Remarks on Harada's Conjecture

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Abstract

An open conjecture by Harada from 1981 gives an easy characterization of the p-blocks of a finite group in terms of the ordinary character table. Kiyota and Okuyama have shown that the conjecture holds for p-solvable groups. In the present work we extend this result using a criterion on the decomposition matrix. In this way we prove Harada's Conjecture for several new families of defect groups and for all blocks of sporadic simple groups. In the second part of the paper we present a dual approach to Harada's Conjecture.

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1 New evidence for Harada's Conjecture

Let G be a finite group, and let p be a prime. We use the notation from [26]. In particular, G^0 is the set of p-regular elements of G. Harada [9] proposed the following conjecture in 1981: If $J \subseteq \operatorname{Irr}(G)$ such that

$$\sum_{\chi \in J} \chi(1)\chi(g) = 0 \qquad \forall g \in G \setminus G^0,$$

then J is a union of p-blocks of G (see also [26, p. 53]). Harada has already observed that it suffices to consider the following blockwise version.

Conjecture (Harada)**.** *Let B be a p-block of G and let $\varnothing \neq J \subseteq \operatorname{Irr}(B)$ such that*

$$\sum_{\chi \in J} \chi(1)\chi(g) = 0 \qquad \forall g \in G \setminus G^0. \tag{1}$$

Then $J = \operatorname{Irr}(B)$.

In his article, Harada verified the conjecture for blocks with cyclic defect groups. After that only a few other cases were considered in the literature and we will mention most of them in the course of this paper.

In this section we give some new evidence for Harada's Conjecture. A cyclic group of order n is denoted by C_n and for convenience let $C_n^m := C_n \times \ldots \times C_n$ (m copies). Let $(\_,\_) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the standard bilinear form. The following lemma is probably well-known. It is related to the notion of residue sets introduced by Ikeda [10, 11, 12, 13, 14, 16].

Lemma 1. *Let B be a block of G with decomposition matrix Q. For $J \subseteq \operatorname{Irr}(B)$ the following assertions are equivalent:*

(i) $\sum_{\chi \in J} \chi(1)\chi(g) = 0$ *for all* $g \in G \setminus G^0$.

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(ii) There exists a vector $a \in \mathbb{Z}^{l(B)}$ such that for every row d_χ of Q we have

$$(d_\chi, a) = \begin{cases} \chi(1) & \text{if } \chi \in J, \\ 0 & \text{if } \chi \notin J. \end{cases}$$

Proof. Assume that (i) is true. By [26, Corollary 2.17] there are integers a_φ such that

$$\sum_{\chi \in J} \chi(1)\chi = \sum_{\varphi \in \mathrm{IBr}(B)} a_\varphi \Phi_\varphi = \sum_{\varphi \in \mathrm{IBr}(B)} a_\varphi \sum_{\chi \in \mathrm{Irr}(B)} d_{\chi\varphi}\chi = \sum_{\chi \in \mathrm{Irr}(B)} \Big(\sum_{\varphi \in \mathrm{IBr}(B)} a_\varphi d_{\chi\varphi} \Big) \chi.$$

Since the irreducible characters of B are linearly independent, it follows that $(a, d_\chi) = \chi(1)$ if $\chi \in J$ and 0 otherwise.

Now assume conversely that (ii) holds. Then we have

$$\sum_{\chi \in J} \chi(1)\chi = \sum_{\chi \in \mathrm{Irr}(B)} \Big(\sum_{\varphi \in \mathrm{IBr}(B)} a_\varphi d_{\chi\varphi} \Big) \chi = \sum_{\varphi \in \mathrm{IBr}(B)} a_\varphi \sum_{\chi \in \mathrm{Irr}(B)} d_{\chi\varphi}\chi = \sum_{\varphi \in \mathrm{IBr}(B)} a_\varphi \Phi_\varphi$$

and the claim follows. □

Suppose that (1) is satisfied with $|J| = 1$. Then by [26, Theorem 3.18], B has defect 0 and therefore $J = \mathrm{Irr}(B)$. The following corollary is a partial generalization.

Corollary 2. *Let B be a counterexample to Harada's Conjecture with $J \subsetneq \mathrm{Irr}(B)$ and $|J| \geq l(B)$. Then the submatrix Q_J of the decomposition matrix corresponding to the characters in J has rank less than $l(B)$.*

Proof. Suppose by way of contradiction that Q_J has full rank. Let $a \in \mathbb{Z}^{l(B)}$ be as in Lemma 1, and let $b := (\varphi(1) : \varphi \in \mathrm{IBr}(B))$. Then $Q_J a = Q_J b$ and therefore $a = b$. However, this gives the contradiction $J = \mathrm{Irr}(B)$. □

In many cases the decomposition matrix of a block can be obtained from a block of a smaller group (for example in the presence of Morita equivalence). Following Brauer, we call a $\mathbb{Z}$-basis of $\mathbb{Z}\,\mathrm{IBr}(B)$ a *basic set* of B (see [26, Definition 7.3]). Replacing $\mathrm{IBr}(B)$ by a basic set transforms the decomposition matrix Q of B into QS where $S \in \mathrm{GL}(l(B), \mathbb{Z})$. It is obvious that the condition in Lemma 1(ii) is invariant under change of basic sets (replace a by $S^{-1}a$). We say that decomposition matrices Q_1 and Q_2 of blocks B_1 and B_2 respectively coincide *up to basic sets* if there exists some $S \in \mathrm{GL}(l(B_1), \mathbb{Z})$ and a permutation matrix T such that $Q_1 S = TQ_2$. It is well-known that this happens if B_1 and B_2 are Morita equivalent (here S is also a permutation matrix). More generally, the relation holds whenever there exists a perfect isometry with positive signs between B_1 and B_2. In this sense, the following proposition extends a well-known result by Kiyota-Okuyama [22].

Proposition 3. *Let B be a block of a finite group G and let B' be a block of a p-solvable group. If B and B' have the same decomposition matrices up to basic sets, then Harada's Conjecture holds for B.*

Proof. Let Q, $J \neq \varnothing$ and a as in Lemma 1. By the remark above, we may assume that Q is the decomposition matrix of a block of a p-solvable group. By the Fong-Swan Theorem ([26, Theorem 10.1]), we can label the characters of B in such a way that the first $l := l(B)$ lines of Q form an identity matrix. This implies that a consists of the degrees of the characters in $X := \{\chi_1, \ldots, \chi_l\} \cap J$ and zeros. In particular, a is non-negative. Now let $b := (\varphi(1) : \varphi \in \mathrm{IBr}(B))$. We claim (after permuting the rows and columns again) that Q has the form

$$Q = \begin{pmatrix} A_J & . \\ . & A_{J'} \end{pmatrix}$$

where $A_J \in \mathbb{Z}^{|J| \times |X|}$ corresponds to the characters in J. For, if d_χ is a row of Q with $\chi \in J$, then we have $(d_\chi, a) = \chi(1) = (d_\chi, b)$. Since d_χ is non-negative, it must be zero whenever a is. Now suppose that $\chi \notin J$. Then $(d_\chi, a) = 0$ which implies that d_χ is zero whenever a is non-zero. This proves the claim. Now by [26, Corollary 3.10], Q is indecomposable and we conclude that $J = \mathrm{Irr}(B)$. □

The Basic Set Conjecture [8] asserts that there is always a basic set for B consisting of restrictions of ordinary irreducible characters. The decomposition matrix Q of B with respect to this basic set contains an identity matrix (hence the situation is similar as in the p-solvable case). It is also known by [31, Proposition 2] that Q is still indecomposable in this context. However, the argument from Proposition 3 eventually fails in this generality, because Q is not necessarily non-negative. For example, the decomposition matrix of the principal 2-block of A_5 is

$$Q = \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

up to basic sets. With $\chi_1, \chi_3 \in \operatorname{Irr}(B)$ the vector $a = (\chi_1(1), 0, \chi_3(1))$ in Lemma 1 seems possible. So we need some additional knowledge of the character degrees in order to see that $\chi_1(1) \neq \chi_3(1)$ and therefore a is not feasible. These ideas are used in the next theorem which extends [21, Theorem 2'].

Theorem 4. *Let B be a 2-block of G with defect group D. Then Harada's Conjecture holds for B provided one of the following conditions is satisfied:*

(i) D is metacyclic or minimal non-abelian,

(ii) D is abelian of rank at most 3,

(iii) $D \cong C_2^4$.

Proof. Suppose first that D is metacyclic. Then by [30, Theorem 8.1] there are only a few cases to consider. If B is nilpotent, then $l(B) = 1$ and the claim follows directly from Lemma 1 (see also [21, Lemma 2]). Now assume that B is not nilpotent. If D is dihedral (including the Klein four-group), semidihedral or quaternion, then the claim has been shown in [21, Theorem 1']. Finally, if D is abelian but not the Klein four-group, then B is Morita equivalent to the principal block of $D \rtimes C_3$ (see [30, Corollary 8.4]). Since $D \rtimes C_3$ is 2-solvable, the claim follows from Proposition 3.

Now assume that D is minimal non-abelian, but not metacyclic. Then by [30, Theorem 12.4] we need to discuss two cases. If $D \cong MNA(r, r)$ for some $r \geq 2$, then again B is Morita equivalent to the principal block of $D \rtimes C_3$ and the claim follows from Proposition 3. It remains to deal with $D \cong MNA(r, 1)$ where $r \geq 2$. Here the decomposition matrix Q of B up to basic sets is given explicitly, i. e. it does not depend on the group G. In particular, B has the same decomposition matrix as the principal block of the 2-solvable group $A_4 \rtimes C_{2^r}$ described in [30, Proposition 12.7]. Thus, the claim follows from Proposition 3.

Next let D be abelian of rank at most 3. Since abelian groups of rank 2 are metacyclic, we may assume that the rank of D equals 3. The possible Morita equivalence classes of B were determined in [5]. Apart from 2-solvable groups only the following groups occur: $A_5 \times C_{2^n}$, $\mathrm{SL}(2, 8)$, J_1 and $\mathrm{Aut}(\mathrm{SL}(2, 8))$. In the first case Q is obtained from the decomposition matrix of the principal block of A_5 by repeating every row 2^n times. It follows that a counterexample to Harada's Conjecture would also give a counterexample for A_5. However, A_5 has a metacyclic Sylow 2-subgroup and we are done by the first part of the proof. In case $\mathrm{SL}(2, 8)$ we have $D \cong C_2^3$, $k(B) = 8$ and $l(B) = 7$. Here the claim follows from [21, Lemma 3]. Finally in the last two cases Q is given as follows (up to basic sets):

$$J_1 : \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \\ 1 & 1 & -1 & . & . \\ 1 & 1 & . & -1 & . \\ 1 & 1 & . & . & -1 \end{pmatrix}, \qquad \mathrm{Aut}(\mathrm{SL}(2, 8)) : \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \\ 1 & -1 & 1 & . & . \\ 1 & -1 & . & 1 & . \\ 1 & -1 & . & . & 1 \end{pmatrix}.$$

In both cases $D \cong C_2^3$. To handle those cases we use the notation from Lemma 1. In the first case we may assume that $\chi_1 \in J$. Suppose first that $\chi_2 \in J$. Then by row 6 of Q, we have $\chi_1(1) + \chi_2(1) - \chi_3(1) = \chi_6(1)$. This forces $\chi_3 \in J$, because otherwise we obtain the contradiction $\chi_1(1) + \chi_2(1) = 0$. By the same argument with row 7 of

Q we obtain $\chi_4 \in J$ and eventually $J = \operatorname{Irr}(B)$. Thus, we assume now that $\chi_2 \notin J$. Then for $i = 3, 4, 5$ we either have ($\chi_i \in J$ and $\chi_1(1) = \chi_i(1)$) or ($\chi_{i+3} \in J$ and $\chi_1(1) = \chi_{i+3}(1)$). Hence, in any case J consists of exactly four characters which have the same degree $\chi_1(1)$. In particular $\chi_1(1)\sum_{\chi\in J}\chi$ vanishes on the 2-singular elements and so does $\sum_{\chi\in J}\chi$. By [26, Corollaries 2.14 and 2.17], we obtain $4\chi_1(1) \equiv 0 \pmod{|G|_2}$. On the other hand, an application of [30, Proposition 1.36] shows that all characters in $\operatorname{Irr}(B)$ have height 0 (this was also proved in general in [20]). In particular, the 2-part of $\chi_1(1)$ equals $|G|_2/8$. This yields the contradiction $4\chi_1(1) \not\equiv 0 \pmod{|G|_2}$. Finally, the case $G = \operatorname{Aut}(\operatorname{SL}(2, 8))$ can be handled analogously and we omit the details.

It remains to prove the claim for $D \cong C_2^4$. The possible Morita equivalence classes were computed by Eaton [4]. Arguing as above, we need to discuss only the classes coming from the principal bocks of $A_4 \times A_5$ and $A_5 \times A_5$. Here, Q is a Kronecker product of the decomposition matrices of the factors (up to basic sets):

$$A_4 \times A_5 : \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & -1 \end{pmatrix}, \qquad A_5 \times A_5 : \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

As usual let $\chi_1 \in J$. Since the principal block of A_5 is not a counterexample, it follows easily that $\chi_1, \chi_2, \chi_3, \chi_4 \in J$. Assume that these are all the characters of J. Then by rows $5, 6, 7, 9, 10, 11$ of Q, the vector a in Lemma 1 has the form $a = (\chi_1(1), \chi_2(1), \chi_3(1), 0, \ldots, 0)$. So the last row of Q yields the contradiction $\chi_4(1) = \chi_1(1) + \chi_2(1) - \chi_3(1) = 0$. Hence, $|J| > 4$. If $\chi_5 \in J$, we also get $\chi_6, \chi_7, \chi_8 \in J$ by looking at the rows $5, \ldots, 8$. In the same way we obtain $J = \operatorname{Irr}(B)$ eventually. □

Our next result extends [14, Proposition 5].

Theorem 5. *Harada's Conjecture holds for sporadic simple groups.*

Proof. Let B be a p-block of a sporadic group G with $l := l(B)$. Suppose that $J \subseteq \operatorname{Irr}(B)$ satisfies the hypothesis of Lemma 1. By [9], we may assume that B has a non-cyclic defect group. As the reader might guess, the proof relies on computer calculations with the character table library in GAP [7]. However, we cannot verify the conjecture directly, because it would take too long to check all subsets of $\operatorname{Irr}(B)$. Instead we do the following. Using the character table of G it is straight-forward to compute the decomposition matrix Q of B up to basic sets. In accordance with the conjecture mentioned above, we first show that B has a basic set consisting of restrictions of ordinary irreducible characters. Finding such a basic set is equivalent to finding l rows of Q such that the corresponding submatrix has determinant ± 1. By construction, Q has lower triangular shape. In many cases we can extract a lower unitriangular matrix from that. In the remaining cases we use a backtracking algorithm to enumerate linearly independent rows. In this way the desired basic set can be found with GAP. We remark that basic sets for most of the sporadic groups were also constructed explicitly by hand in Ikeda's papers [15, 17, 18, 25, 19].

By choosing this basic set, we may assume that Q involves an identity matrix of size $l \times l$. It follows that the vector $a = (a_1, \ldots, a_l)$ in Lemma 1 satisfies $a_i \in \{0, \chi_i(1)\}$ where $\chi_i \in \operatorname{Irr}(B)$ is uniquely determined ($i = 1, \ldots, l$). After replacing J by $\operatorname{Irr}(B) \setminus J$ if necessary, we may assume that $\chi_1 \in J$. Hence, there are 2^{l-1} possibilities for a. If l is relatively small (say $l < 25$), we can check the conjecture by testing all choices of a.

The blocks with $l \geq 25$ are all principal. We list them below:

G	Ly	J_4	Fi_{23}	Co_1			Fi'_{24}		BM			M					
p	5	11	3	2	3	5	2	3	2	3	5	2	3	5	7	11	13
l	35	40	32	26	29	29	33	25	25	71	51	55	83	91	70	45	52

In these special cases we proceed as follows. For a row $d_\chi = (d_1, \ldots, d_l)$ of Q let $F_\chi := \{1 \leq i \leq l : d_i \neq 0\}$. There are always characters χ such that $1 < |F_\chi| < 25$. In order to check the condition $(d_\chi, a) \in \{0, \chi(1)\}$ we only need to consider the entries a_i with $i \in F_\chi$. Hence, there are at most 2^{24} cases to look at. In this way we can show that the coefficients a_i with $i \in F_\chi$ are either all zero or all non-zero. Thus, the set $\{\chi_i : i \in F_\chi\}$ is either disjoint to J or contained in J. In the next step we make $\{\chi_i : i \in F_\chi\}$ bigger by choosing another row d_ψ such that $F_\chi \cap F_\psi \neq \varnothing$. In order to check $(d_\psi, a) \in \{0, \psi(1)\}$ it suffices to consider a_i with $i \in F_\psi \setminus F_\chi$, since we already know that the a_i with $i \in F_\chi$ are not independent of each other. Therefore, the condition $|F_\psi| < 25$ can be replaced by $|F_\psi \setminus F_\chi| < 24$. In this way we show that $\{\chi_i : i \in F_\chi \cup F_\psi\}$ is either disjoint to J or contained in J. Continuing in this way we obtain a covering of $\{1, \ldots, l\}$ and it follows eventually that $\chi_1, \ldots, \chi_l \in J$ (recall that $\chi_1 \in J$). Therefore $a_i \neq 0$ for $i = 1, \ldots, l$ and $J = \mathrm{Irr}(B)$. In this procedure there is only one hard case, namely the Monster in characteristic 5. Here the best choice for χ, ψ gives $|F_\chi| = 12$ and $|F_\chi \setminus F_\psi| = 27$. This is still doable in a matter of minutes. □

For the sake of completeness, we mention that Miyamoto [24] managed to prove Harada's Conjecture in a special case using completely different methods.

2 Dualizing Harada's Conjecture

In this section we consider a dual situation where characters are replaced by conjugacy classes. For motivational purpose, we begin with some well-known results.

Theorem 6 ([26, Theorem 3.19]). *The blocks of G are the connected components of a graph on $\mathrm{Irr}(G)$ such that (χ, ψ) is an edge if and only if*

$$\sum_{g \in G^0} \chi(g)\overline{\psi(g)} \neq 0.$$

Theorem 7 ([6, Lemma IV.6.3(ii)]). *If $\chi, \psi \in \mathrm{Irr}(G)$ lie in different blocks, then*

$$\sum_{g \in S} \chi(g)\overline{\psi(g)} = 0$$

for every p-section S of G

Theorem 8 ([26, Corollary 5.11]). *If $g, h \in G$ lie in different p-sections, then*

$$\sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\overline{\chi(h)} = 0$$

for every block B of G.

The following converse is not so well-known. For the convenience of the reader we provide a proof.

Theorem 9 (Osima [28, Theorem 3]). *Let $J \subseteq \mathrm{Irr}(G)$ such that*

$$\sum_{\chi \in J} \chi(g)\chi(h) = 0 \qquad \forall g \in G^0, h \in G \setminus G^0.$$

Then J is a union of blocks.

Proof. We fix $g \in G^0$. Then, by [26, Theorem 2.13], there are complex numbers a_φ^g such that

$$\sum_{\chi \in J} \chi(g)\chi = \sum_{\varphi \in \mathrm{IBr}(G)} a_\varphi^g \Phi_\varphi.$$

By [26, Corollary 2.14], we have $|G|_p \mid \Phi_\varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. Moreover, [26, Lemma 2.15] implies that

$$a_\varphi^g = \Big[\sum_{\mu \in \mathrm{IBr}(G)} a_\mu^g \Phi_\mu, \varphi\Big]^0 = \Big[\sum_{\chi \in J} \chi(g)\chi, \varphi\Big]^0 = \sum_{\chi \in J} \chi(g)[\chi, \hat{\varphi}] \in \mathbf{R}$$

where $\mathbf{R}$ is the ring of algebraic integers in $\mathbb{C}$. With the notation from [26, p. 16 and 51] we conclude that

$$\sum_{\chi \in J} e_\chi = \frac{1}{|G|} \sum_{\chi \in J} \chi(1) \sum_{g \in G} \chi(g^{-1})g = \sum_{g \in G^0} \Big(\sum_{\chi \in J} \frac{\chi(1)\chi(g^{-1})}{|G|}\Big)g = \sum_{g \in G^0} \Big(\sum_{\varphi \in \mathrm{IBr}(G)} \frac{a_\varphi^{g^{-1}} \Phi_\varphi(1)}{|G|}\Big)g \in \mathrm{Z}(SG).$$

Now the claim follows from [26, Theorem 3.9]. □

Note that Harada's Conjecture is just a strengthening of Theorem 9. We remark also that Theorem 9 does not work if G^0 is replaced by a non-trivial p-section.

In order to interchange characters and conjugacy classes, we need to introduce a block distribution on the classes. This has already been done by several authors (e. g. [3, 27]) in a non-canonically way. Here we mimic Theorem 6 to give a more natural approach. At the same time we need to give up other properties (e. g. the number of blocks of classes is not necessarily the number of blocks of characters).

Definition 10. The *(p-)class blocks* of G are the connected components of the graph on G where (g, h) is an edge if and only if there is a (p-)block B of G such that

$$\sum_{\chi\in\mathrm{Irr}(B)} \chi(g)\overline{\chi(h)} \neq 0.$$

It is clear that class blocks are union of conjugacy classes. Also by Theorem 8, every class block lies in a p-section of G, but the class blocks are usually finer than the p-sections. For example, if G has only one block, then every class block is a conjugacy class by the second orthogonality relation (the converse is also true by Theorem 13 below). On the other hand:

Lemma 11. *If G has a normal p-complement, then the class blocks are the p-sections.*

Proof. Let $x \in G$ be a p-element and let $y, z \in \mathrm{C}_G(x)^0$. We need to show that xy and xz lie in the same class block. By [26, Corollary 6.13] we have $\mathrm{Irr}(B_0) = \mathrm{Irr}(G/\mathrm{O}_{p'}(G))$. It follows that

$$\sum_{\chi\in\mathrm{Irr}(B_0)} \chi(xy)\overline{\chi(xz)} = \sum_{\chi\in\mathrm{Irr}(G/\mathrm{O}_{p'}(G))} \chi(x)\overline{\chi(x)} \neq 0. \qquad \square$$

Here is a less trivial example: The classes (3^2), $(3^2, 2^2)$, $(6, 2)$ and $(6, 4)$ form a (non-trivial) 3-section of the alternating group A_{10}. On the other hand, the first two and the last two form class blocks.

As another remark, the class blocks are generally not invariant under $\mathrm{Aut}(G)$. For example, there are p-groups with automorphisms which do not preserve conjugacy classes. Similarly, the class blocks are not invariant under Galois actions ($g \mapsto g^i$ for some i with $(i, |G|) = 1$).

It happens frequently that a p-element x does not lie in any defect group apart from the whole Sylow p-subgroup. Also, quite often the principal block is the only block with maximal defect. In that case the class blocks in the p-section of x are all just conjugacy classes (see [26, Corollary 5.9]). This draws the focus on the distribution of the p-regular elements into class blocks. Using the character table library in GAP [7] we obtain the following examples:

Example 12. Let G be a simple group whose character table is stored in GAP (for example a sporadic group). Let p be a prime such that G^0 is not a class block. Then one of the following cases occurs:

G	p	comments
M_{11}	3	
M_{22}	2	only one p-block
M_{23}	2	
M_{24}	2	only one p-block
Co_1	2	
Co_2	2	
Co_3	3	
J_4	2	
BM	2	
M	2	

It is quite remarkable that only sporadic groups show up in the table above.

The following generalizes Theorem 7.

Theorem 13. *If $\chi,\psi\in\operatorname{Irr}(G)$ lie in different blocks, then*

$$\sum_{g\in C}\chi(g)\overline{\psi(g)}=0$$

for every class block C of G.

Proof. Let B be a block of G, and let $\operatorname{Irr}(B)=\{\chi_1,\ldots,\chi_r\}$. Let $g_1,\ldots,g_k$ be a set of representatives for the conjugacy classes of G. We may order these elements with respect to their class blocks. Let

$$A:=(|\mathrm{C}_G(g_j)|^{-\frac{1}{2}}\chi_i(g_j))_{i,j}\in\mathbb{C}^{r\times k}$$

(the factors $|\mathrm{C}_G(g_j)|^{-\frac{1}{2}}$ turn the character table into a unitary matrix). Then $A^{\mathrm{t}}\overline{A}$ is a block diagonal matrix where the blocks correspond to the class blocks of G (note that the extra factor in the definition of A does not matter here). On the other hand, the first orthogonality relation implies that $\overline{A}A^{\mathrm{t}}=1_r$ and $\overline{A}A^{\mathrm{t}}\overline{A}=\overline{A}$. Let A_i be the submatrix of A where we only take those columns corresponding to the i-th class block. Then $\overline{A_i}A_i^{\mathrm{t}}\overline{A_i}=\overline{A_i}$ and $A_i^{\mathrm{t}}\overline{A_i}A_i^{\mathrm{t}}=A_i^{\mathrm{t}}$.

In a similar fashion we define

$$M:=(|\mathrm{C}_G(g_j)|^{-\frac{1}{2}}\psi_i(g_j))_{i,j}\in\mathbb{C}^{(k-r)\times k}$$

where $\operatorname{Irr}(G)\setminus\operatorname{Irr}(B)=\{\psi_1,\ldots,\psi_{k-r}\}$. For the corresponding submatrix M_i we also have $\overline{M_i}M_i^{\mathrm{t}}\overline{M_i}=\overline{M_i}$. Our task is to show that $\overline{M_i}A_i^{\mathrm{t}}=0$. The second orthogonality relation gives us $A_i^{\mathrm{t}}\overline{A_i}+M_i^{\mathrm{t}}\overline{M_i}=1_s$ where s is the number of conjugacy classes in the i-th class block. Consequently,

$$\overline{M_i}A_i^{\mathrm{t}}=\overline{M_i}(A_i^{\mathrm{t}}\overline{A_i}+M_i^{\mathrm{t}}\overline{M_i})A_i^{\mathrm{t}}=\overline{M_i}A_i^{\mathrm{t}}\overline{A_i}A_i^{\mathrm{t}}+\overline{M_i}M_i^{\mathrm{t}}\overline{M_i}A_i^{\mathrm{t}}=2\overline{M_i}A_i^{\mathrm{t}},$$

and the claim follows. □

There is a connection to the work of Belonogov [1, 2] which we introduce now. Let D be a union of conjugacy classes of G. Then the *D-blocks* of G are the connected components of the graph on $\operatorname{Irr}(G)$ with vertices (χ,ψ) such that $\sum_{g\in D}\chi(g)\overline{\psi(g)}\neq 0$. Thus, the G^0-blocks are just the p-blocks. Now let D be the class block of the identity element. Then Theorem 13 says that the p-blocks are unions of D-blocks. In fact, judging from examples it seems that the D-blocks are the p-blocks. A similar concept can be found in [23, Section 1].

Now we prove a dual version of Osima's Theorem 9.

Theorem 14. *Let J be a union of conjugacy classes of G such that*

$$\sum_{g\in J}\chi(g)\overline{\psi(g)}=0\qquad\forall\chi,\psi\in\operatorname{Irr}(G)\text{ in different blocks.}$$

Then J is a union of class blocks.

Proof. Let B be a block of G. We define the matrices A and M exactly as in Theorem 13. Moreover, let A_J (respectively M_J) be the submatrix of A (respectively M) whose columns correspond to the classes in J. Similarly, we define $A_{J'}$ and $M_{J'}$ where $J'=G\setminus J$. Then $1=\overline{A}A^{\mathrm{t}}=\overline{A_J}A_J^{\mathrm{t}}+\overline{A_{J'}}A_{J'}^{\mathrm{t}}$. By hypothesis we have $\overline{M_J}A_J^{\mathrm{t}}=0$ and $\overline{A_J}M_J^{\mathrm{t}}=0$. By the first orthogonality relation, also $\overline{M_{J'}}A_{J'}^{\mathrm{t}}=0$. We have to prove that $A_{J'}^{\mathrm{t}}\overline{A_J}=0$. The second orthogonality relation implies that $A_J^{\mathrm{t}}\overline{A_J}+M_J^{\mathrm{t}}\overline{M_J}=1$ and $A_{J'}^{\mathrm{t}}\overline{A_{J'}}+M_{J'}^{\mathrm{t}}\overline{M_{J'}}=1$. We conclude that $\overline{A_J}=\overline{A_J}(A_J^{\mathrm{t}}\overline{A_J}+M_J^{\mathrm{t}}\overline{M_J})=\overline{A_J}A_J^{\mathrm{t}}\overline{A_J}$ and $A_{J'}^{\mathrm{t}}=(A_{J'}^{\mathrm{t}}\overline{A_{J'}}+M_{J'}^{\mathrm{t}}\overline{M_{J'}})A_{J'}^{\mathrm{t}}=A_{J'}^{\mathrm{t}}\overline{A_{J'}}A_{J'}^{\mathrm{t}}$. Now putting things together, we obtain

$$A_{J'}^{\mathrm{t}}\overline{A_J}=A_{J'}^{\mathrm{t}}(\overline{A_J}A_J^{\mathrm{t}}+\overline{A_{J'}}A_{J'}^{\mathrm{t}})\overline{A_J}=2A_{J'}^{\mathrm{t}}\overline{A_J}.$$

The claim follows. □

Comparing Theorem 14 with Theorem 9 one might think it is enough to consider $\chi \in \mathrm{Irr}(B_0)$ and $\psi \in \mathrm{Irr}(G) \setminus \mathrm{Irr}(B_0)$ where B_0 is the principal block of G. However, this is not true in general. A counterexample is given by the group $G = Co_1$ for $p = 5$. Nevertheless, one can adjust the definition of class blocks such that

$$(g,h) \sim \sum_{\chi \in \mathrm{Irr}(B_0)} \chi(g)\overline{\chi(h)} \neq 0.$$

Then the mentioned variation of Theorem 14 would be true, but Theorem 13 would fail.

Eventually, for a dual version of Harada's Conjecture we could set $\chi = 1$ in Theorem 14, but this is again false (even for the modified class block definition above, a counterexample is $G = M_{11}$ for $p = 3$).

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