$ *where* M *has maximal class and* C *is cyclic,*

(vi) $M * C$ *where* M *has maximal class and* C *is cyclic,*

(vii) $D_{2^n} \rtimes C_{2^m}$, $Q_{2^n} \rtimes C_{2^m}$ *and* $D_{2^n}.C_{2^m}$ *as in Theorem 3.1, 3.3 and 3.2,*

(viii) $\texttt{SmallGroup}(32, q)$ *for* $q \in \{11, 22, 28, 29, 33, 34\}$*,*

(ix) *a group which admits only the nilpotent fusion system.*

Then Brauer's $k(B)$*-Conjecture holds for every* 2*-block with defect group* D*.*

Proof. The case (iii) follows from Theorem 4.4. In case (viii) the result follows from the propositions 4.2, 4.7 and 4.8 and [25]. In all other cases it suffices to show $l(B) \le 3$ for every block B with defect group given in the remaining list of the statement. For the abelian defect group $\prod_{i=1}^{n} C_{2^{m_i}}$ where $|\{m_i : i = 1, \ldots, n\}| \ge n-1$ it is easy to see that the inertial index $e(B)$ is at most 3. Thus, results of Puig-Usami [53] imply Alperin's Weight Conjecture in this case. Now $l(B) \le 3$ follows easily. For the remaining cases the claim was shown in [48, 13, 46, 47, 50, 49] and the present paper. □

One can show with GAP that Theorem 4.9 suffices to verify Brauer's $k(B)$-Conjecture for 244 of the 267 defect groups of order 64. Here we also use the following elementary observation: Let $z \in \mathrm{Z}(D)$ such that every fusion system on $D/\langle z \rangle$ is controlled. If $\mathrm{C}_{\mathrm{Aut}(D)}(z)$ is a 2-group, then Brauer's $k(B)$-Conjecture holds for every block with defect group D.

For the group $D \cong \texttt{SmallGroup}(64, 265)$ we can argue even more subtle. Every block B with defect group D fulfills $e(B) \in \{1, 3, 5\}$. In case $e(B) = 3$ we find an element $z \in \mathrm{Z}(D)$ such that $D/\langle z \rangle$ is elementary abelian. Then [53] implies $k(B) \le 64$. On the other hand if $e(B) = 5$, we choose $z \in \mathrm{Z}(D)$ such that $D/\langle z \rangle \cong D_8 * Q_8$. Here the $k(B)$-Conjecture follows from Proposition 4.3.

For the purpose of further research we state all indices q such that Brauer's $k(B)$-Conjecture for the defect group $\texttt{SmallGroup}(64, q)$ is *not* known so far:

$$134, 135, 136, 137, 138, 139, 202, 224, 229, 230, 231, 238, 239, 242, 254, 255, 257, 258, 259, 261, 262, 264, 267.$$

This implies the following corollary.

Corollary 4.10. *Let* B *be a* 2*-block with defect group* D *of order at most* 64*. If* D *is generated by two elements, then Brauer's* $k(B)$*-Conjecture holds for* B*.*

One can also formulate a version of Theorem 4.9 for $k_0(B)$ using Theorem 2.4. Compare also with Theorem 2.5 in [20].

Corollary 4.11. *Let* D *be a* 2*-group containing a cyclic subgroup of index at most* 4*. Then Brauer's* $k(B)$*-Conjecture holds for every block with defect group* D*.*

Proof. We may assume that D is not metacyclic. In particular, $|D|/\exp D = 4$. If D is abelian, the result follows from Corollary 2 in [46]. Hence, let us assume that D is nonabelian. Then D is one of the groups given in Theorem 2 in [34]. We will consider this list of groups case by case and apply Theorem 4.9. We remark that the terms "quasi-dihedral" and "semidihedral" have different meanings in [34].

The group G_1 is metacyclic. For the groups G_2 and G_3 we even know the block invariants precisely. Now consider G_4. Here the element a lies in the center. In particular the group is a cyclic central extension of a group of order 4. The $k(B)$-Conjecture follows. For the group G_5 the element b lies in the center. Moreover, $G_5/\langle b \rangle$ is abelian and has a cyclic subgroup of index 2. Again the claim holds. The groups G_6, G_7, G_8 and G_9 are metacyclic.

The groups G_{10} and G_{11} are cyclic central extensions of metacyclic groups. In G_{12} the subgroup $\langle a\rangle$ is normal; in particular $a^{2^{m-3}} \in \mathrm{Z}(G_{12})$. Moreover, b is central in $G_{12}/\langle a^{2^{m-3}}\rangle$ and $G_{12}/\langle a^{2^{m-3}}\rangle \cong D_{2^{m-2}} \times C_2$. The claim follows. In G_{13} and G_{14} we see that b is central and the corresponding quotient is certainly metacyclic. Next, $a^{2^{m-3}} \in \mathrm{Z}(G_{15})$ and $G_{15}/\langle a^{2^{m-3}}\rangle \cong D_{2^{m-2}} \times C_2$. Exactly the same argument applies to G_{16}. For G_{17} we have $c^{-1}a^2c = abab = a^{2+2^{m-3}}$ and $a^4 \in \mathrm{Z}(G_{17})$. Since $G_{17}/\langle a^4\rangle$ has order 16, the claim follows.

The group G_{18} is slightly more complicated. In general, the core of $\langle a\rangle$ has index at most 8. Thus, $a^{2^{m-3}}$ is always central (in all of these groups). Adjusting notation slightly gives

$$G_{18}/\langle a^{2^{m-3}}\rangle \cong \langle a, b, c \mid a^{2^{m-3}} = b^2 = c^2 = [a, b] = 1,\ cac = a^{-1}b\rangle.$$

We define new elements in this quotient by $\widetilde{v} := a^2b$, $\widetilde{x} := bc$ and $\widetilde{a} := ac$. Then $\widetilde{v}^{2^{m-4}} = 1$, $\widetilde{a}^2 = b$ and $\widetilde{a}^4 = 1$. Moreover, $cbc = c(acac)c = b$. It follows that $\widetilde{x}^2 = 1$ and $\widetilde{x}\widetilde{v}\widetilde{x} = \widetilde{v}^{-1}$. Hence, $\langle \widetilde{v}, \widetilde{x}\rangle \cong D_{2^{m-3}}$. Now $\widetilde{a}\widetilde{v}\widetilde{a}^{-1} = ca^2bc = a^{-2}b = \widetilde{v}^{-1}$ and finally $\widetilde{a}\widetilde{x}\widetilde{a}^{-1} = a^2c = \widetilde{v}\widetilde{x}$. Since $G_{18}/\langle a^{2^{m-3}}\rangle = \langle \widetilde{v}, \widetilde{x}, \widetilde{a}\rangle$, we see that this is precisely the group from Theorem 3.1. The claim follows.

The groups G_{19}, G_{20} and G_{21} are metacyclic. In G_{22} the element a^4 is central and $G_{22}/\langle a^4\rangle$ has order 16. Let us consider G_{23}. Similarly as above we have

$$G_{23}/\langle a^{2^{m-3}}\rangle \cong \langle a, b, c \mid a^{2^{m-3}} = b^2 = c^2 = [a, b] = 1,\ cac = a^{-1+2^{m-4}}b\rangle$$

(observe that the relation $[b, c] \equiv 1 \pmod{\langle a^{2^{m-3}}\rangle}$ follows from $b \equiv a^{1+2^{m-4}}cac$). Here we define $\widetilde{v} := a^{2+2^{m-4}}b$, $\widetilde{x} := bc$ and $\widetilde{a} := ac$. Then again $\langle \widetilde{v}, \widetilde{x}\rangle \cong D_{2^{m-3}}$. Moreover, $\widetilde{a}^2 = a^{2^{m-4}}b$, $\widetilde{a}^4 = 1$ and $\widetilde{a}\widetilde{x}\widetilde{a}^{-1} = bca^{-1}cac = a^{2+2^{m-4}}c = \widetilde{v}\widetilde{x}$. So $G_{23}/\langle a^{2^{m-3}}\rangle$ is the group from Theorem 3.1. Now it is easy to see that $G_{24}/\langle a^{2^{m-3}}\rangle \cong G_{25}/\langle a^{2^{m-3}}\rangle \cong G_{23}/\langle a^{2^{m-3}}\rangle$. Finally the group G_{26} has order 32; so also here the $k(B)$-Conjecture holds. This completes the proof. □

For every integer $n \geq 6$ there are exactly 33 groups of order 2^n satisfying the hypothesis of Corollary 4.11.

5 Olsson's Conjecture

We have seen in [20] that Olsson's Conjecture holds for all controlled 2-blocks of defect at most 5. Using the table above, we remove the controlled condition.

Theorem 5.1. *Olsson's Conjecture holds for all* 2*-blocks of defect at most* 5.

Proof. By the remark above it suffices to consider only the defect groups $D := \texttt{SmallGroup}(32, m)$ where $m \in \{27, 43, 44, 46\}$. Let B be a block with defect group D and fusion system $\mathcal{F}$. Then we can find (with GAP) an element $u \in D$ such that $|\mathrm{C}_D(u)| = |D : D'|$. Moreover, we can choose u such that every element $v \in D$ of the same order also satisfies $|\mathrm{C}_D(u)| = |D : D'|$. Hence, the subgroup $\langle u\rangle$ is fully $\mathcal{F}$-centralized. In particular $\mathrm{C}_D(u)$ is a defect group of the block b_u. Now the claim follows from Proposition 2.5(ii) in [20]. □

In [20] we also verified Olsson's Conjecture for defect groups of p-rank 2 provided $p > 3$. We use the opportunity to explore the case $p = 3$ in more detail.

Theorem 5.2. *Let B be a* 3*-block of a finite group G with defect group D. Assume that D has* 3*-rank* 2*, but not maximal class. Then Olsson's Conjecture holds for B.*

Proof. By Theorem 5.6 in [20] we may assume that the fusion system $\mathcal{F}$ of B is not controlled. Then $|D| \geq 3^4$, since D does not have maximal class. By Theorem 4.1 and 4.2 in [11] it remains to handle the groups $D = G(3, r; \epsilon)$ of order 3^r where $r \geq 5$ and $\epsilon \in \{\pm 1\}$ as in Theorem 4.7 in [11] (by Remark A.3 in [11], $G(3, 4; \epsilon)$ has maximal class). Assume that D is given by generators and relations as in Theorem A.1 of the same paper. Consider the element $x := ac$. By Lemma A.8 in [11], x is not contained in the unique $\mathcal{F}$-essential ($\mathcal{F}$-Alperin) subgroup $C(3, r-1) = \langle a, b, c^3\rangle$. In particular, $\langle x\rangle$ is fully $\mathcal{F}$-centralized, and the block b_x of the subsection (x, b_x) has defect group $\mathrm{C}_D(x)$. It is easy to see that $D' = \langle b, c^{3^{r-3}}\rangle \cong C_p \times C_p$. It follows that $x^{3^{r-4}} \equiv c^{3^{r-4}} \not\equiv 1$

(mod D') and $|\langle x\rangle| \ge 3^{r-3}$. As usual we have $|\mathrm{C}_D(x)| \ge |D:D'| = 3^{r-2}$. In case $|\mathrm{C}_D(x)| \ge 3^{r-1}$ we get the contradiction $b \in D' \subseteq \mathrm{C}_D(x)$. Hence, $|\mathrm{C}_D(x)| = |D:D'|$ and $\mathrm{C}_D(x)/\langle x\rangle$ is cyclic. Now Olsson's Conjecture for B follows from Proposition 2.5 in [20]. □

Theorem 5.3. *Let B be a 3-block of a finite group with defect group D of order at least 3^4. Assume that D has maximal class, but is* not *isomorphic to the group*

$$B(3,r;0,0,0) = \langle s, s_1, \ldots, s_{r-1} \mid s^3 = s_{r-2}^3 = s_{r-1}^3 = [s_1,s_2] = \ldots = [s_1,s_{r-1}] = \\ = s_1^3 s_2^3 s_3 = \ldots = s_{r-3}^3 s_{r-2}^3 s_{r-1} = 1,\ s_i = [s_{i-1}, s] \text{ for } i = 2,\ldots,r-1\rangle$$

of order 3^r. Then Olsson's Conjecture holds for B.

Proof. By Theorem 5.6 in [20] we may assume that the fusion system $\mathcal{F}$ of B is not controlled. Then $\mathcal{F}$ is given as in Theorem 5.10 in [11][1]. In particular $D = B(3,r;0,\gamma,0)$ is given by generators and relations as in Theorem A.2 in [11] where $\gamma \in \{1,2\}$. Let D_1 as in Definition III.14.3 in [21]. Observe that in the notation of [11, 4] we have $D_1 = \gamma_1(D)$. Proposition A.9 in [11] shows $x := ss_1 \notin D_1$. Moreover, we have $x^3 \ne 1$ also by Proposition A.9 in [11]. Then by Lemma A.15 in [11], x does not lie in one of the centric subgroups D_1, E_i or V_i for $i \in \{-1,0,1\}$. This, shows that x is not $\mathcal{F}$-conjugate to an element in D_1. By Satz III.14.17 in [21], D is not an exceptional group. In particular, Hilfssatz III.14.13 in [21] implies $|\mathrm{C}_D(y)| = 9 = |D:D'|$ for all $y \in D\setminus D_1$. Hence, $\langle x\rangle$ is fully $\mathcal{F}$-centralized. Thus, the block b_x of the subsection (x,b_x) has defect group $\mathrm{C}_D(x)$. Now Olsson's Conjecture follows from Proposition 2.5 in [20]. □

We remark that the method in Theorem 5.3 does not work for the groups $B(3,r;0,0,0)$. For example, every block of a subsection of the principal 3-block of ${}^3D_4(2)$ has defect at least 3 (here $r=4$). However, $|D:D'| = 3^2$ for every 3-group of maximal class.

Acknowledgment

This work is supported by the German Academic Exchange Service (DAAD) and the German Research Foundation (DFG) and was written partly in Santa Cruz, USA. I thank the University of California for their hospitality.

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