2-Blocks with minimal nonabelian defect groups III

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Abstract

We prove that two 2-blocks of (possibly different) finite groups with a common minimal nonabelian defect group and the same fusion system are isotypic (and therefore perfectly isometric) in the sense of Broué. This continues former work by [Cabanes-Picaronny, 1992], [Sambale, 2011] and [Eaton-Külshammer-Sambale, 2012].

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1 Introduction

Since its appearance in 1990, Broué's Abelian Defect Conjecture gained much attention among representation theorists. On the level of characters it predicts the existence of a perfect isometry between a block with abelian defect group and its Brauer correspondent. These blocks have a common defect group and the same fusion system. Although Broué's Conjecture is false for nonabelian defect groups (see [4]), one can still ask if perfect isometries or even isotypies exist. We affirmatively answer this question for $p = 2$ and minimal nonabelian defect groups (see Theorem 9 below). These are the nonabelian defect groups such that any proper subgroup is abelian. Doing so, we verify the character-theoretic version of Rouquier's Conjecture [17, A.2] in this special case (see Corollary 10 below). At the same time we provide a new infinite family of defect groups supporting a blockwise Z^*-Theorem.

By Rédei's classification of minimal nonabelian p-groups, one has to consider three distinct families of defect groups. For two of these families the result already appeared in the literature (see [3, 19, 5]). Hence, it suffices to handle the remaining family which we will do in the next section. The proof of the main result is an application of Horimoto-Watanabe [10, Theorem 2]. The last section of the present paper also contains a related result for the nonabelian defect group of order 27 and exponent 9.

Our notation is fairly standard. We consider blocks B of finite groups with respect to a p-modular system $(K, \mathcal{O}, F)$ where $\mathcal{O}$ is a complete discrete valuation ring with quotient field K of characteristic 0 and field of fractions F of characteristic p. As usual, we assume that K is "large" enough and F is algebraically closed. The number of irreducible ordinary characters (resp. Brauer characters) of B is denoted by $k(B)$ (resp. $l(B)$). Moreover, $k_i(B)$ is the number of those irreducible characters of B which have height $i \geq 0$. For other results on block invariants and fusion systems we often refer to [20]. Moreover, for the definition and construction of perfect isometries we follow [1, 3]. A cyclic group of order $n \in \mathbb{N}$ is denoted by C_n.

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2 A class of minimal nonabelian defect groups

Let B be a non-nilpotent 2-block of a finite group G with defect group

$$D = \langle x, y \mid x^{2^r} = y^2 = [x,y]^2 = [x,x,y] = [y,x,y] = 1\rangle \cong C_2^2 \rtimes C_{2^r} \tag{1}$$

where $r \geq 2$, $[x,y] := xyx^{-1}y^{-1}$ and $[x,x,y] := [x,[x,y]]$.

We have already investigated some properties of B in [19], and later gave simplified proofs in [20, Chapter 12]. For the convenience of the reader we restate some of these results.

Lemma 1 ([20, Lemma 12.3])**.** *Let $z := [x,y]$. Then the following holds:*

(i) $\Phi(D) = \mathrm{Z}(D) = \langle x^2, z\rangle \cong C_{2^{r-1}} \times C_2$.

(ii) $D' = \langle z \rangle \cong C_2$.

(iii) $|\mathrm{Irr}(D)| = 5 \cdot 2^{r-1}$.

Recall that a (saturated) fusion system $\mathcal{F}$ on a p-group P determines the following subgroups:

$$\begin{aligned} \mathrm{Z}(\mathcal{F}) &:= \{x \in P : x \text{ is fixed by every morphism in } \mathcal{F}\}, \\ \mathfrak{foc}(\mathcal{F}) &:= \langle f(x)x^{-1} : x \in Q \leq P,\ f \in \mathrm{Aut}_{\mathcal{F}}(Q)\rangle, \\ \mathfrak{hyp}(\mathcal{F}) &:= \langle f(x)x^{-1} : x \in Q \leq P,\ f \in \mathrm{O}^p(\mathrm{Aut}_{\mathcal{F}}(Q))\rangle. \end{aligned}$$

Lemma 2. *The fusion system $\mathcal{F}$ of B is the constrained fusion system of the finite group $A_4 \rtimes C_{2^r}$ where C_{2^r} acts as a transposition in $\mathrm{Aut}(A_4) \cong S_4$. In particular, B has inertial index 1 and $Q := \langle x^2, y, z\rangle \cong C_{2^{r-1}} \times C_2^2$ is the only $\mathcal{F}$-essential subgroup of D. Moreover, $\mathrm{Aut}_{\mathcal{F}}(Q) \cong S_3$. Without loss of generality, $\mathrm{Z}(\mathcal{F}) = \langle x^2\rangle$ and $\mathfrak{hyp}(B) = \mathfrak{foc}(B) = \mathfrak{foc}(\mathcal{F}) = \langle y, z\rangle$.*

Proof. We have seen in [20, Proposition 12.7] that $\mathcal{F}$ is constrained and coincides with the fusion system of $A_4 \rtimes C_{2^r}$. The construction of the semidirect product $A_4 \rtimes C_{2^r}$ is slightly different in [20], but it is easy to see that both constructions give isomorphic groups. The remaining claims follow from the proof of [20, Proposition 12.7]. □

By a result of Watanabe [25, Theorem 3 and Lemma 3], the hyperfocal subgroup of a 2-block is trivial or non-cyclic. Hence, our situation with a Klein-four (hyper)focal subgroup represents the first non-trivial example in some sense. Recall that a *B-subsection* is a pair (u, b_u) such that $u \in D$ and b_u is a Brauer correspondent of B in $\mathrm{C}_G(u)$.

Lemma 3. *The set $\mathcal{R} := \mathrm{Z}(D) \cup \{x^i y^j : i,j \in \mathbb{Z},\ i \text{ odd}\}$ is a set of representatives for the $\mathcal{F}$-conjugacy classes of D with $|\mathcal{R}| = 2^{r+1}$. For $u \in \mathcal{R}$ let (u, b_u) be a B-subsection. Then b_u has defect group $\mathrm{C}_D(u)$. Moreover, $l(b_u) = 1$ whenever $u \in \mathcal{R} \setminus \langle x^2\rangle$.*

Proof. By Lemma 2, it is easy to see that $\mathcal{R}$ is in fact a set of representatives for the $\mathcal{F}$-conjugacy classes of D. Observe that $\langle u\rangle$ is fully $\mathcal{F}$-normalized for all $u \in \mathcal{R}$. Hence, by [20, Lemma 1.34], b_u has defect group $\mathrm{C}_D(u)$ and fusion system $\mathrm{C}_{\mathcal{F}}(\langle u\rangle)$. It is easy to see that $\mathrm{C}_{\mathcal{F}}(\langle u\rangle)$ is trivial unless $u \in \mathrm{Z}(\mathcal{F}) = \langle x^2\rangle$. This shows $l(b_u) = 1$ for $u \in \mathcal{R} \setminus \langle x^2\rangle$. □

Theorem 4 ([20, Theorem 12.4])**.** *We have $k(B) = 5 \cdot 2^{r-1}$, $k_0(B) = 2^{r+1}$, $k_1(B) = 2^{r-1}$ and $l(B) = 2$.*

Proof. By Lemma 2, we have $|D : \mathfrak{foc}(B)| = 2^r$. In particular, $2^r \mid k_0(B)$ by [16, Theorem 1]. Moreover, [11, Theorem 1.1] implies $2^{r+1} \leq k_0(B)$. By Lemma 3 we have $l(b_x) = 1$. Thus, we obtain $k_0(B) = 2^{r+1}$ by a result of Robinson (see [20, Theorem 4.12]). In order to determine $l(B)$, we use induction on r. Let $u := x^2$. Then b_u dominates a block $\overline{b_u}$ of $\mathrm{C}_G(u)/\langle u\rangle$ with defect group $\overline{D} := D/\langle u\rangle \cong D_8$ and fusion system $\overline{\mathcal{F}} := \mathcal{F}/\langle u\rangle$. By [13, Theorem 6.3], $\langle x^2, y, z\rangle/\langle u\rangle \cong C_2^2$ is the only $\overline{\mathcal{F}}$-essential subgroup of $\overline{D}$. Therefore, a result of Brauer (see [20, Theorem 8.1]) shows that $l(b_u) = l(\overline{b_u}) = 2$. By Lemma 3 and [20, Theorem 1.35] it follows that

$k(B) > k_0(B)$. Since $|\mathrm{Z}(D) : \mathrm{Z}(D) \cap \mathfrak{foc}(B)| = 2^{r-1}$, we have $2^{r-1} \mid k_i(B)$ for $i \geq 1$ by [16, Theorem 2]. Thus, by [15, Theorem 3.4] we obtain

$$2^{r+2} \leq k_0(B) + 4(k(B) - k_0(B)) \leq \sum_{i=0}^{\infty} k_i(B) 2^{2i} \leq |D| = 2^{r+2}.$$

This gives $k_1(B) = 2^{r-1}$ and $k(B) = k_0(B) + k_1(B) = 5 \cdot 2^{r-1}$. In case $r = 2$, [20, Theorem 1.35] implies

$$l(B) = k(B) - \sum_{1 \neq u \in \mathcal{R}} l(b_u) = 10 - 8 = 2.$$

Now let $r \geq 3$ and $1 \neq \langle u \rangle < \langle x^2 \rangle$. Then $\overline{b_u}$ as above has the same type of defect group as B except that r is smaller. Hence, induction gives $l(b_u) = l(\overline{b_u}) = 2$. Now the claim $l(B) = 2$ follows again by [20, Theorem 1.35]. □

In the following results we denote the set of irreducible characters of B of height i by $\mathrm{Irr}_i(B)$.

Proposition 5 ([20, Proposition 12.9])**.** *The set $\mathrm{Irr}_0(B)$ contains four 2-rational characters and two families of 2-conjugate characters of size 2^i for every $i = 1, \ldots, r-1$. The characters of height 1 split into two 2-rational characters and one family of 2-conjugate characters of size 2^i for every $i = 2, \ldots, r-2$.*

Proposition 6. *There are 2-rational characters $\chi_i \in \mathrm{Irr}(B)$ for $i = 1, 2, 3$ such that*

$$\begin{aligned} \mathrm{Irr}_0(B) &= \{\chi_i * \lambda : i = 1, 2,\ \lambda \in \mathrm{Irr}(D/\mathfrak{foc}(B))\}, \\ \mathrm{Irr}_1(B) &= \{\chi_3 * \lambda : \lambda \in \mathrm{Irr}(\mathrm{Z}(D)\mathfrak{foc}(B)/\mathfrak{foc}(B))\}. \end{aligned}$$

In particular, the characters of height 1 have the same degree and $|\{\chi(1) : \chi \in \mathrm{Irr}_0(B)\}| \leq 2$.

Proof. We have already seen in the proof of Theorem 4 that the action of $D/\mathfrak{foc}(B)$ on $\mathrm{Irr}_0(B)$ via the $*$-construction has two orbits, and the action of $\mathrm{Z}(D)\mathfrak{foc}(B)/\mathfrak{foc}(B)$ on $\mathrm{Irr}_1(B)$ is regular. By Proposition 5 we can choose 2-rational representatives for these orbits. Notice that we identify the sets $\mathrm{Irr}(D/\mathfrak{foc}(B))$ and $\mathrm{Irr}(\mathrm{Z}(D)\mathfrak{foc}(B)/\mathfrak{foc}(B))$ with subsets of $\mathrm{Irr}(D)$ in an obvious manner. □

In the situation of Proposition 6 it is conjectured that $\chi_1(1) \neq \chi_2(1)$ (see [14]).

Proposition 7 ([20, Proposition 12.8])**.** *The Cartan matrix of B is given by*

$$2^{r-1} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

up to basic sets.

Observe that Proposition 7 also gives the Cartan matrix for the defect group D_8 and the corresponding fusion system (this would be the case $r = 1$).

Now we are in a position to obtain the generalized decomposition matrix of B. This completes partial results in [19, Section 3.3].

Proposition 8. *Let $\mathcal{R}$ and χ_i be as in Lemma 3 and Proposition 6 respectively. Then there are basic sets for b_u ($u \in \mathcal{R}$) and signs $\epsilon, \sigma \in \{\pm 1\}$ such that the generalized decomposition numbers of B have the following form*

u	x^{2i}	$x^{2i}z$	x^{2i+1}	$x^{2i+1}y$
$d^u_{\chi_1 \varphi}$	$(1, 0)$	1	1	1
$d^u_{\chi_2 \varphi}$	$(0, \epsilon)$	ϵ	ϵ	$-\epsilon$
$d^u_{\chi_3 \varphi}$	(σ, σ)	-2σ	0	0

.

Proof. Since the Galois group of $\mathbb{Q}(e^{2\pi i/2^r})$ over $\mathbb{Q}$ acts on the columns of the generalized decomposition matrix (cf. Proposition 5), we only need to determine the numbers $d^u_{\chi_i\varphi}$ for $u\in\{x,xy,x^{2^j},x^{2^j}z\}$ $(i=1,2,3,$ $j=1,\ldots,r)$. First let $u=x$. Then the orthogonality relations show that

$$2^r|d^x_{\chi_1\varphi}|^2+2^r|d^x_{\chi_2\varphi}|^2+2^{r-1}|d^x_{\chi_3\varphi}|^2=2^{r+1}.$$

Since χ_1 and χ_2 have height 0, we have $d^x_{\chi_1\varphi}\ne 0\ne d^x_{\chi_2\varphi}$ (see [20, Proposition 1.36]). It follows that $d^x_{\chi_i\varphi}=\pm1$ for $i=1,2$ and $d^x_{\chi_3\varphi}=0$, because χ_i is 2-rational. By replacing φ with $-\varphi$ if necessary (i. e. changing the basic set for b_x), we may assume that $d^x_{\chi_1\varphi}=1$. We set $d^x_{\chi_2\varphi}=:\epsilon_0$. Similarly, we obtain $d^{xy}_{\chi_1\varphi}=1$, $d^{xy}_{\chi_2\varphi}=\pm1$ and $d^{xy}_{\chi_3\varphi}=0$. Now since the columns d^x and d^{xy} of the generalized decomposition matrix are orthogonal, we obtain $d^{xy}_{\chi_2\varphi}=-\epsilon_0$.

Now let $u:=x^{2^j}$ for some $j\in\{1,\ldots,r\}$. Let $\mathrm{IBr}(b_u)=\{\varphi_1,\varphi_2\}$ (see proof of Theorem 4). Then by Proposition 7 we get

$$\begin{aligned}2^r|d^u_{\chi_1\varphi_1}|^2+2^r|d^u_{\chi_2\varphi_1}|^2+2^{r-1}|d^u_{\chi_3\varphi_1}|^2&=3\cdot2^{r-1},\\ 2^r|d^u_{\chi_1\varphi_2}|^2+2^r|d^u_{\chi_2\varphi_2}|^2+2^{r-1}|d^u_{\chi_3\varphi_2}|^2&=3\cdot2^{r-1},\\ 2^rd^u_{\chi_1\varphi_1}\overline{d^u_{\chi_1\varphi_2}}+2^rd^u_{\chi_2\varphi_1}\overline{d^u_{\chi_2\varphi_2}}+2^{r-1}d^u_{\chi_3\varphi_1}\overline{d^u_{\chi_3\varphi_2}}&=2^{r-1}.\end{aligned}$$

Obviously, $d^u_{\chi_1\varphi_1}d^u_{\chi_2\varphi_1}=0$ and we may assume that $(d^u_{\chi_1\varphi_1},d^u_{\chi_1\varphi_2})=(1,0)$ and $(d^u_{\chi_2\varphi_1},d^u_{\chi_2\varphi_2})=(0,\epsilon_j)$ for a sign $\epsilon_j\in\{\pm1\}$. Moreover, $d^u_{\chi_3\varphi_1}=d^u_{\chi_3\varphi_2}=:\sigma_j\in\{\pm1\}$. Now let $u:=x^{2^j}z$. Then we have

$$2^r|d^u_{\chi_1\varphi}|^2+2^r|d^u_{\chi_2\varphi}|^2+2^{r-1}|d^u_{\chi_3\varphi}|^2=2^{r+2}.$$

It is known that $2\mid d^u_{\chi_3\varphi}\ne0$, since b_u is major (see [20, Proposition 1.36]). This gives $d^u_{\chi_1\varphi}=1$, $d^u_{\chi_2\varphi}=\pm1$ and $d^u_{\chi_3\varphi}=\pm2$. By the orthogonality to $d^{x^{2^j}}$ we obtain that $d^u_{\chi_3\varphi}=-2\sigma_j$ and $d^u_{\chi_2\varphi}=\epsilon_j$.

It remains to show that the signs ϵ_j and σ_j do not depend on j. For this we consider characters $\lambda,\psi\in\mathrm{Irr}(D)$ whose values are given as follows

	x^{2^j}	$x^{2^j}z$	x	xy
λ	1	1	1	-1
ψ	2	-2	0	0

Observe that ψ is the inflation of the irreducible character of $D/\langle x^2\rangle\cong D_8$ of degree 2. It is easy to see that $(\lambda+\psi)(x^{2k}y)=-1=1-2=(\lambda+\psi)(x^{2k}z)$ for every $k\in\mathbb{Z}$. It follows that $\lambda+\psi$ is $\mathcal{F}$-stable, i. e. $(\lambda+\psi)(u)=(\lambda+\psi)(v)$ whenever u and v are $\mathcal{F}$-conjugate. By Broué-Puig [1], $\chi_1*(\lambda+\psi)$ is a generalized character of B. In particular, the scalar product $(\chi_1*(\lambda+\psi),\chi_3)_G$ is an integer. This number can be computed by using the so-called contribution numbers $m^u_{\chi_1\chi_3}:=d^u_{\chi_1}C_u^{-1}\overline{d^u_{\chi_3}}^{\mathrm{T}}$ where C_u is the Cartan matrix of b_u and $d^u_{\chi_i}$ is the row of the generalized decomposition matrix corresponding to (u,b_u) and χ_i. In case $u=x^{2^j}$ we have

$$C_u^{-1}=2^{-r-2}\begin{pmatrix}3&-1\\-1&3\end{pmatrix}$$

by Proposition 7. This gives $m^u_{\chi_1\chi_3}=2^{-r-1}\sigma_j$. Similarly, $m^u_{\chi_1\chi_3}=-2^{-r-1}\sigma_j$ for $u=x^{2^j}z$. Thus, we obtain

$$\begin{aligned}(\chi_1*(\lambda+\psi),\chi_3)_G&=\sum_{u\in\mathcal{R}}(\lambda+\psi)(u)m^u_{\chi_1\chi_3}=\sum_{u\in\mathrm{Z}(D)}(\lambda+\psi)(u)m^u_{\chi_1\chi_3}\\&=(3+1)\Big(2^{-r-1}\sigma_r+2^{-r-1}\sum_{j=1}^{r-1}\sigma_j2^{r-j-1}\Big)=2^{-r+1}\sigma_r+\sum_{j=1}^{r-1}\sigma_j2^{-j}.\end{aligned}$$

If $\sigma_1=\sigma_j$ for some $j\ne1$, then it follows immediately that $\sigma_1=\ldots=\sigma_r$ (otherwise the scalar product above is not an integer). Now suppose that $-\sigma_1=\sigma_2=\ldots=\sigma_r$. In this case we replace χ_3 by the 2-rational character $\chi_3*\tau$ where $\tau\in\mathrm{Irr}(\mathrm{Z}(D)\mathfrak{foc}(B)/\mathfrak{foc}(B))$ such that $\tau(x^2)=-1$. This changes σ_1, but does not affect σ_j for $j>1$.

A similar argument with the scalar product $(\chi_2*(\lambda+\psi),\chi_3)_G$ implies that $\epsilon_1=\ldots=\epsilon_r$. In case $\epsilon_0=-\epsilon_1$, we replace χ_2 by $\chi_2*\tau$ where $\tau\in\mathrm{Irr}(D/\mathfrak{foc}(B))$ such that $\tau(x)=-1$. Observe again that this changes ϵ_0, but keeps ϵ_j for $j>0$. This completes the proof. □

3 The main result

Theorem 9. *Let B and $\widetilde{B}$ be 2-blocks of (possibly different) finite groups with a common minimal nonabelian defect group and the same fusion system. Then B and $\widetilde{B}$ are isotypic (and therefore perfectly isometric).*

Proof. We may assume that B is not nilpotent by Broué-Puig [2]. Let D be a defect group of B and $\widetilde{B}$. If $|D| = 8$, then the claim follows from [3]. Now suppose that D is given as in (1). We will attach a tilde to everything associated with $\widetilde{B}$. By Proposition 8 and [10, Theorem 2] there is a perfect isometry $I : \mathrm{CF}(G, B) \to \mathrm{CF}(\widetilde{G}, \widetilde{B})$ where $\mathrm{CF}(G, B)$ denotes the space of class functions with basis $\mathrm{Irr}(B)$ over K. It remains to show that I is also an isotypy. In order to do so, we follow [3, Section V.2]. For each $u \in D$ let $\mathrm{CF}(\mathrm{C}_G(u)_{2'}, b_u)$ be the space of class functions on $\mathrm{C}_G(u)$ which vanish on the p-singular classes and are spanned by $\mathrm{IBr}(b_u)$. The decomposition map $d_G^u : \mathrm{CF}(G, B) \to \mathrm{CF}(\mathrm{C}_G(u)_{2'}, b_u)$ is defined by

$$d_G^u(\chi)(s) := \chi(e_{b_u} us) = \sum_{\varphi \in \mathrm{IBr}(b_u)} d_{\chi\varphi}^u \varphi(s)$$

for $\chi \in \mathrm{Irr}(B)$ and $s \in \mathrm{C}_G(u)_{2'}$ where e_{b_u} is the block idempotent of b_u over $\mathcal{O}$. Then I determines isometries

$$I^u : \mathrm{CF}(\mathrm{C}_G(u)_{2'}, b_u) \to \mathrm{CF}(\mathrm{C}_{\widetilde{G}}(u)_{2'}, \widetilde{b_u})$$

by the equation $d_{\widetilde{G}}^u \circ I = I^u \circ d_G^u$. Note that I^1 is the restriction of I. We need to show that I^u can be extended to a perfect isometry $\widehat{I^u} : \mathrm{CF}(\mathrm{C}_G(u), b_u) \to \mathrm{CF}(\mathrm{C}_{\widetilde{G}}(u), \widetilde{b_u})$. Suppose first that b_u is nilpotent. Then by Proposition 8, $d_G^u(\chi_1) = \epsilon\varphi$ and $d_{\widetilde{G}}^u(I(\chi_1)) = \widetilde{\epsilon}\widetilde{\varphi}$ where $\mathrm{IBr}(b_u) = \{\varphi\}$ and $\mathrm{IBr}(\widetilde{b_u}) = \{\widetilde{\varphi}\}$ for some signs $\epsilon, \widetilde{\epsilon} \in \{\pm 1\}$. It follows that $I^u(\varphi) = \epsilon\widetilde{\epsilon}\widetilde{\varphi}$. Let $\psi \in \mathrm{Irr}_0(b_u)$ and $\widetilde{\psi} \in \mathrm{Irr}_0(\widetilde{b_u})$ be 2-rational characters. Then it is well-known that $\varphi = d_{\mathrm{C}_G(u)}^1(\psi)$ and $\mathrm{Irr}(b_u) = \{\psi * \lambda : \lambda \in \mathrm{Irr}(D)\}$ (see [2]). Therefore, we may define $\widehat{I^u}$ by $\widehat{I^u}(\psi * \lambda) := \epsilon\widetilde{\epsilon}\widetilde{\psi} * \lambda$ for $\lambda \in \mathrm{Irr}(D)$. Then $\widehat{I^u}$ is a perfect isometry and

$$\widehat{I^u}(\varphi) = \widehat{I^u}(d_{\mathrm{C}_G(u)}^1(\psi)) = d_{\mathrm{C}_{\widetilde{G}}(u)}^1(\widehat{I^u}(\psi)) = \epsilon\widetilde{\epsilon} d_{\mathrm{C}_{\widetilde{G}}(u)}^1(\widetilde{\psi}) = \epsilon\widetilde{\epsilon}\widetilde{\varphi} = I^u(\varphi).$$

Hence, $\widehat{I^u}$ extends I^u. Moreover, $\widehat{I^u}$ does not depend on the generator of $\langle u \rangle$, since the signs ϵ and $\widetilde{\epsilon}$ were defined by means of 2-rational characters.

Assume next that b_u is non-nilpotent. Then $u \in \langle x^2 \rangle$ and b_u has defect group D. By Proposition 8, we can choose basic sets φ_1, φ_2 (resp. $\widetilde{\varphi_1}$, $\widetilde{\varphi_2}$) for b_u (resp. $\widetilde{b_u}$) such that $\varphi_i = d_G^u(\chi_i)$ and $\widetilde{\varphi}_i = d_{\widetilde{G}}^u(I(\chi_i))$ for $i = 1, 2$. Then $I^u(\varphi_i) = \widetilde{\varphi}_i$ for $i = 1, 2$. Since the Cartan matrix of b_u with respect to the basic set φ_1, φ_2 is already fixed (and given by Proposition 7), we find 2-rational characters $\psi_i \in \mathrm{Irr}_0(b_u)$ such that $d_{\mathrm{C}_G(u)}^1(\psi_i) = \epsilon_i\varphi_i$ with $\epsilon_i \in \{\pm 1\}$ for $i = 1, 2$ (see proof of Proposition 8). Similarly, one has $\widetilde{\psi}_i \in \mathrm{Irr}_0(\widetilde{b_u})$ such that $d_{\mathrm{C}_{\widetilde{G}}(u)}^1(\widetilde{\psi}_i) = \widetilde{\epsilon}_i\widetilde{\varphi}_i$. Then, by what we have already shown, there exists a perfect isometry $\widehat{I^u} : \mathrm{CF}(\mathrm{C}_G(u), b_u) \to \mathrm{CF}(\mathrm{C}_{\widetilde{G}}(u), \widetilde{b_u})$ sending ψ_i to $\epsilon_i\widetilde{\epsilon}_i\widetilde{\psi}_i$ for $i = 1, 2$. We have

$$\widehat{I^u}(\varphi_i) = \epsilon_i\widehat{I^u}(d_{\mathrm{C}_G(u)}^1(\psi_i)) = \epsilon_i d_{\mathrm{C}_{\widetilde{G}}(u)}^1(\widehat{I^u}(\psi_i)) = \widetilde{\epsilon}_i d_{\mathrm{C}_{\widetilde{G}}(u)}^1(\widetilde{\psi}_i) = \widetilde{\varphi}_i = I^u(\varphi_i)$$

for $i = 1, 2$. This shows that $\widehat{I^u}$ extends I^u. Moreover, it is easy to see that $\widehat{I^u}$ does not depend on the generator of $\langle u \rangle$.

Altogether we have proved the theorem if D is given as in (1). By [20, Theorem 12.4] it remains to handle the case

$$D \cong \langle x, y \mid x^{2^r} = y^{2^r} = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle$$

where $r \geq 2$. Here B and $\widetilde{B}$ are Morita equivalent and therefore perfectly isometric. However, a Morita equivalence does not automatically provide an isotypy. Nevertheless, in this special case the Morita equivalence is a composition of various "natural" equivalences (namely Fong reductions, Külshammer-Puig reduction and Külshammer's reduction for blocks with normal defect groups, see [5, proof of Theorem 1]). In particular, the generalized decomposition matrices of B and $\widetilde{B}$ coincide up to signs (see [24]). Now we can use the same methods as above in order to construct an isotypy. In fact, for every B-subsection (u, b_u) one has that b_u is nilpotent or $u = [x, y]$ and b_u Morita equivalent to B (see proof of [19, Proposition 4.3]). We omit the details. □

Corollary 10. *Let B be a 2-block of a finite group G with minimal nonabelian defect group $D \not\cong D_8$. Then B is isotypic to a Brauer correspondent in $\mathrm{N}_G(\mathfrak{hyp}(B))$.*

Proof. Let b_D be a Brauer correspondent of B in $D\,\mathrm{C}_G(D)$. Since $D\,\mathrm{C}_G(D) \subseteq \mathrm{N}_G(\mathfrak{hyp}(B))$, the Brauer correspondent $b := b_D^{\mathrm{N}_G(\mathfrak{hyp}(B))}$ of B has defect group D. By Theorem 9, it suffices to show that B and b have the same fusion system. Observe that $\mathrm{N}_G(D, b_D) \subseteq \mathrm{N}_G(\mathfrak{hyp}(B))$. In particular, B and b have the same inertial quotient. If there is only the trivial fusion system on D, then we are done (this applies if D is metacyclic of order at least 16). In case $D \cong Q_8$, B is a controlled block (see e. g. [3]). Since B and b have the same inertial quotient, it follows that these blocks also have the same fusion system. It remains to consider the two other families of defect groups (see [20, Theorem 12.4]). For one of these families the fusion system is again controlled (see [20, Proposition 12.7]). Finally, if D is given as in (1), then the fusion system is constrained and the automorphisms of the essential subgroup (if it exists) also act on $\mathfrak{hyp}(B)$. Hence, B is nilpotent if and only if b is nilpotent. Again the claim follows from Theorem 9. □

We remark that Corollary 10 would be false in case $D \cong D_8$. The principal 2-block of $\mathrm{GL}(3, 2)$ gives a counterexample. If B is a block of a finite group G with defect group as given in (1), then B is also isotypic to a Brauer correspondent in $\mathrm{C}_G(u)$ where $u \in \mathrm{Z}(\mathcal{F})$. This resembles Glauberman's Z^*-Theorem.

In the situation of Theorem 9 (or Corollary 10) it is desirable to extend the isotypies to Morita equivalences (as we did in [5]). This is not always possible if $|D| = 8$, since for example the principal 2-blocks of the symmetric groups S_4 and S_5 are not Morita equivalent. Nevertheless, the possible Morita equivalence classes in case $|D| = 8$ are known by Erdmann's classification of tame algebra [6] (at least over F, cf. [9]). In view of [5] one may still ask if two non-nilpotent 2-blocks with isomorphic defect groups as in Section 2 are Morita equivalent. We will see that the answer is again negative.

Consider the groups $G_1 := A_4 \rtimes C_{2^r}$ and $G_2 := A_5 \rtimes C_{2^r}$ constructed similarly as in Lemma 2. Then $G_1/\mathrm{Z}(G_1) \cong S_4$ and $G_2/\mathrm{Z}(G_2) \cong S_5$. Let B_i be the principal 2-block of G_i, and let $\overline{B_i}$ be the principal 2-block of $G_i/\mathrm{Z}(G_i)$ for $i = 1, 2$. Then the Cartan matrix of B_i is just the Cartan matrix of $\overline{B_i}$ multiplied by $|\mathrm{Z}(G_i)| = 2^{r-1}$. It is known that the Cartan matrices of $\overline{B_1}$ and $\overline{B_2}$ do not coincide (regardless of the labeling of the simple modules). Therefore, B_1 and B_2 are not Morita equivalent.

Nevertheless, the structure of a finite group G with a minimal nonabelian Sylow 2-subgroup P as given in (1) is fairly restricted. More precisely, Glauberman's Z^*-Theorem implies $x^2 \in \mathrm{Z}^*(G)$, and the structure of $G/\mathrm{Z}^*(G)$ follows from the Gorenstein-Walter Theorem [7]. In particular, G has at most one nonabelian composition factor by Feit-Thompson.

We use the opportunity to present a related result for $p = 3$ which extends [20, Theorem 8.15].

Theorem 11. *Let B and $\widetilde{B}$ be non-nilpotent blocks of (possibly different) finite groups both with defect group $C_9 \rtimes C_3$. Then B and $\widetilde{B}$ are isotypic.*

Proof. As in the proof of Theorem 9, we will make use of [10, Theorem 2]. Let

$$D := \langle x, y \mid x^9 = y^3 = 1,\ yxy^{-1} = x^4\rangle$$

be a defect group of B, and let $\mathcal{F}$ be the fusion system of B. By Stancu [21], B is controlled with inertial index 2, and we may assume that x and x^{-1} are $\mathcal{F}$-conjugate (see proof of [20, Theorem 8.8]). Then $\mathcal{R} := \{1, x, x^3, y, y^2, xy, xy^2\}$ is a set of representatives for the $\mathcal{F}$-conjugacy classes of D (see proof of [20, Theorem 8.15]). It suffices to show that the generalized decomposition numbers of B are essentially unique (up to basic sets and signs and permutations of rows). Since the Galois group of $\mathbb{Q}(e^{2\pi i/9})$ over $\mathbb{Q}$ acts on the columns of the generalized decomposition matrix, we only need to determine the numbers $d^u_{\chi\varphi}$ for $u \in \{x, x^3, y, xy\}$. By [20, Theorem 8.15] there are four 3-rational characters $\chi_i \in \mathrm{Irr}(B)$ $(i = 1, \ldots, 4)$ such that χ_1, χ_2, χ_3 have height 0 and χ_4 has height 1. Since $\mathfrak{foc}(B) = \langle x\rangle$, we see that

$$\mathrm{Irr}(B) = \{\chi_i * \lambda : i = 1, 2, 3,\ \lambda \in \mathrm{Irr}(D/\mathfrak{foc}(B))\} \cup \{\chi_4\}.$$

Let $u := x^3$. Then $\mathrm{IBr}(b_u) = \{\varphi\}$ and $d^u_{\chi_i\varphi}$ are non-zero (rational) integers. Moreover, $d^u_{\chi_4\varphi} \equiv 0 \pmod 3$. After permuting χ_1, χ_2 and χ_3 and changing the basic set for b_u if necessary, we may assume that $d^u_{\chi_1\varphi} = 2$,

$d^u_{\chi_2\varphi} =: \epsilon_1 \in \{\pm 1\}$, $d^u_{\chi_3\varphi} =: \epsilon_2 \in \{\pm 1\}$ and $d^u_{\chi_4\varphi} = 3\epsilon_3 \in \{\pm 3\}$. Now let $u := x$. Then $d^u_{\chi_i\varphi} = \pm 1$ for $i = 1, 2, 3$ and $d^u_{\chi_4\varphi} = 0$. We may choose a basic set for b_u such that $d^u_{\chi_1\varphi} = 1$. Then by the orthogonality relations, $d^u_{\chi_2\varphi} = -\epsilon_1$ and $d^u_{\chi_3\varphi} = -\epsilon_2$. Next let $u := y$. Then b_u dominates a block of $\mathrm{C}_G(u)/\langle u\rangle$ with cyclic defect group $\mathrm{C}_D(u)/\langle u\rangle \cong C_3$ and inertial index 2. This yields $\mathrm{IBr}(b_u) = \{\varphi_1, \varphi_2\}$ and the Cartan matrix of b_u is given by

$$3\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

(not only up to basic sets, but this is not important here). We can choose a basic set such that $(d^u_{\chi_1\varphi_1}, d^u_{\chi_1\varphi_2}) = (1, 1)$, $(d^u_{\chi_2\varphi_1}, d^u_{\chi_2\varphi_2}) = (\sigma_1, 0)$, $(d^u_{\chi_3\varphi_1}, d^u_{\chi_3\varphi_2}) = (0, \sigma_2)$ and $(d^u_{\chi_4\varphi_1}, d^u_{\chi_4\varphi_2}) = (0, 0)$ for some signs $\sigma_1, \sigma_2 \in \{\pm 1\}$. Finally for $u := xy$ we obtain $d^u_{\chi_1\varphi} = 1$, $d^u_{\chi_i\varphi} = -\sigma_{i-1}$ for $i = 2, 3$ and $d^u_{\chi_4\varphi} = 0$ after changing the basic set if necessary. The following table summarizes the results

u	x^3	x	y	xy
$d^u_{\chi_1\varphi}$	2	1	$(1,1)$	1
$d^u_{\chi_2\varphi}$	ϵ_1	$-\epsilon_1$	$(\sigma_1,0)$	$-\sigma_1$
$d^u_{\chi_3\varphi}$	ϵ_2	$-\epsilon_2$	$(0,\sigma_2)$	$-\sigma_2$
$d^u_{\chi_4\varphi}$	$3\epsilon_3$	0	$(0,0)$	0

It suffices to show that $\epsilon_i = \sigma_i$ for $i = 1, 2$ (observe that we do not need the ordinary decomposition numbers in order to apply [10, Theorem 2]). For this, let $\lambda \in \mathrm{Irr}(D/\langle x^3\rangle)$ such that $\lambda(x) = e^{2\pi i/3}$ and $\lambda(y) = 1$. Then the generalized character $\psi := \lambda + \overline{\lambda} - 2 \cdot 1_D$ of D is constant on $\langle x\rangle \setminus \langle x^3\rangle$ and thus $\mathcal{F}$-stable. By [1], $\chi_1 * \psi$ is a generalized character of B and $(\chi_1 * \psi, \chi_2)_G \in \mathbb{Z}$. As in the proof of Theorem 9, we compute

$$(\chi_1 * \psi, \chi_2)_G = \sum_{u\in\mathcal{R}} \psi(u) m^u_{\chi_1\chi_2} = \psi(x) m^x_{\chi_1\chi_2} + \psi(xy) m^{xy}_{\chi_1\chi_2} + \psi(xy^2) m^{xy^2}_{\chi_1\chi_2} = \frac{1}{3}\epsilon_1 + \frac{2}{3}\sigma_1.$$

This shows $\epsilon_1 = \sigma_1$. Similarly, one gets $\epsilon_2 = \sigma_2$ by computing $(\chi_1 * \psi, \chi_3)_G$. Hence, [10, Theorem 2] gives a perfect isometry $I : \mathrm{CF}(G, B) \to \mathrm{CF}(\widetilde{G}, \widetilde{B})$. In order to show that I is also an isotypy, we make use of the notation introduced in the proof of Theorem 9. Let $u \in D$ such that b_u is nilpotent. Then by the table above, we have $\mathrm{IBr}(b_u) = \{\pm d^u_G(\chi_2)\}$. Thus, one can extend I^u just as in Theorem 9. Now suppose that b_u is non-nilpotent and thus $u = y$ (up to inversion). We choose a basic set φ_1, φ_2 for b_u as above such that $d^u_G(\chi_i) = \varphi_{i-1}$ for $i = 2, 3$. Now we have to determine the ordinary decomposition numbers of b_u with respect to φ_1, φ_2. The defect group of b_u is $\mathrm{C}_D(y) = \langle x^3, y\rangle \cong C_3 \times C_3$ and $\mathfrak{foc}(b_u) = \langle x^3\rangle$. By Kiyota [12], $k(b_u) = 9$. Therefore, there are 3-rational characters $\psi_i \in \mathrm{Irr}(b_u)$ such that

$$\mathrm{Irr}(b_u) = \{\psi_i * \lambda : i = 1, 2, 3,\ \lambda \in \mathrm{Irr}(\langle x^3, y\rangle/\langle x^3\rangle)\}.$$

By the Cartan matrix of b_u given above (with respect to φ_1, φ_2), it follows immediately that $d^1_{\mathrm{C}_G(u)}(\psi_i) = \epsilon_i\varphi_i$ with $\epsilon_i \in \{\pm 1\}$ for $i = 1, 2$ after a suitable permutation of ψ_1, ψ_2, ψ_3. Similarly, $d^1_{\mathrm{C}_{\widetilde{G}}(u)}(\widetilde{\psi}_i) = \widetilde{\epsilon}_i\widetilde{\varphi}_i$. By a result of Usami [22], there is a perfect isometry $\mathrm{CF}(\mathrm{C}_G(u), b_u) \to \mathrm{CF}(\mathrm{C}_{\widetilde{G}}(u), \widetilde{b}_u)$. However, we need the additional information that ψ_i is mapped to $\pm\widetilde{\psi}_i$. In order to show this, we use [10, Theorem 2] again. Observe that $d^u_{\mathrm{C}_G(u)}(\psi_i) = \zeta_i d^1_{\mathrm{C}_G(u)}(\psi_i) = \zeta_i\epsilon_i\varphi_i$ for a cube root of unity ζ_i. But since $d^u_{\psi_i\varphi_i}$ is rational, we have $\zeta_i = 1$. Now an elementary application of the orthogonality relations shows that the generalized decomposition matrix of b_u (in $\mathrm{C}_G(u)$) is determined by

v	1	y	x^3	x^3y
$d^v_{\psi_1\varphi}$	$(\epsilon_1,0)$	$(\epsilon_1,0)$	ϵ_1	ϵ_1
$d^v_{\psi_2\varphi}$	$(0,\epsilon_2)$	$(0,\epsilon_2)$	ϵ_2	ϵ_2
$d^v_{\psi_3\varphi}$	(ϵ_3,ϵ_3)	(ϵ_3,ϵ_3)	$-\epsilon_3$	$-\epsilon_3$

It follows that there is a perfect isometry $\widehat{I^u} : \mathrm{CF}(\mathrm{C}_G(u), b_u) \to \mathrm{CF}(\mathrm{C}_{\widetilde{G}}(u), \widetilde{b}_u)$ such that $\widehat{I^u}(\psi_i) = \epsilon_i\widetilde{\epsilon}_i\widetilde{\psi}_i$ for $i = 1, 2$. Therefore $\widehat{I^u}$ extends I^u. As in the proof of Theorem 9, it is also clear that $\widehat{I^u}$ is independent of the choice of the generator of $\langle u\rangle$. This finishes the proof. $\square$

The proof method of Theorem 11 also works for other defect groups. In fact, Watanabe [23] showed independently (using more complicated methods) that two p-blocks ($p > 2$) with a common metacyclic, minimal nonabelian defect group and the same fusion system are perfectly isometric. Again, this gives evidence for the character-theoretic version of Rouquier's Conjecture (see [25, Theorem 2]). As another remark, Holloway-Koshitani-Kunugi [8, Example 4.3] constructed a perfect isometry between the principal 3-block of $G := \mathrm{Aut}(\mathrm{SL}(2,8)) \cong {}^2G_2(3)$ and its Brauer correspondent. Since G has a Sylow 3-subgroup isomorphic to $C_9 \rtimes C_3$, this is a special case of Theorem 11. Note that in the introduction of Ruengrot [18] it is erroneously stated that these blocks are *not* perfectly isometric.

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