FINITE DIMENSIONAL ALGEBRAS NOT ARISING AS BLOCKS OF GROUP ALGEBRAS

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Abstract. We develop new techniques to classify basic algebras of blocks of finite groups over algebraically closed fields of prime characteristic. We apply these techniques to simplify and extend previous classifications by Linckelmann, Murphy and Sambale. In particular, we fully classify blocks with 16-dimensional basic algebra.

1. Introduction

Linckelmann [15] instigated the study of small dimensional symmetric basic algebras over an algebraically closed field of prime characteristic, in the context of enumerating which of them could be the basic algebra of a block of a finite group. In that paper, he gave a complete classification up to dimension twelve, except for one case of an algebra of dimension nine; see Section 2.9 of that paper. The paper of Linckelmann and Murphy [16] eliminated that 9-dimensional algebra using some fairly sophisticated group representation theory. We provide a proof of this elimination that is completely different from the one in that paper, by examining the Auslander–Reiten quiver.

The second author [26] took Linckelmann's methods further, up to dimension fourteen. We provide alternative proofs for some of the difficult cases that occurred in that paper. By a recent paper of Macgregor [17], it became clear that the classification of tame basic algebras in dimension 14 might be incomplete. Hence, the list given in [26] might be incomplete as well. We comment on the details in Section 6 below.

By way of preparation, we prove some theorems that dispose of a number of possible Cartan matrices, which occur for block algebras. The most interesting of these is the following theorem, whose proof can be found in Section 5.

Theorem 1.1. *Suppose that A is a finite dimensional indecomposable symmetric algebra over an algebraically closed field. Suppose that A has a simple module S whose Cartan invariant $c_{S,S}$ is equal to 3 and all the other $c_{S,T}$ are either one or zero. Then A is not Morita equivalent to a block of wild representation type of a finite group algebra in prime characteristic.*

It has been observed in [26] that in dimension 15 there might be a 13-block of defect 1, which is not known to exist. Leaving this open case aside, we show that there is only one more block in this dimension.

Theorem 1.2. *Let B be a block of a finite group with defect group D and basic algebra A of dimension 15. Then one of the following holds:*

(1) $D \cong C_{19}$ and A is Morita equivalent to the principal 19-block of $\mathrm{GL}(3,7)$.

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(2) $D \cong C_{13}$ and A is a Brauer tree algebra with Cartan matrix

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Finally, we extend the classification of basic algebras to dimension 16 as follows.

Theorem 1.3. *Let B be a block of a finite group with defect group D and basic algebra A of dimension 16. Then one of the following holds:*

(1) *$|D| = 16$ and A is isomorphic to the group algebra of D.*

(2) *$D \cong C_2^4$ and A is Morita equivalent to a non-principal block of $H \cong D \rtimes 3_+^{1+2}$ with $H/Z(H) \cong A_4^2$.*

(3) *$D \cong C_{23}$ and A is Morita equivalent to the principal 23-block of $\mathrm{PSL}(2, 137)$.*

(4) *$D \cong C_5$ and A is Morita equivalent to the principal 5-block of S_5 or $\mathrm{Sz}(8)$.*

(5) *$D \cong C_{13}$ and A is Morita equivalent to the principal 13-block of $\mathrm{GL}(4, 5)$.*

(6) *$D \cong D_8$ and A is Morita equivalent to the principal 2-block of $\mathrm{GL}(3, 2)$.*

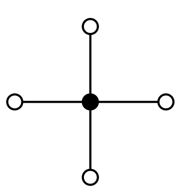
In total there are 20 Morita equivalence classes.

2. Preliminaries

Throughout, we work with a finite group G over an algebraically closed field k of characteristic p dividing $|G|$. We fix a block B of kG with defect group D. Then the basic algebra A of B is a finite dimensional symmetric algebra. Recall that A and B have isomorphic centres. The dimension of this centre coincides with the number $k(B)$ of irreducible characters in B. The number of simple modules of B (and A) is denoted by $l(B)$. The dimension of A itself is the sum of the entries of the Cartan matrix C of B. The determinant of C is a power of p, which severely restricts the possibilities. The largest elementary divisor of C is $|D|$ and it occurs with multiplicity one. If $\det(C) = p$, we conclude that D is cyclic of order p. In this case, A is a Brauer tree algebra and of finite representation type. This further limits the possibilities for C. We extend Proposition 2 in [26] as follows (the proof is the same).

Proposition 2.1. *Let B be a block with cyclic defect group D, $l(B) = 4$ and multiplicity $m := \frac{|D|-1}{4}$. Then the possible Brauer trees are given as follows:*

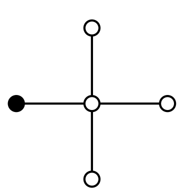
(1)



$$C = \begin{pmatrix} m+1 & m & m & m \\ m & m+1 & m & m \\ m & m & m+1 & m \\ m & m & m & m+1 \end{pmatrix} \qquad \dim A = 16m + 4 = 4|D|$$

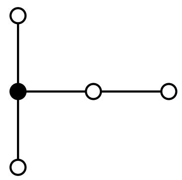
This occurs for $B = k[D \rtimes C_4]$ provided $4 \mid p - 1$.

(2)



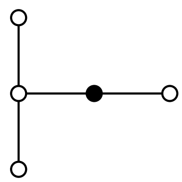
$$C = \begin{pmatrix} m+1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \qquad \dim A = m + 19 = \frac{|D| + 75}{4}$$

(3)



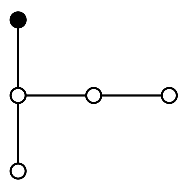
$$C=\begin{pmatrix} m+1 & m & m & . \\ m & m+1 & m & . \\ m & m & m+1 & 1 \\ . & . & 1 & 2 \end{pmatrix} \qquad \dim A = 9m+7 = \frac{9|D|+19}{4}$$

(4)



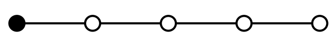
$$C=\begin{pmatrix} m+1 & m & 1 & . \\ m & m+1 & 1 & . \\ 1 & 1 & 2 & 1 \\ . & . & 1 & 2 \end{pmatrix} \qquad \dim A = 4m+16 = |D|+15$$

(5)



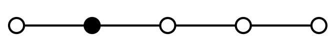
$$C=\begin{pmatrix} m+1 & 1 & 1 & . \\ 1 & 2 & 1 & . \\ 1 & 1 & 2 & 1 \\ . & . & 1 & 2 \end{pmatrix} \qquad \dim A = m+15 = \frac{|D|+59}{4}$$

(6)



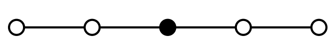
$$C=\begin{pmatrix} m+1 & 1 & . & . \\ 1 & 2 & 1 & . \\ . & 1 & 2 & 1 \\ . & . & 1 & 2 \end{pmatrix} \qquad \dim A = m+13 = \frac{|D|+51}{4}.$$

(7)



$$C=\begin{pmatrix} m+1 & m & . & . \\ m & m+1 & 1 & . \\ . & 1 & 2 & 1 \\ . & . & 1 & 2 \end{pmatrix} \qquad \dim A = 4m+10 = |D|+9.$$

(8)



$$C=\begin{pmatrix} 2 & 1 & . & . \\ 1 & m+1 & m & . \\ . & m & m+1 & 1 \\ . & . & 1 & 2 \end{pmatrix} \qquad \dim A = 4m+10 = |D|+9.$$

Sometimes a Cartan matrix leads to a Brauer graph algebra, which is the same as a symmetric special biserial algebra. They all have finite or tame representation type by Wald–Waschbüsch [27].

Blocks of finite group algebras with tame (but not finite) representation type only occur in characteristic two, and the defect groups in this case are dihedral, semidihedral or generalised quaternion. These algebras were first investigated by Erdmann [10]. By a recent paper of Macgregor [17], all Cartan matrices of tame blocks are known and we list them for the convenience of the reader (this includes the degenerate case $D \cong C_2^2$, although it is not listed in [17]):

Theorem 2.2. *Let B be a non-nilpotent tame block with defect group D of order 2^n and Cartan matrix C. Then one of the following holds:*

(1) *$D \cong D_{2^n}$ and C is one of the following:*

$$\begin{pmatrix} 2^n & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 \\ 2 & 2^{n-2}+1 \end{pmatrix}, \quad \begin{pmatrix} 2^n & 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+1 & 2^{n-2} \\ 2^{n-1} & 2^{n-2} & 2^{n-2}+1 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2^{n-2}+1 & 2^{n-2} \\ 1 & 2^{n-2} & 2^{n-2}+1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2^{n-2}+1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

(2) *$D \cong Q_{2^n}$ and C is one of the following:*

$$2\begin{pmatrix} 2^{n-1} & 2^{n-2} \\ 2^{n-2} & 2^{n-3}+1 \end{pmatrix}, \quad 2\begin{pmatrix} 4 & 2 \\ 2 & 2^{n-3}+1 \end{pmatrix}, \quad 2\begin{pmatrix} 2^{n-1} & 2^{n-2} & 2^{n-2} \\ 2^{n-2} & 2^{n-3}+1 & 2^{n-3} \\ 2^{n-2} & 2^{n-3} & 2^{n-3}+1 \end{pmatrix},$$
$$2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2^{n-3}+1 & 2^{n-3} \\ 1 & 2^{n-3} & 2^{n-3}+1 \end{pmatrix}, \quad 2\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2^{n-3}+1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

(3) *$D \cong SD_{2^n}$ and C is one of the following:*

$$2\begin{pmatrix} 2^{n-1} & 2^{n-2} \\ 2^{n-2} & 2^{n-3}+1 \end{pmatrix}, \quad 2\begin{pmatrix} 4 & 2 \\ 2 & 2^{n-3}+1 \end{pmatrix}, \quad \begin{pmatrix} 2^n & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix},$$
$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2^{n-2}+1 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2^n & 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+1 & 2^{n-2} \\ 2^{n-1} & 2^{n-2} & 2^{n-2}+2 \end{pmatrix}, \quad \begin{pmatrix} 2^{n-2}+1 & 2^{n-2}-1 & 2^{n-2} \\ 2^{n-2}-1 & 2^{n-2}+1 & 2^{n-2} \\ 2^{n-2} & 2^{n-2} & 2^{n-2}+2 \end{pmatrix},$$
$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 6 \end{pmatrix}.$$

We stress that in the situation of Theorem 2.2, the Cartan matrix does not necessarily determine the Morita equivalence class of the block, although no concrete example of this phenomenon is known (see [17, first two cases in Theorem 2.2]). We apply the previous theorem to list all Cartan matrices of 2-blocks with defect at most three.

Proposition 2.3. *Let B be a non-nilpotent 2-block with defect group D and Cartan matrix C. If $|D| \leqslant 8$, then one of the following holds:*

(1) *B is tame and C is one of the following matrices:*

$$\begin{pmatrix} 8 & 4 \\ 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 8 & 4 & 4 \\ 4 & 3 & 2 \\ 4 & 2 & 3 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \quad 2\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad 2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(2) *$D \cong C_2^3$ and C is one of the following matrices:*

$$2\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix},\quad 2\begin{pmatrix}4&2&2\\2&2&1\\2&1&2\end{pmatrix},\quad \begin{pmatrix}8&6&2&2&2\\6&8&2&2&2\\2&2&4&.&.\\2&2&.&4&.\\2&2&.&.&4\end{pmatrix},\quad \begin{pmatrix}8&4&4&4&4\\4&4&3&3&1\\4&3&4&2&2\\4&3&2&4&2\\4&1&2&2&4\end{pmatrix},$$

$$\begin{pmatrix}8&4&4&4&3\\4&4&2&2&2\\4&2&4&2&2\\4&2&2&4&2\\3&2&2&2&2\end{pmatrix},\quad \begin{pmatrix}4&2&2&2&2&2&2\\2&4&2&2&2&2&2\\2&2&4&2&2&2&2\\2&2&2&4&2&2&2\\2&2&2&2&4&2&2\\2&2&2&2&2&4&2\\2&2&2&2&2&2&4\end{pmatrix},\quad \begin{pmatrix}8&4&4&4&2&2&2\\4&4&2&2&.&2&1\\4&2&4&2&1&.&2\\4&2&2&4&2&1&.\\2&.&1&2&2&.&.\\2&2&.&1&.&2&.\\2&1&2&.&.&.&2\end{pmatrix}.$$

Proof. If $D \cong \{1, C_2, C_4, C_8, C_4 \times C_2\}$, then B is nilpotent since $\mathrm{Aut}(D)$ is a 2-group. If $D \cong \{C_2^2, D_8, Q_8\}$, then B is tame and the claim follows from Theorem 2.2 (note that $l(B) = 2$ is only possible for $D \cong D_8$). Finally, if $D \cong C_2^3$, then the claim follows from Eaton [6]. □

3. Auslander–Reiten theory

In this section, we collect some well-known results from Auslander–Reiten theory for blocks of finite groups, that we shall use in this paper.

Theorem 3.1. *The tree class of every component of the Auslander–Reiten quiver of B is either the Dynkin diagram A_n, in which case B has cyclic defect, or a Euclidean diagram, or one of three infinite trees, A_∞, D_∞ or A_∞^∞.*

Proof. This is Theorem A of Webb [28]. Because the field k is algebraically closed, the infinite trees B_∞ and C_∞ do not occur. □

Theorem 3.2. *If the tree class of a component of the Auslander–Reiten quiver of B is a Euclidean diagram, then B has Klein four defect group, and the tree class is $\tilde{A}_{1,2}$ or $\tilde{A}_5$.*

Proof. This is Theorem 1.1 of Bessenrodt [2]. Since k is algebraically closed, $\tilde{B}_3$ does not occur. □

Theorem 3.3. *Every Auslander–Reiten component of a block B of wild representation type has tree class A_∞. If B has an Auslander–Reiten component of tree class D_∞, then B is a tame block with semidihedral defect groups.*

Proof. This is Theorem 1 of Erdmann [11]. □

Theorem 3.4. *If P is a projective indecomposable module in the block B, then the radical modulo the socle, $\mathsf{Rad}(P)/\mathsf{Soc}(P)$ has at most two direct summands.*

Proof. This follows from the fact that there is an almost split sequence of the form

$$0 \to \mathsf{Rad}(P) \to P \oplus \mathsf{Rad}(P)/\mathsf{Soc}(P) \to P/\mathsf{Soc}(P) \to 0.$$

If $\mathsf{Rad}(P)/\mathsf{Soc}(P)$ has more than two direct summands, then by Theorems 3.1 and 3.2, the only way for this to be part of an Auslander–Reiten component is for the tree class to be D_∞. By Theorem 3.3, this implies that B has semidihedral defect groups. Examining Erdmann [8, 9], there are no examples with semidihedral defect groups where $\mathsf{Rad}(P)/\mathsf{Soc}(P)$ has more than two summands. ☐

Remark 3.5. Without assuming that the field k is algebraically closed, there are examples where $\mathsf{Rad}(P)/\mathsf{Soc}(P)$ has three summands. But this only happens when the defect group is a Klein four group, k does not have a primitive cube root of unity, and B is Morita equivalent to the principal block of A_4 or A_5 (see Bessenrodt [2]).

4. Small symmetric local algebras

We shall need the following facts about small symmetric local algebras.

Proposition 4.1. *If A be a symmetric local k-algebra with $\dim_k A \leqslant 7$, then*

(1) *A is commutative.*
(2) *If $\mathsf{Rad}(A)/\mathsf{Soc}(A)$ is indecomposable, then one of the following is true.*
 (a) *$A \cong k[x]/(x^n)$ for some $n \leqslant 7$,*
 (b) *$A \cong k[x,y]/(x^3,y^2)$, of dimension 6,*
 (c) *k has characteristic two and $A \cong k[x,y]/(x^3, y^2+x^2y)$ of dimension 6, or*
 (d) *k has characteristic three and $A \cong k[x,y]/(x^3+x^2y, y^2)$ of dimension 6.*

Proof.

(1) Let Z be the centre of A. Külshammer [14] proved that if $\dim_k Z \leqslant 4$, then A is commutative. This was extended by Chlebowitz and Külshammer [5], where it is proved that if $\dim_k Z = 5$, then A has dimension 5 or 8. It cannot happen that $\dim_k Z = 6$ and $\dim_k A = 7$, so this proves that A is commutative.
(2) Let X denote $\mathsf{Rad}(A)/\mathsf{Soc}(A)$. Poonen [20] lists the commutative local algebras of dimension up to six. Among these, the ones that are Gorenstein with X indecomposable are those listed above (note that the algebras listed in cases (c) and (d) are isomorphic to $k[x,y]/(x^3,y^2)$ in other characteristics). It remains to deal with dimension seven. If the radical layers of A have dimensions $[1,1,1,1,1,1,1]$ we are in case (a). If they have dimensions $[1,2,1,1,1,1]$, then $\mathsf{Soc}(X)$ has dimension two while $\mathsf{Rad}^3(X)$ has dimension one. An element of $\mathsf{Soc}(X)$ that is not in $\mathsf{Rad}^3(X)$ spans a 1-dimensional summand of X, so X decomposes. Similarly, in the case $[1,3,1,1,1]$, an element of $\mathsf{Soc}(X)$ that is not in $\mathsf{Rad}^2(X)$ spans a 1-dimensional summand. In the cases $[1,4,1,1]$ and $[1,3,2,1]$, an element of $\mathsf{Soc}(X)$ that is not in $\mathsf{Rad}(X)$ spans a 1-dimensional summand. In the case $[1,5,1]$, X is semisimple, and decomposes as a direct sum of five 1-dimensional summands. In the remaining case $[1,2,2,1,1]$, $A/\mathsf{Soc}(A)$ is a 6-dimensional algebra with radical layers $[1,2,2,1]$ and socle layers $[1,1,2,2]$. Again examining Poonen's list [20], the possibilities for $A/\mathsf{Soc}(A)$ are $k[x,y]/(x^2,xy^2,y^4)$ and $k[x,y]/(x^2+y^3,xy^2,y^4)$. In both these cases, the quotient $\mathsf{Rad}(A)/(\mathsf{Soc}(A),xA)$ of X is uniserial of length three spanned by the powers of y, but X has no uniserial submodule of length three. This contradicts the fact that X is supposed to be self-dual, since A is Gorenstein. ☐

5. Cartan invariants

In this section, we prove some theorems about Cartan invariants of blocks of group algebras.

Proof of Theorem 1.1. Suppose that A is as in the theorem, and that A has wild representation type. We examine the structure of the projective cover P_S of S. It follows from Theorem 3.4 that $\mathsf{Rad}(P_S)/\mathsf{Soc}(P_S)$ has at most two direct summands. Since S occurs with multiplicity three in P_S, there has to be a nilpotent endomorphism of P_S whose image lies in the radical but not in the socle. Since each other composition factor occurs with multiplicity one, they must all be in the kernel of such an endomorphism. It follows that $\mathsf{Ext}^1_A(S,S)$ is 1-dimensional, and so $\mathsf{Rad}(P_S)/\mathsf{Soc}(P_S)$ has a direct summand isomorphic to S. Write $\mathsf{Rad}(P_S)/\mathsf{Soc}(P_S) = S \oplus X$. If $X = 0$, then A has finite representation type, so we have $X \neq 0$. Thus the component of the Auslander–Reiten quiver containing S has the following shape.

$$\begin{tikzcd}
 & & & \cdots & & & \\
 & & \Omega^{-1}X \arrow[dr] & & \Omega X \arrow[dr] & & \\
 & \Omega^2 S \arrow[ur] \arrow[dr] & & S \arrow[ur] \arrow[dr] & & \Omega^{-2}S & \\
\cdots & & \Omega S \arrow[ur] \arrow[dr] \arrow[r, dotted] & P_S \arrow[r, dotted] & \Omega^{-1}S \arrow[ur] \arrow[dr] & & \cdots \\
 & \Omega^2 X \arrow[ur] & & X \arrow[ur] & & \Omega^{-2}X & \\
 & & & \cdots & & &
\end{tikzcd}$$

The automorphism Ω sends S to ΩS and therefore acts as an automorphism of the stable part of this Auslander–Reiten component. It is a glide reflection with a horizontal axis, and its square is the translation. It is easy to check that A_∞ does not have an automorphism fitting this description, so this component does not have type A_∞. It now follows from Theorem 3.3 that A is not Morita equivalent to a block of a finite group algebra in prime characteristic. □

Theorem 5.1. *No block of wild representation type of a finite group has Cartan matrix*

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$

with $2 \leqslant a, b \leqslant 7$.

Proof. Let the simple modules be S and T. Then S and T have to extend each other, and so the structures of their projective covers are

$$\begin{tikzcd}[column sep=small]
 & S \arrow[dl, dash] \arrow[dr, dash] & & & & T \arrow[dl, dash] \arrow[dr, dash] & \\
\hat{S} \arrow[dr, dash] & & T \arrow[dl, dash] & & S \arrow[dr, dash] & & \hat{T}, \arrow[dl, dash] \\
 & S & & & & T &
\end{tikzcd}$$

where $\hat{S}$ and $\hat{T}$ are modules with $a-2$, respectively $b-2$ composition factors, all isomorphic to S, respectively T. By Theorem 3.4, $\hat{S}$ and $\hat{T}$ are either zero or indecomposable. The algebras $\mathsf{End}_A(P_S)$ and $\mathsf{End}_A(P_T)$ are symmetric local algebras of dimension at most seven. So by Proposition 4.1, they are commutative, and either uniserial or 6-dimensional. If both are uniserial, then A is a Brauer tree algebra, and therefore either of finite representation type or tame biserial. On the other hand, if either a or b is equal to six, then the determinant is either prime or 35. In the former case the block has cyclic defect, while in the latter case the determinant is not a prime power, so there is no block of this form. $\square$

Theorem 5.2. *There is no block of a finite group with Cartan matrix*

$$\begin{pmatrix} a & 1 & 1 \\ 1 & b & . \\ 1 & . & c \end{pmatrix}$$

with $a \geqslant 3$, $b \geqslant 2$, and $c \geqslant 2$.

Proof. The structure of the projectives has to be

$$\begin{array}{ccc} & S & \\ \diagup & | & \diagdown \\ T & U & \hat{S} \\ \diagdown & | & \diagup \\ & S & \end{array} \qquad \begin{array}{ccc} & T & \\ \diagup & & \diagdown \\ S & & \hat{T} \\ \diagdown & & \diagup \\ & T & \end{array} \qquad \begin{array}{ccc} & U & \\ \diagup & & \diagdown \\ S & & \hat{U} \\ \diagdown & & \diagup \\ & U & \end{array}$$

where $\hat{S}$ has $a-2$ composition factors, all isomorphic to S, $\hat{T}$ has $b-2$ composition factors, all isomorphic to T, and $\hat{U}$ has $c-2$ composition factors, all isomorphic to U. Note that $\hat{T}$ and $\hat{U}$ are allowed to be zero. But since $a \geqslant 3$, $\hat{S}$ is not zero, and $\mathsf{Rad}(P_S)/\mathsf{Soc}(P_S)$ is forced to have at least three direct summands, contradicting Theorem 3.4. $\square$

The following theorem was heavily used in [26].

Theorem 5.3. *Let Q be the decomposition matrix of B. Then $C = Q^{\mathrm{t}}Q$ is the Cartan matrix of B. Let $M := |D|QC^{-1}Q^{\mathrm{t}}$. Then M is an integer matrix. The number $k_0(B)$ of irreducible height zero characters of B coincides with the number of diagonal entries of M, which are coprime to p. In particular, D is abelian if and only if all diagonal entries of M are coprime p.*

Proof. The equation $C = Q^{\mathrm{t}}Q$ is well-known. The second claim follows from Lemma 4.1 of [25]. The last claim is a consequence of the recent solution of Brauer's height zero conjecture [18]. $\square$

6. Small dimensional basic algebras

Dimensions one to seven caused no problems in Linckelmann's analysis [15].

Dimension eight. For the Cartan matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

of determinant 8, Linckelmann resorts to knowledge of blocks with defect groups of order 8. This case can be eliminated more directly using Theorem 1.1.

Dimension nine. In dimension 9, the Cartan matrix that causes difficulty is

$$\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}.$$

of determinant 9. This case was not resolved in Linckelmann's paper, but Theorem 5.1 of Linckelmann and Murphy [16] shows that this is only possible for a block with cyclic defect. There, it was proved using some fairly deep results from block theory. We eliminate it directly as a special case of Theorem 5.1.

Dimension ten. This did not cause any trouble in Linckelmann's analysis.

Dimension eleven. The Cartan matrix

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & . \\ 1 & . & 2 \end{pmatrix}$$

was eliminated by Linckelmann using Okuyama's analysis of blocks of Loewy length three. We can instead apply Theorem 1.1 to eliminate this case.

Dimension twelve. Again, this did not cause any trouble in Linckelmann's analysis.

Dimension thirteen. In dimension 13, there are two Cartan matrices we wish to comment on. The first case we consider is

$$\begin{pmatrix} 7 & 1 \\ 1 & 4 \end{pmatrix}.$$

The determinant is 27, so we are in characteristic three. Theorem 5.1 shows that this cannot happen for a block of wild representation type. But if this is the Cartan matrix of a Brauer tree algebra, then there are two exceptional vertices, so the algebra is tame biserial. This cannot happen in odd characteristic, so this is ruled out.

The second case we consider is the Cartan matrix

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 2 & . \\ 1 & . & 2 \end{pmatrix},$$

of determinant 16. By Theorem 5.2, no block of a finite group can have this Cartan matrix.

Dimension fourteen. In Proposition 3 of [26], the first author stated that the Cartan matrix $\left(\begin{smallmatrix} 5 & 2 \\ 2 & 4 \end{smallmatrix}\right)$ belongs to two tame blocks of $\mathrm{PGL}(2,7)$ or $3.M_{10}$. However, Case $(*)$ in Theorem 2.3 of Macgregor [17] states that there might be other Morita equivalences of tame blocks with this Cartan matrix. Hence, this case remains open.

Another Cartan matrix of interest to us is

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & . \\ 1 & . & 2 \end{pmatrix}.$$

This has determinant 25, so we are in characteristic five. Again this violates Theorem 5.2, so no block of a finite group can have this Cartan matrix.

Dimension fifteen. We begin by formulating an easy lemma that we shall use here, and again in the case of dimension sixteen.

Lemma 6.1. *The dimension d of the basic algebra of a block B is at least $4l(B)-2$, so $l(B) \leqslant (d+2)/4$.*

Proof. Since the Cartan matrix C is positive definite and indecomposable, its trace is at least $2l(B)$. On the other hand, there must be at least $2l(B)-2$ positive entries off the diagonal, so the sum of the entries of C is at least $4l(B)-2$. $\square$

Proof of Theorem 1.2. Since 15 is not a prime power, we have $l(B) > 1$. So using Lemma 6.1, we have $2 \leqslant l(B) \leqslant 4$. By [26, Proposition 2 and its proof] and Proposition 2.1, we find the two stated blocks of defect 1. We may now assume that $\det(C)$ is not a prime. For $l(B) = 2$ there is only one potential Cartan matrix left:

$$C = \begin{pmatrix} 6 & 3 \\ 3 & 3 \end{pmatrix}.$$

But C has elementary divisors 3, 3 and therefore cannot arise from a block. Thus, let $l(B) = 3$. Since there are at least four positive entries off the diagonal, the trace of C is bounded by 12. An individual entry on the diagonal can therefore be at most 8. The entries off the diagonal are bounded by 2 since otherwise one gets a non-positive minor. This makes it easy to enumerate all feasible Cartan matrices. Afterwards we remove those which differ only by permuting the simple modules. This leaves only the matrices

$$\begin{pmatrix} 4 & . & 2 \\ . & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 6 & . & 1 \\ . & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

with determinant 8 and 27 respectively. This first case is excluded by Proposition 2.3. In the second case, Theorem 1.1 implies that D is cyclic. But then $l(B) \leqslant p-1 = 2$.

Finally, if $l(B) = 4$, then C is one of the following matrices:

$$\begin{pmatrix} 2 & . & 1 & . \\ . & 2 & 1 & . \\ 1 & 1 & 2 & 1 \\ . & . & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & . & 1 & . \\ . & 2 & 1 & 1 \\ 1 & 1 & 2 & . \\ . & 1 & . & 3 \end{pmatrix}.$$

In the first case, $|D| = 8$ and this cannot happen again by Proposition 2.3. In the second case $|D| = 9$ and D must be cyclic by Theorem 1.1. But then $l(B) \leqslant 2$, a contradiction. $\square$

Dimension sixteen. This is postponed to Section 7 below.

Dimension seventeen. A case of interest in dimension 17 is the Cartan matrix

$$\begin{pmatrix} 6 & . & 1 \\ . & 5 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The determinant is 49, so we are in characteristic seven. Applying Proposition 4.1, we see that the heart of each projective indecomposable has two summands and these are all uniserial, of length one, three, or four. The algebra is therefore tame biserial, and since the characteristic is not two, this therefore cannot be a block algebra.

There are also some Cartan matrices with four simples, that need to be considered in dimension 17, e. g.

$$\begin{pmatrix} 2 & . & 1 & . \\ . & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ . & 1 & 1 & 3 \end{pmatrix}.$$

They can all be ruled out easily with Theorem 1.1.

On the other hand, the principal 2-block of $\mathrm{PGL}(2, 47)$ has defect group D_{32} and Cartan matrix

$$\begin{pmatrix} 9 & 2 \\ 2 & 4 \end{pmatrix}.$$

Using Theorem 5.3 and [17], one can show that this is the only Morita equivalence class for this Cartan matrix. The principal 5-block of $\mathrm{PSL}(2, 13)$ has defect group C_{25} and Cartan matrix

$$\begin{pmatrix} 13 & 1 \\ 1 & 2 \end{pmatrix},$$

but here there might be other blocks with non-cyclic defect group and the same Cartan matrix. Similarly, the Cartan matrix

$$\begin{pmatrix} 9 & 3 \\ 3 & 2 \end{pmatrix}$$

occurs (at least) for a non-principal block of $2.S_6$ with defect group C_3^2. This is the first non-local wild block that we have encountered.

7. Basic algebras of dimension 16

The proof of Theorem 1.3 requires the following lemma about a specific 2-block of defect four.

Lemma 7.1. *Let B be a block with defect group $D \cong D_8 \times C_2$ and $l(B) = 2$. Then B is perfectly isometric to the principal block of $S_4 \times C_2$.*

Proof. The blocks with defect group $D_8 \times C_2$ were investigated in Section 9.1 of [22]. Let

$$D = \langle x, y \mid x^4 = y^2 = 1,\ yxy^{-1} = x^{-1} \rangle \times \langle z \mid z^2 = 1 \rangle \cong D_8 \times C_2.$$

Let $\mathcal{F}$ be the fusion system of B on D. Since $l(B) = 2$, we are in case (ab) or (ba) of Lemma 9.3 and Theorem 9.7 in [22]. Replacing y by xy if necessary, we may assume case (ab), i. e. $E := \langle x^2, xy, z \rangle$ is the only $\mathcal{F}$-essential subgroup and $\mathrm{Aut}_{\mathcal{F}}(E) \cong S_3$. Replacing z by $x^2 z$ if necessary, we may assume that $Z(\mathcal{F}) = \langle z \rangle$ is the centre of $\mathcal{F}$ and $\mathfrak{foc}(B) = \langle x^2, xy \rangle$ is the focal subgroup of B. Moreover, Theorem 9.7 in [22] shows that

$$k(B) = k_0(B) + k_1(B) = 8 + 2 = 10$$

where $k_i(B)$ denotes the number of irreducible characters of height i in B.

Now observe that the principal block B_0 of $S_4 \times C_2$ has the same defect group and the same fusion system as B. By Theorem 6.1 of [25], it suffices to show that B and B_0 have the same generalised decomposition matrix up to signs and basic sets. By Lemma 9.5 in [22],

$$\mathcal{R} := \{1, x, x^2, y, z, xz, x^2 z, yz\}$$

is a set of representatives for the $\mathcal{F}$-conjugacy classes of D. We fix B-subsections (u, b_u) for $u \in \mathcal{R}$ such that b_u has defect group $C_D(u)$. We now make use of the Broué–Puig [4] $*$-construction. By a theorem of Robinson [21], there exist $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}(B)$ such that

$$\begin{aligned}\mathrm{Irr}_0(B) &= \{\lambda * \chi_i : \lambda \in \mathrm{Irr}(D/\mathfrak{foc}(B)), i = 1, 2\},\\ \mathrm{Irr}_1(B) &= \{\lambda * \chi_3 : \lambda \in \mathrm{Irr}(Z(\mathcal{F}))\}\end{aligned}$$

where $\mathrm{Irr}_i(B)$ is the set of irreducible characters of height i of B. By Lemma 10 in [24], the generalised decomposition numbers fulfil $d^u_{\lambda*\chi,\varphi} = \lambda(u) d^u_{\chi,\varphi}$ for $u \in \mathcal{R}$ and $\varphi \in \mathrm{IBr}(b_u)$. Hence, it suffices to determine $d^u_{\chi_i,\varphi}$ for $i = 1, 2, 3$. Since D is a rational group, these numbers are integers. For $i = 1, 2$ we have $d^u_{\chi_i,\varphi} \neq 0$ by Proposition 1.36 of [22]. Let $u \in \mathcal{R} \setminus Z(\mathcal{F})$. Then b_u is nilpotent and $l(b_u) = 1$. By the orthogonality relations of generalised decomposition numbers (see Theorem 1.14 in [22]), we have $d^u_{\chi_i,\varphi} = \pm 1$ for $i = 1, 2$. We may choose basic sets such that $d^u_{\chi_1,\varphi} = 1$ for all $u \in \mathcal{R} \setminus Z(\mathcal{F})$. If $u \in \{x, y, xz, yz\}$, then b_u has defect 3 and $d^u_{\chi_3,\varphi} = 0$. We may choose χ_2 such that $d^x_{\chi_2,\varphi} = 1 = d^{xz}_{\chi_2,\varphi}$. The orthogonality between x, y and xz, yz shows that $d^y_{\chi_2,\varphi} = -1 = d^{yz}_{\chi_2,\varphi}$.

It remains to consider $u \in \{x^2, z, x^2 z\}$. Replacing χ_3 by $-\chi_3$ if necessary, we may assume that $d^{x^2}_{\chi_3,\varphi} = 2$. Recall that b_z dominates a unique block $\overline{b_z}$ of $C_G(z)/\langle z\rangle$ with defect group $D/\langle z\rangle \cong D_8$. The Cartan matrix of $\overline{b_z}$ is $\overline{C_z} := \left(\begin{smallmatrix}3&1\\1&3\end{smallmatrix}\right)$ up to basic sets by Proposition 2.3. Hence, the Cartan matrix of b_z is $2\overline{C_z}$ up to basic sets. We may choose a basic set and $\alpha = \pm 1$ such that $d^z_{\chi_1,.} = (1, 0)$, $d^z_{\chi_2,0} = (0, \alpha)$ and $d^z_{\chi_3,.} = (1, 1)$ (interchanging χ_3 and $\lambda * \chi_3$ if necessary). The orthogonality between z and $x^2 z$ implies $d^{x^2z}_{\chi_3,\varphi} = -2$ and $d^{x^2z}_{\chi_2,\varphi} = \alpha$. In order to determine α and $\beta := d^{x^2}_{\chi_2,\varphi}$ we use the contribution matrices

$$(m^u_{\chi,\psi})_{\chi,\psi\in\mathrm{Irr}(B)} := 16 Q_u C_u^{-1} Q_u^{\mathrm{t}} \in \mathbb{Z}^{10\times 10},$$

where $Q_u = (d^u_{\chi,\varphi})$ and $C_u = Q_u^{\mathrm{t}} Q_u$ is the Cartan matrix of b_u. We compute

$$m^u_{\chi_1,\chi_2} = \begin{cases} 2 & \text{if } u \in \{x, xz\},\\ -2 & \text{if } u \in \{y, yz\},\\ \alpha & \text{if } u = x^2 z,\\ \beta & \text{if } u = x^2.\end{cases}$$

By restricting a generalised character of $S_4 \times C_2$, we obtain an $\mathcal{F}$-invariant generalised character λ of D such that $\lambda(1) = \lambda(z) = 0$, $\lambda(x) = \lambda(xz) = \lambda(x^2) = \lambda(x^2 z) = 4$ and $\lambda(y) = \lambda(yz) = 2$. Then

$$0 = \sum_{u\in\mathcal{R}} \lambda(u) m^u_{\chi_1,\chi_2} = 4(2 + 2 + \alpha + \beta) + 2(-2 - 2)$$

by [1, p. 684]. It follows that $\alpha = \beta = -1$. This completely determines the generalised decomposition matrices for non-trivial subsections as follows:

$$
\begin{array}{c|cccccccc}
 & x & y & xz & yz & x^2 & x^2z & \multicolumn{2}{c}{z} \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & . \\
 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & . \\
 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & . \\
 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & . \\
\chi_2 & 1 & -1 & 1 & -1 & -1 & -1 & . & -1 \\
 & -1 & 1 & -1 & 1 & -1 & -1 & . & -1 \\
 & 1 & -1 & -1 & 1 & -1 & 1 & . & 1 \\
 & -1 & 1 & 1 & -1 & -1 & 1 & . & 1 \\
\hline
\chi_3 & . & . & . & . & 2 & -2 & 1 & 1 \\
 & . & . & . & . & 2 & 2 & -1 & -1
\end{array}
$$

Now the claim follows from Theorem 6.1 of [25]. $\square$

Proof of Theorem 1.3. As before, let C be the Cartan matrix of B.

Case 1: $l(B) = 1$.
Here, $|D| = 16$. If B is nilpotent, then $A \cong kD$ by Puig's theorem (see Theorem 1.30 in [22]). By partial solutions on the modular isomorphism problem, these algebras are pairwise non-isomorphic (see Lemma 14.2.7 in [19]). If B is non-nilpotent, then D must be elementary abelian and the inertial index of B is 9 (see Theorem 13.2 and the proof of Theorem 13.6 in [22]). By Eaton's classification [7], A is Morita equivalent to a non-principal block of H as given in the statement. In total we obtain 15 isomorphism types of basic algebras with $l(B) = 1$.

Case 2: $l(B) = 2$.
By Theorem 2.2, B cannot be a tame block. Using Proposition 2 of [26], it is easy to see that C is one of the following matrices

$$
\begin{array}{cccccc}
C & \begin{pmatrix} 12 & 1 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 11 & 1 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix} & \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} & \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \\
|D| & 23 & 32 & 8 & 16 & 16
\end{array}
$$

The first case occurs for the principal 23-block of $\mathrm{PSL}(2, 137)$. Here A is uniquely determined as a Brauer tree algebra. In the second case, B must have finite representation type by Theorem 1.1. Then $D \cong C_{32}$ and B would be nilpotent since $\mathrm{Aut}(D)$ is a 2-group. In the third case, $D \cong D_8$ since otherwise $l(B) \neq 2$. But then B would be tame. Now consider the fifth case. The possible decomposition matrices of B are

$$
\begin{pmatrix} 2 & 1 \\ . & 1 \\ . & 1 \\ . & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 1 & . \\ 1 & . \\ 1 & . \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & . \\ 1 & . \\ . & 1 \\ . & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.
$$

By Theorem 5.3 we obtain $k_0(B) = 4$ in all cases. Since B satisfies the Alperin–McKay conjecture (see Theorem 13.6 of [22]), we have $k_0(B_D) = 4$ where B_D is the Brauer correspondent of B in $N_G(D)$. Recall that B_D dominates a block $\overline{B_D}$ of $N_G(D)/D'$ with abelian defect group D/D'. Hence,

$$k(\overline{B_D}) = k_0(\overline{B_D}) \leqslant k_0(B_D) = 4$$

and $|D/D'| = 4$ by Theorem 1.31 in [22]. But then D has maximal nilpotency class and B would be tame.

It remains to deal with the Cartan matrix $C = \left(\begin{smallmatrix} 6 & 2 \\ 2 & 6 \end{smallmatrix}\right)$, where $|D| = 16$. Since B is not tame, we have $k_0(B) > 4$ as seen above. The possible decomposition matrices of B are:

$$\begin{pmatrix} 2 & 1 \\ 1 & . \\ 1 & . \\ . & 1 \\ . & 1 \\ . & 1 \\ . & 1 \\ . & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . \\ 1 & . \\ 1 & . \\ 1 & . \\ . & 1 \\ . & 1 \\ . & 1 \\ . & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In the first case $k_0(B) = k(B)$ and D is abelian by Theorem 5.3. However, there is no such block with $l(B) = 2$. Therefore, the second matrix must be the decomposition matrix of B. In particular, $k(B) = k_0(B) + k_1(B) = 8 + 2 = 10$. Using the results in [22, Chapters 8, 9], one can exclude metacyclic defect groups and $Q_8 \times C_2$ or $Q_8 * C_4$ for D. The remaining cases are $D \cong D_8 \times C_2$ or the minimal non-abelian group $\texttt{SmallGroup}(16, 3)$. By Lemma 7.1 and Theorem 9 in [23], B is perfectly isometric to principal block of $H_1 := S_4 \times C_2$ or $H_2 := \texttt{SmallGroup}(48, 30) \cong A_4 \rtimes C_4$. In particular, $Z(A) \cong Z(B) \cong Z(kH_i)$ for $i = 1$ or 2 (see Theorem 4.4 in [25]). One can show with MAGMA [3] that

$$Z(kH_1) \cong Z(kH_2) \cong k[w, x, y, z]/(w^2x, w^2y, w^2 + z^2, x^2, xy, xz, y^2, yz, z^3) \tag{7.2}$$

with basis 1, w, x, y, z, $w^2 = z^2$, wx, wy, wz, $w^3 = wz^2$. The centre $Z(A)$ is the subset of $\mathsf{End}(P_S) \times \mathsf{End}(P_T)$ consisting of the elements that annihilate the homomorphisms from P_S to P_T and from P_T to P_S. Now $\mathsf{End}(P_S) \times \mathsf{End}(P_T)$ has dimension 12. The idempotents ε_S and ε_T are not in $Z(A)$ but their sum is. Since $Z(A)$ has dimension 10, it follows that the radical $J(Z(A))$ has codimension one in $J(\mathsf{End}(P_S)) \times J(\mathsf{End}(P_T))$. By (7.2), $J(Z(A)$ is indecomposable, the projections $Z(A) \to \mathsf{End}(P_S)$ and $Z(A) \to \mathsf{End}(P_T)$ are surjective and $Z(A) \to \mathsf{End}(P_S) \times \mathsf{End}(P_T)$ is injective. This implies that $\mathsf{End}(P_S)$ and $\mathsf{End}(P_T)$ are 6-dimensional commutative Gorenstein rings. Using (7.2), we claim that the only 6-dimensional Gorenstein quotient of $Z(A)$ is $Z(A)/(x, y) \cong k[w, z]/(w^2 + z^2, z^3)$. For if x or y has non-zero image in a Gorenstein quotient, then the socle has to be divisible by x or y. This forces the Loewy length to be three. Since the socle of a Gorenstein ring is 1-dimensional, this means that the second Loewy layer has to be 4-dimensional. It is easy to see that there is no such Gorenstein quotient.

Case 3: $l(B) = 3$.
By Proposition 2 in [26], there are no such blocks of defect 1. By Theorem 2.2 and Theorem 2.1 in Macgregor [17], there is just one Morita equivalence class of tame blocks. Here $D \cong D_8$ and A is Morita equivalent to the principal block of $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,2)$. Now suppose that B is not tame. As explained in the proof of Theorem 1.2, it is easy to make a list of potential Cartan matrices:

$$\begin{array}{cccccc} C & \begin{pmatrix} 5 & 1 & . \\ 1 & 3 & 2 \\ . & 2 & 2 \end{pmatrix} & \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 5 & 1 & 2 \\ 1 & 3 & . \\ 2 & . & 2 \end{pmatrix} & \begin{pmatrix} 6 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 7 & . & 1 \\ . & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \\ |D| & 4 & 8 & 8 & 16 & 32 \end{array}$$

The first three candidates are excluded by Proposition 2.3. In the fifth case, B has finite representation type by Theorem 1.1. But then B would be nilpotent since $p = 2$. Now consider case four, where $|D| = 16$. The possible decomposition matrices are:

$$\begin{pmatrix} 2 & . & . \\ 1 & 1 & 1 \\ . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & . \\ 1 & . & 1 \\ . & 1 & 1 \\ 1 & . & . \\ 1 & . & . \\ 1 & . & . \\ 1 & . & . \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ . & 1 & . \\ . & . & 1 \\ 1 & . & . \\ 1 & . & . \\ 1 & . & . \\ 1 & . & . \\ 1 & . & . \end{pmatrix}$$

The first two cases yield $k_0(B) = 4$. But then B must be a tame block by the arguments above for $l(B) = 2$. In the final case, $k_0(B) = k(B) = 8$ and D is abelian. Here D cannot be of type $C_4 \times C_2^2$, because this would yield an elementary divisor 2 of C. Consequently, D is elementary abelian. However, this is excluded by Eaton [7].

Case 4: $l(B) = 4$.
By Proposition 2.1, there exist three (potential) blocks of defect 1: two 5-blocks with multiplicity $m = 1$ and a 13-block with multiplicity 3. The 5-blocks occur in S_5 and $\mathrm{Sz}(8)$ by [13]. The 13-block is Morita equivalent to the principal block of $\mathrm{GL}(4,5)$ by [12]. In the general case we enumerate the possibilities for C. There are at least six positive off-diagonal entries of C and therefore the trace is bounded by 10. The diagonal entries are bounded by 4 while the off-diagonal entries can be at most 3. This only leaves the case

$$C = \begin{pmatrix} 2 & 1 & . & 1 \\ 1 & 2 & 1 & . \\ . & 1 & 3 & . \\ 1 & . & . & 3 \end{pmatrix}$$

where $|D| = 16$. Here B is not tame since $l(B) = 4$. Hence, this block is excluded again by Theorem 1.1 since $p = 2$.

Case 5: $l(B) \geqslant 5$.
This cannot happen, by Lemma 6.1. □

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References

[1] C. G. Ardito and B. Sambale, *Cartan matrices and Brauer's $k(B)$-Conjecture V*, J. Algebra **606** (2022), 670–699.

[2] C. Bessenrodt, *The Auslander–Reiten quiver of a modular group algebra revisited*, Math. Zeit. **206** (1991), 25–34.

[3] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[4] M. Broué and L. Puig, *Characters and local structure in G-algebras*, J. Algebra **63** (1980), 306–317.

[5] M. Chlebowitz and B. Külshammer, *Symmetric local algebras with 5-dimensional center*, Trans. Amer. Math. Soc. **329** (1992), no. 2, 715–731.

[6] C. W. Eaton, *Morita equivalence classes of 2-blocks of defect three*, Proc. Amer. Math. Soc. **144** (2016), 1961–1970.

[7] C. W. Eaton, *Morita equivalence classes of blocks with elementary abelian defect groups of order 16*, arXiv:1612.03485v4.

[8] K. Erdmann, *Algebras and semidihedral defect groups I*, Proc. London Math. Soc. (3) **57** (1988), no. 1, 109–150.

[9] ______, *Algebras and semidihedral defect groups II*, Proc. London Math. Soc. (3) **60** (1990), no. 1, 123–165.

[10] ______, *Blocks of tame representation type and related algebras*, Lecture Notes in Mathematics, vol. 1428, Springer-Verlag, Berlin/New York, 1990.

[11] ______, *On Auslander–Reiten components for group algebras*, J. Pure & Applied Algebra **104** (1995), 149–160.

[12] P. Fong and B. Srinivasan, *Blocks with cyclic defect groups in $GL(n, q)$*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), 1041–1044.

[13] G. Hiss and K. Lux, *Brauer trees of sporadic groups*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1989.

[14] B. Külshammer, *On the structure of block ideals in group algebras of finite groups*, Comm. Algebra **8** (1980), no. 19, 1867–1872.

[15] M. Linckelmann, *Finite dimensional algebras arising as blocks of finite group algebras*, Representations of Algebras (G. Leuschke, F. Bleher, R. Schiffler, and D. Zacharia, eds.), Contemp. Math., vol. 705, American Math. Society, 2018, pp. 155–188.

[16] M. Linckelmann and W. Murphy, *A nine-dimensional algebra which is not a block of a finite group*, Quarterly J. Math. (Oxford) **72** (2021), 1077–1088.

[17] N. Macgregor, *Morita equivalence classes of tame blocks of finite groups*, J. Algebra **608** (2022), 719–754.

[18] G. Malle, G. Navarro, A. A. Schaeffer Fry and P. H. Tiep, *Brauer's Height Zero Conjecture*, arXiv:2209.04736v1

[19] D. S. Passman, *The algebraic structure of group rings*, Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1985.

[20] B. Poonen, *Isomorphism types of commutative algebras of finite rank over an algebraically closed field*, Computational arithmetic geometry (K. E. Lauter and K. A. Ribet, eds.), Contemp. Math., vol. 463, 2008, pp. 111–120.

[21] G. R. Robinson, *On the focal defect group of a block, characters of height zero, and lower defect group multiplicities*, J. Algebra **320** (2008), 2624–2628.

[22] B. Sambale, *Blocks of finite groups and their invariants*, Springer Lecture Notes in Math., Vol. 2127, Springer-Verlag, Cham, 2014.
[23] B. Sambale, 2-*Blocks with minimal nonabelian defect groups III*, Pacific J. Math. **280** (2016), 475–487.
[24] B. Sambale, *Cartan matrices and Brauer's* $k(B)$*-Conjecture IV*, J. Math. Soc. Japan **69** (2017), 735–754.
[25] B. Sambale, *Survey on perfect isometries*, Rocky Mountain J. Math. **50** (2020), 1517–1539.
[26] B. Sambale, *Blocks with small-dimensional basic algebra*, Bull. Austral. Math. Soc. **103** (2021), 461–474.
[27] B. Wald and J. Waschbüsch, *Tame biserial algebras*, J. Algebra **95** (1985), no. 2, 480–500.
[28] P. J. Webb, *The Auslander-Reiten quiver of a finite group*, Math. Zeit. **179** (1982), 97–121.

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