A note on Olsson's Conjecture

Lászlo Héthelyi, Burkhard Külshammer and Benjamin Sambale

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Dedicated to Geoffrey Robinson on the occasion of his 60th birthday

Abstract

For a p-block B of a finite group G with defect group D Olsson conjectured that $k_0(B) \le |D : D'|$, where $k_0(B)$ is the number of characters in B of height 0 and D' denotes the commutator subgroup of D. Brauer deduced Olsson's Conjecture in the case where D is a dihedral 2-group using the fact that certain algebraically conjugate subsections are also conjugate in G. We generalize Brauer's argument for arbitrary primes p and arbitrary defect groups. This extends two results by Robinson. For $p > 3$ we show that Olsson's Conjecture is satisfied for defect groups of p-rank 2 and for minimal non-abelian defect groups.

1 Introduction

In order to state Olsson's Conjecture we need some notations. Let $\mathbf{R}$ be a complete discrete valuation ring with quotient field $\mathbf{K}$ of characteristic 0. Moreover, let (π) be the maximal ideal of $\mathbf{R}$ and $\mathbf{F} := \mathbf{R}/(\pi)$. We assume that $\mathbf{F}$ is algebraically closed of characteristic $p > 0$. We fix a finite group G, and assume that $\mathbf{K}$ contains all $|G|$-th roots of unity. Let B be a p-block of $\mathbf{R}G$ (or simply of G) with defect group D. We denote the set of irreducible ordinary characters by $\mathrm{Irr}(B)$ and its cardinality by $k(B)$. These characters split in $k_i(B)$ characters of height $i \in \mathbb{N}_0$. Here the *height* of a character χ in B is the largest integer $h(\chi) \ge 0$ such that $p^{h(\chi)}|G : D|_p \bigm| \chi(1)$, where $|G : D|_p$ denotes the highest p-power dividing $|G : D|$. We set $\mathrm{Irr}_0(B) := \{\chi \in \mathrm{Irr}(B) : h(\chi) = 0\}$. Finally, let $\mathrm{IBr}(B)$ be the set of irreducible Brauer characters and $l(B) := |\mathrm{IBr}(B)|$.

In the situation above, Olsson conjectured in 1975 that we always have $k_0(B) \le |D : D'|$, where D' denotes the derived subgroup of D (see [42]). This conjecture has been verified in some cases, but remains open in general. For example it was shown in [30] that Olsson's Conjecture for B would follow from the Alperin-McKay Conjecture for B (see also [56, 21]). Recall that the Alperin-McKay Conjecture predicts that $k_0(B) = k_0(b)$, where b is the Brauer correspondent of B in $\mathbf{R}\,\mathrm{N}_G(D)$. In particular Olsson's Conjecture holds for p-solvable, symmetric or alternating groups by [41, 44, 36]. If D is abelian, Olsson's Conjecture follows from Brauer's $k(B)$-Conjecture $k(B) \le |D|$. Moreover, Olsson's Conjecture is satisfied if D is metacyclic (see [55, 61]) or if $p = 2$ and D is minimal non-abelian (see [52]). Hendren verified Olsson's Conjecture for some, but not all p-blocks with a non-abelian defect group of order p^3 (see [24, 23]).

This paper is organized as follows. In the second section we introduce two results by Robinson and extend them in some sense using ideas of [53, 54]. In the third and fourth sections we generalize an argument of Brauer regarding a Galois action on subsections. In Section 5 we show that Olsson's Conjecture is fulfilled for controlled blocks with certain defect groups. In the last section we use the classification of finite simple groups to prove Olsson's Conjecture for defect groups of p-rank 2 and for minimal non-abelian defect groups if $p > 3$ (in both cases). In particular, our results here settle most of the cases of Olsson's Conjecture left open in Hendren's papers [24, 23].

2 Subsection

The notion of B-subsections provides one tool in order to attack Olsson's Conjecture. Here a *B-subsection* is a pair (u, b_u), where $u \in D$ and b_u is a Brauer correspondent of B in $\mathbf{R}\,\mathrm{C}_G(u)$. Robinson showed the following proposition (see [47]).

Proposition 2.1 (Robinson). *If b_u has defect d, then $k_0(B) \le p^d\sqrt{l(b_u)}$.*

We mention another result by Robinson which will be improved later (see Theorem 3.4 in [46]). Recall that a B-subsection (u, b_u) is called *major* if b_u and B have the same defect.

Proposition 2.2 (Robinson). *If (u, b_u) is a major B-subsection such that $l(b_u) = 1$, then*

$$\sum_{i=0}^{\infty} p^{2i} k_i(B) \le |D|.$$

In order to make these propositions clearer, we introduce the fusion system $\mathcal{F}$ of B. For this we use the notation of [43, 34], and we assume that the reader is familiar with these articles. Let b_D be a Brauer correspondent of B in $\mathbf{R}D\,\mathrm{C}_G(D)$. Then for every subgroup $Q \le D$ there is a unique block b_Q of $\mathbf{R}Q\,\mathrm{C}_G(Q)$ such that $(Q, b_Q) \le (D, b_D)$. We denote the inertial group of b_Q in $\mathrm{N}_G(Q)$ by $\mathrm{N}_G(Q, b_Q)$. Then $\mathrm{Aut}_{\mathcal{F}}(Q) \cong \mathrm{N}_G(Q, b_Q)/\,\mathrm{C}_G(Q)$.

The fusion of subsections is given by the following proposition (see [51]).

Proposition 2.3. *Let $\mathcal{R}$ be a set of representatives for the $\mathcal{F}$-conjugacy classes of elements of D such that $\langle u\rangle$ is fully $\mathcal{F}$-normalized for $u \in \mathcal{R}$ ($\mathcal{R}$ always exists). Then*

$$\big\{(u, b_u) : u \in \mathcal{R}\big\}$$

is a set of representatives for the G-conjugacy classes of B-subsections, where $b_u := b_{\langle u\rangle}$ has defect group $\mathrm{C}_D(u)$.

Brauer proved Olsson's Conjecture for 2-blocks with dihedral defect groups using a Galois action on the generalized decomposition numbers (see [10]). We provide the necessary definitions for that purpose. Let p^k be the order of u, and let $\zeta := \zeta_{p^k}$ be a primitive p^k-th root of unity. Then the generalized decomposition numbers $d^u_{\chi\varphi}$ for $\chi \in \mathrm{Irr}(B)$ and $\varphi \in \mathrm{IBr}(b_u)$ lie in the ring of integers $\mathbb{Z}[\zeta]$. Hence, there exist integers $a_i^{\varphi} := (a_i^{\varphi}(\chi))_{\chi\in\mathrm{Irr}(B)} \in \mathbb{Z}^{k(B)}$ such that

$$d^u_{\chi\varphi} = \sum_{i=0}^{\varphi(p^k)-1} a_i^{\varphi}(\chi)\zeta^i \tag{2.1}$$

(see Satz I.10.2 in [39]). Here $\varphi(p^k)$ denotes Euler's totient function.

Let $\mathcal{G}$ be the Galois group of the cyclotomic field $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$. Then $\mathcal{G} \cong \mathrm{Aut}(\langle u\rangle) \cong (\mathbb{Z}/p^k\mathbb{Z})^{\times}$ and we will often identify these groups. We will also interpret the elements of $\mathcal{G}$ as integers in $\{1, \ldots, p^k\}$ by a slight abuse of notation. Then (u^{γ}, b_u) for $\gamma \in \mathcal{G}$ is also a (*algebraically conjugate*) subsection and

$$\gamma(d^u_{\chi\varphi}) = d^{u^{\gamma}}_{\chi\varphi} = \sum_{i=0}^{\varphi(p^k)-1} a_i^{\varphi}(\chi)\zeta^{i\gamma}.$$

We use the opportunity to present a slight generalization of Lemma 1 in [54]. Here we call two matrices $A, B \in \mathbb{Z}^{l\times l}$ *equivalent* if there exists a matrix $S \in \mathrm{GL}(l, \mathbb{Z})$ with $A = S^{\mathrm{T}}BS$, where S^{T} denotes the transpose of S. This is just Brauer's notion of basic sets.

Theorem 2.4. *Let B be a p-block of G, and let (u, b_u) be a B-subsection. Let $C_u = (c_{ij})$ be the Cartan matrix of b_u up to equivalence. Then for every positive definite, integral quadratic form $q(x_1, \ldots, x_{l(b_u)}) = \sum_{1\le i\le j\le l(b_u)} q_{ij}x_ix_j$ we have*

$$k_0(B) \le \sum_{1\le i\le j\le l(b_u)} q_{ij}c_{ij}.$$

In particular

$$k_0(B) \le \sum_{i=1}^{l(b_u)} c_{ii} - \sum_{i=1}^{l(b_u)-1} c_{i,i+1}.$$

If (u, b_u) is major, we can replace $k_0(B)$ by $k(B)$ in these formulas.

Proof. [1] First of all assume, that C_u is the Cartan matrix of b_u (not only up to equivalence!). Let $\varphi_1, \ldots, \varphi_l$ ($l := l(b_u)$) be the irreducible Brauer characters of b_u. Then we have rows $d_\chi := (d^u_{\chi\varphi_1}, \ldots, d^u_{\chi\varphi_l})$ for $\chi \in \mathrm{Irr}(B)$. Let $Q = (\widetilde{q}_{ij})_{i,j=1}^{l}$ with

$$\widetilde{q}_{ij} := \begin{cases} q_{ij} & \text{if } i = j, \\ q_{ij}/2 & \text{if } i \neq j \end{cases}.$$

Then we have

$$\sum_{1\le i\le j\le l} q_{ij}c_{ij} = \sum_{1\le i,j\le l} \widetilde{q}_{ij}c_{ij} = \sum_{1\le i,j\le l}\ \sum_{\chi\in\mathrm{Irr}(B)} \widetilde{q}_{ij} d^u_{\chi i}\overline{d^u_{\chi j}}$$
$$= \sum_{\chi\in\mathrm{Irr}(B)} d_\chi Q \overline{d_\chi}^{\mathrm{T}} \ge \sum_{\chi\in\mathrm{Irr}_0(B)} d_\chi Q \overline{d_\chi}^{\mathrm{T}},$$

since Q is positive definite. Thus, it suffices to show

$$\sum_{\chi\in\mathrm{Irr}_0(B)} d_\chi Q \overline{d_\chi}^{\mathrm{T}} \ge k_0(B).$$

For this, let p^n be the order of u, and let $f := p^{n-1}(p-1) - 1$. We fix a character $\chi \in \mathrm{Irr}_0(B)$ and set $d := d_\chi$. Then there are integral rows $a_m \in \mathbb{Z}^l$ ($m = 0, \ldots, f$) such that $d = \sum_{m=0}^{f} a_m\zeta^m$. By Corollary 2 in [11] at least one of the rows a_m does not vanish.

It is known that for every $\gamma \in \mathcal{G}$ there is a character $\chi' \in \mathrm{Irr}(B)$ such that $\gamma(d) = d_{\chi'}$. Thus, it suffices to show

$$\sum_{\gamma\in\mathcal{G}} \gamma(d)Q\overline{\gamma(d)}^{\mathrm{T}} = \sum_{\gamma\in\mathcal{G}} \gamma(dQ\overline{d}^{\mathrm{T}}) \ge |\mathcal{G}| = f + 1.$$

We have

$$\sum_{\gamma\in\mathcal{G}} \gamma(dQ\overline{d}^{\mathrm{T}}) = \sum_{\gamma\in\mathcal{G}} \gamma\left(\sum_{i=0}^{f} a_iQa_i^{\mathrm{T}} + \sum_{j=1}^{f}\sum_{m=0}^{f-j} a_mQa_{m+j}^{\mathrm{T}}(\zeta^j + \overline{\zeta}^j)\right)$$
$$= (f+1)\sum_{i=0}^{f} a_iQa_i^{\mathrm{T}} + 2\sum_{j=1}^{f}\sum_{m=0}^{f-j} a_mQa_{m+j}^{\mathrm{T}} \sum_{\gamma\in\mathcal{G}}\gamma(\zeta^j).$$

The p^m-th cyclotomic polynomial Φ_{p^m} has the form

$$\Phi_{p^m} = X^{p^{m-1}(p-1)} + X^{p^{m-1}(p-2)} + \ldots + X^{p^{m-1}} + 1.$$

This gives

$$\sum_{\gamma\in\mathcal{G}} \gamma(\zeta^j) = \begin{cases} -p^{n-1} & \text{if } p^{n-1} \mid j \\ 0 & \text{otherwise} \end{cases}$$

for $j \in \{1, \ldots, f\}$. It follows that

$$\sum_{\gamma\in\mathcal{G}} \gamma(dQ\overline{d}^{\mathrm{T}}) = (f+1)\sum_{i=0}^{f} a_iQa_i^{\mathrm{T}} - 2p^{n-1}\sum_{j=1}^{p-2}\ \sum_{m=0}^{f-jp^{n-1}} a_mQa_{m+p^{n-1}j}^{\mathrm{T}}$$
$$= p^{n-1}\left((p-1)\sum_{i=0}^{f} a_iQa_i^{\mathrm{T}} - 2\sum_{j=1}^{p-2}\ \sum_{m=0}^{f-jp^{n-1}} a_mQa_{m+p^{n-1}j}^{\mathrm{T}}\right). \tag{2.2}$$

[1] Proof corrected on March 31, 2013

For $p = 2$ the claim follows immediately, since then $f + 1 = 2^{n-1}$. Thus, suppose $p > 2$. Then we have

$$\{0, 1, \ldots, f - jp^{n-1}\} \,\dot\cup\, \{(p-1-j)p^{n-1}, (p-1-j)p^{n-1}+1, \ldots, f\} = \{0, 1, \ldots, f\}$$

for all $j \in \{1, \ldots, p-2\}$. This shows that every row a_m occurs exactly $p-2$ times in the second sum of (2.2). Hence,

$$\sum_{\gamma\in\mathcal{G}} \gamma(dQ\overline{d}^{\mathrm{T}}) = p^{n-1}\left(\sum_{i=0}^{f} a_i Q a_i^{\mathrm{T}} + \sum_{j=1}^{p-2} \sum_{m=0}^{f-jp^{n-1}} (a_m - a_{m+jp^{n-1}})Q(a_m - a_{m+jp^{n-1}})^{\mathrm{T}}\right).$$

Now assume that a_m does not vanish for some $m \in \{0, \ldots, f\}$. Then we have $a_m Q a_m^{\mathrm{T}} \geq 1$, since Q is positive definite. Again, a_m occurs exactly $p-2$ times in the second sum. Let $a_m - a_{m'}$ (resp. $a_{m'} - a_m$) be such an occurrence. Then we have

$$a_{m'} Q a_{m'}^{\mathrm{T}} + (a_m - a_{m'})Q(a_m - a_{m'})^{\mathrm{T}} \geq 1.$$

Now the first inequality of the theorem follows easily.

The result does not depend on the basic set for C_u, since changing the basic set is essentially the same as taking another quadratic form q (see [32]). For the second claim we take the quadratic form corresponding to the Dynkin diagram of type A_l for q. If (u, b_u) is major, then all rows d_χ for $\chi \in \mathrm{Irr}(B)$ do not vanish (see Theorem V.9.5 in [18]). Hence, we can replace $k_0(B)$ by $k(B)$. □

We present an application.

Proposition 2.5. *Let (u, b_u) be a B-subsection such that b_u has defect group Q. Then the following hold:*

(i) If $Q/\langle u\rangle$ is cyclic, we have

$$k_0(B) \leq \left(\frac{|Q/\langle u\rangle| - 1}{l(b_u)} + l(b_u)\right)|\langle u\rangle| \leq |Q|.$$

(ii) If $|Q/\langle u\rangle| \leq 9$, we have $k_0(B) \leq |Q|$.

(iii) Suppose $p = 2$. If $Q/\langle u\rangle$ is metacyclic or minimal non-abelian or isomorphic to $C_4 \wr C_2$, we have $k_0(B) \leq |Q|$.

Proof.

(i) It is well-known that b_u dominates a block $\overline{b_u}$ of $\mathrm{C}_G(u)/\langle u\rangle$ with cyclic defect group $Q/\langle u\rangle$ and $l(\overline{b_u}) = l(b_u)$. By [14, 48] the Cartan matrix b_u has the form $|\langle u\rangle|(m + \delta_{ij})_{1\leq i,j\leq l(b_u)}$ up to equivalence, where $m := (|Q/\langle u\rangle| - 1)/l(b_u)$ is the multiplicity of $\overline{b_u}$. Now the claim follows from Theorem 2.4.

(ii) See Theorem 1 in [54].

(iii) If $Q/\langle u\rangle$ is metacyclic, the claim follows as in Theorem 2 of [53]. If $Q/\langle u\rangle$ is minimal non-abelian, the claim can easily deduced from the results in [52, 16]. Finally, for $D/\langle u\rangle \cong C_4 \wr C_2$ the result follows from [29]. □

Since $u \in \mathrm{Z}(Q)$ in Proposition 2.5(i), the condition implies that Q is abelian of rank at most 2. It is known that the number $l(b_u)$ in Proposition 2.5(i) equals the inertial index of $\overline{b_u}$ (see [14]).

3 The case $p = 2$

Let $p = 2$, and let (u, b_u) be a B-subsection for a block B of G. Then by Proposition 2.3 we may assume that $\langle u\rangle$ is fully $\mathcal{F}$-normalized, where $\mathcal{F}$ is the fusion system of B. By Proposition 2.5 in [34] $\langle u\rangle$ is also fully $\mathcal{F}$-centralized and

$$\operatorname{Aut}_{\mathcal{F}}(\langle u\rangle) = \operatorname{Aut}_D(\langle u\rangle) = \mathrm{N}_D(\langle u\rangle)\,\mathrm{C}_G(u)/\,\mathrm{C}_G(u) \cong \mathrm{N}_D(\langle u\rangle)/\,\mathrm{C}_D(u).$$

Hence, Theorem 2.4(ii) in [33] implies that $\mathrm{C}_D(u)$ is a defect group of b_u.

Theorem 3.1. *Let B be a 2-block of a finite group G with defect group D and fusion system $\mathcal{F}$, and let (u, b_u) be a B-subsection such that $\langle u\rangle$ is fully $\mathcal{F}$-normalized and b_u has Cartan matrix $C_u = (c_{ij})$. Let $\operatorname{IBr}(b_u) = \{\varphi_1, \ldots, \varphi_{l(b_u)}\}$ such that $\varphi_1, \ldots, \varphi_m$ are stable under $\mathrm{N}_D(\langle u\rangle)$ and $\varphi_{m+1}, \ldots, \varphi_{l(b_u)}$ are not. Then $m \geq 1$. Suppose further that u is conjugate to u^{-5^n} for some $n \in \mathbb{Z}$ in D. Then*

$$k_0(B) \leq \frac{|\mathrm{N}_D(\langle u\rangle)/\,\mathrm{C}_D(u)|}{\varphi\big(|\langle u\rangle|\big)} \sum_{1\leq i\leq j\leq m} q_{ij}c_{ij} \tag{3.1}$$

for every positive definite, integral quadratic form $q(x_1, \ldots, x_m) = \sum_{1\leq i\leq j\leq m} q_{ij}x_ix_j$. In particular if $l(b_u) = 1$, we get

$$k_0(B) \leq \frac{|\mathrm{N}_D(\langle u\rangle)|}{\varphi\big(|\langle u\rangle|\big)}. \tag{3.2}$$

If $l(b_u) = 2$, we may replace C_u by an equivalent matrix such that $|\mathrm{C}_D(u)|c_{11}/\det C_u$ is even and as small as possible. In this case (with the hypothesis above) we have

$$k_0(B) \leq \frac{|\mathrm{N}_D(\langle u\rangle)/\,\mathrm{C}_D(u)|c_{11}}{\varphi\big(|\langle u\rangle|\big)} \leq \frac{|\mathrm{N}_D(\langle u\rangle)|}{\varphi\big(|\langle u\rangle|\big)}. \tag{3.3}$$

Proof. Let $\chi \in \operatorname{Irr}_0(B)$ and $|\langle u\rangle| = 2^k$ for some $k \geq 0$. We write $d^u_\chi := (d^u_{\chi\varphi_1}, \ldots, d^u_{\chi\varphi_l})$, where $l := l(b_u)$. Then we have $|\mathrm{C}_D(u)|m^{(u,b_u)}_{\chi\chi} = d^u_\chi|\mathrm{C}_D(u)|C_u^{-1}\overline{d^u_\chi}$ for the contribution $m^{(u,b_u)}_{\chi\chi}$ (see Eq. (5.2) in [9]). By Corollary 2 in [11] it follows that

$$|\mathrm{C}_D(u)|m^{(u,b_u)}_{\chi\chi} = |\mathrm{C}_D(u)|\big(\chi^{(u,b_u)}, \chi\big)_G \not\equiv 0 \pmod{(\pi)}.$$

Since $\zeta \equiv 1 \pmod{(\pi)}$, we see that

$$d^u_{\chi\varphi_i} \equiv \gamma(d^u_{\chi\varphi_i}) \equiv \sum_{j=0}^{\varphi(2^k)-1} a^i_j(\chi) \pmod{(\pi)}$$

for $\gamma \in \mathcal{G}$. In particular $d^u_{\chi\varphi_i} \equiv \overline{d^u_{\chi\varphi_i}} \pmod{(\pi)}$. We write $|\mathrm{C}_D(u)|C_u^{-1} = (\widetilde{c}_{ij})$. Then it follows that

$$\begin{aligned}
0 \not\equiv |\mathrm{C}_D(u)|m^{(u,b_u)}_{\chi\chi} &\equiv \sum_{1\leq i,j\leq l} \widetilde{c}_{ij}d^u_{\chi\varphi_i}\overline{d^u_{\chi\varphi_j}} \equiv \sum_{1\leq i\leq l} \widetilde{c}_{ii}(d^u_{\chi\varphi_i})^2 \\
&\equiv \sum_{1\leq i\leq l} \widetilde{c}_{ii} \sum_{j=0}^{\varphi(2^k)-1} a^i_j(\chi)^2 \equiv \sum_{1\leq i\leq l} \widetilde{c}_{ii} \sum_{j=0}^{\varphi(2^k)-1} a^i_j(\chi) \pmod{(\pi)}
\end{aligned}$$

Now every $g \in \mathrm{N}_D(\langle u\rangle)$ induces a permutation on $\operatorname{IBr}(b_u)$. Let P_g be the corresponding permutation matrix. Then g also acts on the rows $d^u_i := (d^u_{\chi\varphi_i} : \chi \in \operatorname{Irr}(B))$ for $i = 1, \ldots, l$, and it follows that $C_uP_g = P_gC_u$. Hence, we also have $C_u^{-1}P_g = P_gC_u^{-1}$ for all $g \in \mathrm{N}_D(\langle u\rangle)$. If $\{\varphi_{m_1}, \ldots, \varphi_{m_2}\}$ $(m < m_1 < m_2 \leq l)$ is an orbit under $\mathrm{N}_D(\langle u\rangle)$, it follows that $d^u_{\chi\varphi_{m_1}} \equiv \ldots \equiv d^u_{\chi\varphi_{m_2}} \pmod{(\pi)}$ and $\widetilde{c}_{m_1m_1} = \ldots = \widetilde{c}_{m_2m_2}$. Since the length of this orbit is even, we get

$$\sum_{1\leq i\leq m} \widetilde{c}_{ii} \sum_{j=0}^{\varphi(2^k)-1} a^i_j(\chi) \not\equiv 0 \pmod{2}.$$

In particular $m \geq 1$. In case $|\langle u\rangle| \leq 2$ this simplifies to

$$\sum_{1 \leq i \leq m} \widetilde{c}_{ii} a_0^i(\chi) \not\equiv 0 \pmod 2.$$

We show that this holds in general. Thus, let $k \geq 2$ and $i \in \{1, \ldots, m\}$. Since (u, b_u) is conjugate to (u^{-5^n}, b_u) and φ_i is stable, we have

$$\sum_{j=0}^{\varphi(2^k)-1} a_j^i(\chi)\zeta^j = d_{\chi\varphi_i}^u = d_{\chi\varphi_i}^{u^{-5^n}} = \sum_{j=0}^{2^{k-1}-1} a_j^i(\chi)\zeta^{-5^n j}.$$

Moreover, for every $j \in \{0, \ldots, \varphi(2^k) - 1\}$ there is some $j_1 \in \{0, \ldots, \varphi(2^k) - 1\}$ such that $\zeta^{-5^n j} = \pm\zeta^{j_1}$. In order to compare coefficients observe that

$$\zeta^j = \zeta^{-5^n j} \Longrightarrow j \equiv -5^n j \pmod{2^k} \Longrightarrow 1 \equiv -5^n \pmod{2^k/\gcd(2^k, j)} \Longrightarrow j = 0.$$

Hence, the set $\{\pm\zeta^j : j = 1, \ldots, \varphi(2^k) - 1\}$ splits under the action of $\langle -5^n + 2^k\mathbb{Z}\rangle$ into orbits of even length. This shows $\sum_{j=0}^{\varphi(2^k)-1} a_j^i(\chi) \equiv a_0^i(\chi) \pmod 2$. Hence,

$$\sum_{1 \leq i \leq m} \widetilde{c}_{ii} a_0^i(\chi) \not\equiv 0 \pmod 2 \tag{3.4}$$

for every $\chi \in \mathrm{Irr}_0(B)$. In particular, there is an $i \in \{1, \ldots, m\}$ such that $a_0^i(\chi) \neq 0$. This gives

$$k_0(B) \leq \sum_{1 \leq i \leq j \leq m} q_{ij}(a_0^i, a_0^j)$$

(see proof of Theorem 2.4).

Now let k again be arbitrary. Observe that $a_0^i = \varphi(2^k)^{-1} \sum_{\gamma\in\mathcal{G}} \gamma(d_i^u)$ for $i \in \{1, \ldots, m\}$. By the orthogonality relations for generalized decomposition numbers we have $(d_i^{u^\gamma}, d_j^{u^\delta}) = c_{ij}$ for $\gamma, \delta \in \mathcal{G}$ if u^γ and u^δ are conjugate under $\mathrm{N}_D(\langle u\rangle)$ (see Theorem 5.4.11 in [37] for example). Otherwise we have $(d_i^{u^\gamma}, d_j^{u^\delta}) = 0$. This implies

$$(a_0^i, a_0^j) = \frac{1}{\varphi(2^k)^2} \sum_{\gamma,\delta\in\mathcal{G}} (d_i^{u^\gamma}, d_j^{u^\delta}) = \frac{|\mathrm{N}_D(\langle u\rangle)/\,\mathrm{C}_D(u)|}{\varphi(2^k)} c_{ij},$$

and (3.1) follows. In case $l = 1$ we have $C = (|\mathrm{C}_D(u)|)$, and (3.2) is also clear.

Now assume $l = 2$. Here we can use (3.4) in a stronger sense. We have $m = 2$. Since $|\mathrm{C}_D(u)|$ occurs as elementary divisor of C_u exactly once, we see that the rank of $\frac{|\mathrm{C}_D(u)|}{\det C_u} C_u \pmod 2$ is 1. Hence, $\frac{|\mathrm{C}_D(u)|}{\det C_u} C_u \pmod 2$ has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pmod 2, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2, \qquad \text{or} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \pmod 2.$$

Now it is easy to see that we may replace C_u by an equivalent matrix (still denoted by $C_u = (c_{ij})$) such that $|\mathrm{C}_D(u)|c_{11}/\det C_u$ is even and as small as possible. Then we also have to replace the rows d_1^u and d_2^u by linear combinations of each other. This gives rows $\widehat{d}_i^u$ and $\widehat{a}_j^i$ for $i = 1, 2$ and $j = 0, \ldots, \varphi(2^k) - 1$. Observe that the contributions do not depend on the representative of the equivalence class of C_u. Moreover, $\widetilde{c}_{11}$ is odd and $\widetilde{c}_{22}$ is even. Hence, (3.4) takes the form

$$\widehat{a}_0^1(\chi) \not\equiv 0 \pmod 2$$

for all $\chi \in \mathrm{Irr}_0(B)$. Since both φ_1 and φ_2 are stable under $\mathrm{N}_D(\langle u\rangle)$, we have $\gamma(\widehat{d}_1^u) = \widehat{d}_1^u$ for all $\gamma \in \mathrm{Aut}_{\mathcal{F}}(\langle u\rangle)$. Hence,

$$k_0(B) \leq (\widehat{a}_0^1, \widehat{a}_0^1) = \frac{|\mathrm{N}_D(\langle u\rangle)/\,\mathrm{C}_D(u)|c_{11}}{\varphi(2^k)}$$

as above. It remains to show that $c_{11} \leq |\mathrm{C}_D(u)|$. The reduction theory of quadratic forms gives an equivalent matrix $C_u' = (c_{ij}')$ such that $0 \leq 2c_{12}' \leq \min\{c_{11}', c_{22}'\}$ (see [12] for example). In case $c_{12}' = 0$ we may assume

$c_{11} \le c'_{11} = |\mathrm{C}_D(u)|$, since $|\mathrm{C}_D(u)|$ is the largest elementary divisor of C'_u. Hence, let $c'_{12} > 0$. Since the entries of C_u and thus also of C'_u are divisible by $\alpha := \det C_u/|\mathrm{C}_D(u)|$, we even have $c'_{12} \ge \alpha$. It follows that

$$3\alpha^2 \le 3(c'_{12})^2 \le c'_{11}c'_{22} - (c'_{12})^2 = \det C'_u \le \frac{|\mathrm{C}_D(u)|^2}{2}$$

and $\alpha \le |\mathrm{C}_D(u)|/4$. It was shown in the proof of Theorem 1 of [53] that

$$\max\{c'_{11}, c'_{22}\} \le c'_{11} + c'_{22} - c'_{12} \le c'_{11} + c'_{22} - \alpha \le \alpha\frac{|\mathrm{C}_D(u)|/\alpha + 3}{2} = \frac{|\mathrm{C}_D(u)| + 3\alpha}{2} \le |\mathrm{C}_D(u)|.$$

If $\alpha^{-1}c'_{11}$ or $\alpha^{-1}c'_{22}$ is even, the result follows from the minimality of c_{11}. Otherwise we replace C'_u by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} C'_u \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} c'_{11} + c'_{22} - 2c'_{12} & c'_{12} - c'_{22} \\ c'_{12} - c'_{22} & c'_{22} \end{pmatrix}.$$

Then $c_{11} \le c'_{11} + c'_{22} - 2c'_{12} \le |\mathrm{C}_D(u)|$. This finishes the proof. □

In the situation of Theorem 3.1 we have $u \in \mathrm{Z}(\mathrm{C}_G(u))$. Hence, all Cartan invariants c_{ij} are divisible by $|\langle u\rangle|$. This shows that the right hand side of (3.1) is always an integer. It is also known that $k_0(B)$ is divisible by 4 unless $|D| \le 2$.

Observe that the subsection (u, b_u) in Theorem 3.1 cannot be major unless $|\langle u\rangle| \le 2$, since then u would be contained in $\mathrm{Z}(D)$.

If $m = l(b_u)$ in Theorem 3.1, it suffices to know the Cartan matrix C_u only up to equivalence. For, replacing C_u by an equivalent matrix is essentially the same as taking another quadratic form q. However, for $m < l(b_u)$ we really have to use the "exact" Cartan matrix C_u which is unknown in most cases. For $p > 2$ there are not always stable characters in $\mathrm{IBr}(b_u)$ (see Proposition (2E)(ii) and the example following it in [28]).

We give an example. Let D be a (non-abelian) 2-group of maximal class. Then there is an element $x \in D$ such that $|D : \langle x\rangle| = 2$ and x is conjugate to x^{-5^n} for some $n \in \{0, |\langle x\rangle|/8\}$ under D. Since $\langle x\rangle \trianglelefteq D$, the subgroup $\langle x\rangle$ is fully $\mathcal{F}$-normalized, and b_x has cyclic defect group $\mathrm{C}_D(x) = \langle x\rangle$. Thus, Dade's Theorem on blocks with cyclic defect groups implies $l(b_x) = 1$. Hence, Theorem 3.1 shows Olsson's Conjecture $k_0(B) \le 4 = |D : D'|$. This was already proved in [10, 42].

On the other hand, we cannot improve Theorem 3.1 or Theorem 2.4 if u is not conjugate to u^{-5^n} in D. Indeed, if D a modular 2-group and $x \in D$ such that $|D : \langle x\rangle| = 2$, then B is nilpotent (see [17]) and $k_0(B) = |D : D'| = |D|/2 = |\mathrm{C}_D(x)|$.

We give a more general example.

Proposition 3.2. *Let D be a 2-group and $x \in D$ such that $|D : \langle x\rangle| \le 4$, and suppose that one of the following holds:*

(i) x is conjugate to x^{-5^n} in D for some $n \in \mathbb{Z}$,

(ii) $\langle x\rangle \trianglelefteq D$.

Then Olsson's Conjecture holds for all blocks with defect group D.

Proof. Let B be a block with defect group D and fusion system $\mathcal{F}$. By [55] we may assume that D is non-metacyclic.

(i) By hypothesis, x is conjugate to x^{-5^n} in $\mathcal{F}$. This condition is preserved if we replace x by an $\mathcal{F}$-conjugate. Hence, we may assume that $\langle x\rangle$ is fully $\mathcal{F}$-normalized. Then x is conjugate to x^{-5^n} in D. In particular $|\mathrm{C}_D(x)/\langle x\rangle| \le 2$. Hence, b_x dominates a block of $\mathrm{C}_G(x)/\langle x\rangle$ with cyclic defect group $\mathrm{C}_D(x)/\langle x\rangle$. This shows $l(b_x) = 1$. Now we can apply Theorem 3.1 which gives $k_0(B) \le 8$. In case $|D : D'| = 4$ a theorem of Taussky (see for example Proposition 1.6 in [8]) implies that D has maximal class which was excluded.

(ii) We consider the order of $\mathrm{C}_D(x)$.

Case 1: $\mathrm{C}_D(x) = \langle x\rangle$.
Since D is non-metacyclic, $D/\langle x\rangle$ is non-cyclic. Hence, we are in case (i).

Case 2: $x \in \mathrm{Z}(D)$.
If D is abelian, the result follows from Theorem 2 in [53]. Thus, we may assume that D is non-abelian. Then every conjugacy class of D has length at most 2. By a result of Knoche (see for example Aufgabe III.24b in [25]) this is equivalent to $|D'| = 2$. Let $y \in D \setminus \mathrm{Z}(D)$. Then $\mathrm{C}_D(y)$ is non-cyclic. After replacing y by xy if necessary, we have $|\langle x\rangle| = |\langle y\rangle|$. By Proposition 2.5 it suffices to show that $\langle y\rangle$ is fully $\mathcal{F}$-normalized. By Alperin's Fusion Theorem (see [34]) every $\mathcal{F}$-isomorphism on $\langle y\rangle$ is a composition of automorphisms of $\mathcal{F}$-essential subgroups containing y or of D itself. Assume that $E < D$ is $\mathcal{F}$-essential such that $\langle y\rangle \le E$. Since E is metacyclic and $\mathrm{Aut}(E)$ is not a 2-group, Lemma 1 in [35] implies $E \cong Q_8$ or $E \cong C_2 \times C_2$; in particular $|D| \le 16$. Moreover, Proposition 1.8 and Proposition 10.17 in [8] imply that D has maximal class, because every $\mathcal{F}$-essential subgroup is self-centralizing. This contradiction shows that there are no $\mathcal{F}$-essential subgroups containing y. Then of course $\langle y\rangle$ is fully $\mathcal{F}$-normalized.

Case 3: $|\mathrm{C}_D(x)/\langle x\rangle| = 2$.
Let $y \in \mathrm{C}_D(x) \setminus \langle x\rangle$ be of order 2. If $z \in D \setminus \mathrm{C}_D(x)$, we may assume that $\langle x, z\rangle$ is a modular 2-group by (i). In particular we have $|\langle z\rangle| = 2$ after replacing z by zx^m for some $m \in \mathbb{Z}$ if necessary. Let $|\langle x\rangle| = 2^r$ for some $r \in \mathbb{N}$. Since $\langle x\rangle \trianglelefteq D$, we have $zyz^{-1} \in \{y, yx^{2^{r-1}}\}$. In case $zyz^{-1} = yx^{2^{r-1}}$ it is easy to see that $|D : \langle xy\rangle| = 4$ and $xy \in \mathrm{Z}(D)$. Then we are done by Case 2. Thus, we may assume that $zyz^{-1} = y$ and $y \in \mathrm{Z}(D)$. Then D is given as follows:

$$D = \langle x, z\rangle \times \langle y\rangle \cong M_{2^{r+1}} \times C_2,$$

where $M_{2^{r+1}}$ denotes the modular 2-group of order 2^{r+1} and C_2 denotes a cyclic group of order 2. Now we have $|D'| = 2$ and the claim follows from Proposition 2.5 applied to the subsection (x, b_x). Here observe that $\langle x\rangle$ is fully $\mathcal{F}$-normalized, since $\langle x\rangle \trianglelefteq D$. □

We like to point out that every subgroup of D is fully $\mathcal{F}$-normalized whenever $\mathcal{F}$ is controlled by $\mathrm{Aut}_{\mathcal{F}}(D)$. The groups in Proposition 3.2 were given explicitly by generators and relations in [40].

By the propositions in [54] it is easy to see that Olsson's Conjecture holds for 2-blocks with defect at most 4. For defect groups D of order 32 one can show with GAP [19] that there is always an element $x \in D$ such that $|\mathrm{C}_D(x)| = |D : D'|$. If in addition D is abelian, Olsson's Conjecture follows from Corollary 2 in [54] for every block with defect group D. If D is non-abelian, then $|\mathrm{C}_D(x)/\langle x\rangle| \le 8$. Thus, by Proposition 2.5(ii) Olsson's Conjecture also holds for controlled 2-blocks of defect 5.

4 The case $p > 2$

Now we turn to the case where B is a p-block of G for an odd prime p. We fix some notations for this section. As before (u, b_u) is a B-subsection such that $|\langle u\rangle| = p^k$. Moreover, $\zeta \in \mathbb{C}$ is a primitive p^k-th root of unity. Since the situation is more complicated for odd primes, we assume further that $l(b_u) = 1$. We write $\mathrm{IBr}(b_u) = \{\varphi_u\}$. Then the generalized decomposition numbers $d^u_{\chi\varphi_u}$ for $\chi \in \mathrm{Irr}(B)$ form a column $d(u)$. Let d be the defect of b_u. Since $u \in \mathrm{Z}(\mathrm{C}_G(u))$, u is contained in every defect group of b_u. In particular, $k \le d$. As in the case $p = 2$ we can write

$$d(u) = \sum_{i=0}^{\varphi(p^k)-1} a^u_i \zeta^i$$

with $a^u_i \in \mathbb{Z}^{k(B)}$ (change of notation!). We define the following matrix

$$A := \big(a^u_i(\chi) : i = 0, \ldots, \varphi(p^k) - 1,\ \chi \in \mathrm{Irr}(B)\big) \in \mathbb{Z}^{\varphi(p^k) \times k(B)}.$$

The next lemma uses the same idea as in case $p = 2$.

Lemma 4.1. *Let (u, b_u) be a B-subsection with $|\langle u\rangle| = p^k$ and $l(b_u) = 1$.*

(i) For $\chi \in \mathrm{Irr}_0(B)$ we have

$$\sum_{i=0}^{\varphi(p^k)-1} a_i^u(\chi) \not\equiv 0 \pmod{p}.$$

(ii) If (u, b_u) is major and $\chi \in \mathrm{Irr}(B)$, then $p^{h(\chi)} \mid a_i^u(\chi)$ for $i = 0, \ldots, \varphi(p^k) - 1$ and

$$\sum_{i=0}^{\varphi(p^k)-1} a_i^u(\chi) \not\equiv 0 \pmod{p^{h(\chi)+1}}.$$

Proof.

(i) Since $l(b_u) = 1$, we have $p^d m_{\chi\chi}^{(u,b_u)} = d_{\chi\varphi_u}^u \overline{d_{\chi\varphi_u}^u}$ for the contribution $m_{\chi\chi}^{(u,b_u)}$ (see Eq. (5.2) in [9]). By Corollary 2 in [11] it follows that

$$p^d m_{\chi\chi}^{(u,b_u)} = p^d \big(\chi^{(u,b_u)}, \chi\big)_G \not\equiv 0 \pmod{(\pi)}$$

and $d_{\chi\varphi_u}^u \not\equiv 0 \pmod{(\pi)}$. Since $\zeta \equiv 1 \pmod{(\pi)}$, the claim follows from (2.1).

(ii) Let $\psi \in \mathrm{Irr}_0(B)$. Then (5G) in [9] implies

$$h(\chi) = \nu\big(|D| m_{\chi\psi}^{(u,b_u)}\big) = \nu(d_{\chi\varphi_u}^u) + \nu(\overline{d_{\psi\varphi_u}^u}),$$

where ν is the p-adic valuation. Thus, $h(\chi) = \nu(d_{\chi\varphi_u}^u)$ follows from (i). Now the claim is easy to see. □

The proof of the main theorem of this section is an application of the next proposition.

Proposition 4.2. *For every positive definite, integral quadratic form $q(x_1, \ldots, x_{\varphi(p^k)}) = \sum_{1 \le i \le j \le \varphi(p^k)} q_{ij} x_i x_j$ we have*

$$k_0(B) \le \sum_{1 \le i \le j \le \varphi(p^k)} q_{ij} (a_{i-1}^u, a_{j-1}^u). \tag{4.1}$$

If (in addition) (u, b_u) is major, we can replace $k_0(B)$ by $\sum_{i=0}^{\infty} p^{2i} k_i(B)$ in (4.1).

Proof. By Lemma 4.1(i) every column $a^u(\chi)$ of A corresponding to a character χ of height 0 does not vanish. Hence, we have

$$k_0(B) \le \sum_{\chi \in \mathrm{Irr}(B)} q(a^u(\chi)) = \sum_{\chi \in \mathrm{Irr}(B)} \sum_{1 \le i \le j \le \varphi(p^k)} q_{ij} a_{i-1}^u(\chi) a_{j-1}^u(\chi) = \sum_{1 \le i \le j \le \varphi(p^k)} q_{ij} (a_{i-1}^u, a_{j-1}^u).$$

If (u, b_u) is major and $\chi \in \mathrm{Irr}(B)$, then $p^{-h(\chi)} a^u(\chi)$ is a non-vanishing integral column by Lemma 4.1(ii). In this case we have

$$\sum_{i=0}^{\infty} p^{2i} k_i(B) \le \sum_{\chi \in \mathrm{Irr}(B)} p^{2h(\chi)} q(p^{-h(\chi)} a^u(\chi)) = \sum_{1 \le i \le j \le \varphi(p^k)} q_{ij} (a_{i-1}^u, a_{j-1}^u).$$

The second claim follows. □

Notice that we have used only a weak version of Lemma 4.1 in the proof above.

In order to find a suitable quadratic form it is often very useful to replace A by UA for some integral matrix $U \in \mathrm{GL}(\varphi(p^k), \mathbb{Q})$ (observe that the argument in the proof of Proposition 4.2 remains correct).

However, we need a more explicit expression of the scalar products (a_i^u, a_j^u). For this reason we introduce an auxiliary lemma about inverses of Vandermonde matrices. Let $\mathcal{G} = \{\sigma_1, \ldots, \sigma_{\varphi(p^k)}\}$. For an integer $i \in \mathbb{Z}$ there is $i' \in \{1, \ldots, p^{k-1}\}$ such that $-i \equiv i' \pmod{p^{k-1}}$. We will use this notation for the rest of the paper.

Lemma 4.3. *The inverse of the Vandermonde matrix* $V := \big(\sigma_i(\zeta)^{j-1}\big)_{i,j=1}^{\varphi(p^k)}$ *is given by*

$$V^{-1} = p^{-k}\big(\sigma_j(t_{i-1})\big)_{i,j=1}^{\varphi(p^k)},$$

where $t_i = \zeta^{-i} - \zeta^{i'}$.

Proof. For $i, j \in \{0, \ldots, \varphi(p^k) - 1\}$ we have

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(t_i)\sigma_l(\zeta)^j = \sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j-i} - \zeta^{j+i'}).$$

Assume first that $i = j$. Then $\zeta^{j-i} = 1$ and $j + i' = i + i'$ is divisible by p^{k-1} but not by p^k. Hence, $\zeta^{j+i'}$ is a primitive p-th root of unity. Since the second coefficient of the p-th cyclotomic polynomial $\Phi_p(X) = X^{p-1} + X^{p-2} + \ldots + X + 1$ is 1, we get $\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j+i'}) = -p^{k-1}$. This shows that

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(1 - \zeta^{i+i'}) = \varphi(p^k) + p^{k-1} = p^k.$$

Now let $i \neq j$. Then $j-i \not\equiv 0 \pmod{p^k}$ and $j+i' \not\equiv 0 \pmod{p^k}$. Moreover, $j-i \equiv j+i' \pmod{p^{k-1}}$, since $i+i' \equiv 0 \pmod{p^{k-1}}$. Assume first that $j-i \not\equiv 0 \pmod{p^{k-1}}$. Then ζ^{j-i} is a primitive p^s-th root of unity for some $s \geq 2$. Since the second coefficient of the p^s-th cyclotomic polynomial $\Phi_{p^s}(X) = X^{(p-1)p^{s-1}} + X^{(p-2)p^{s-1}} + \ldots + X^{p^{s-1}} + 1$ (see Lemma I.10.1 in [39]) is 0, we have $\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j-i}) = 0$. The same holds for $j + i'$. Finally let $j - i \equiv 0 \pmod{p^{k-1}}$. Then we have (as in the first part of the proof)

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j-i} - \zeta^{j+i'}) = -p^{k-1} + p^{k-1} = 0.$$

This proves the claim. □

Now let $\mathcal{A} := \mathrm{Aut}_{\mathcal{F}}(\langle u\rangle) \leq \mathcal{G}$. The next proposition shows that the scalar products (a_i^u, a_j^u) only depend on p, $k - d$ and $\mathcal{A}$.

Proposition 4.4. *We have*

$$\begin{aligned} p^{k-d}(a_i^u, a_j^u) = |\{\tau \in \mathcal{A} : p^k \mid i - j\tau\}| - |\{\tau \in \mathcal{A} : p^k \mid i + j'\tau\}| + \\ |\{\tau \in \mathcal{A} : p^k \mid i' - j'\tau\}| - |\{\tau \in \mathcal{A} : p^k \mid i' + j\tau\}|. \end{aligned} \tag{4.2}$$

Proof. Let $W := \big(d_{\chi\varphi_u}^{\sigma_i(u)} : i = 1, \ldots, \varphi(p^k),\ \chi \in \mathrm{Irr}(B)\big)$ be a part of the generalized decomposition matrix. If V is the Vandermonde matrix in Lemma 4.3, we have $VA = W$ and $A = V^{-1}W$. This shows

$$\big((a_{i-1}^u, a_{j-1}^u)\big)_{i,j=1}^{\varphi(p^k)} = AA^{\mathrm{T}} = V^{-1}WW^{\mathrm{T}}V^{-\mathrm{T}} = V^{-1}W\overline{W}^{\mathrm{T}}\overline{V}^{-\mathrm{T}}.$$

Now let $S := (s_{ij})_{i,j=1}^{\varphi(p^k)}$, where

$$s_{ij} := \begin{cases} 1 & \text{if } \sigma_i\sigma_j^{-1} \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the orthogonality relations (see proof of Theorem 3.1) imply $W\overline{W}^{\mathrm{T}} = p^d S$. It follows that

$$\begin{aligned} p^{2k-d}(a_i^u, a_j^u) &= \sum_{l=1}^{\varphi(p^k)} \sigma_l(t_i) \sum_{m=1}^{\varphi(p^k)} s_{lm}\sigma_m(\overline{t_j}) = \sum_{l=1}^{\varphi(p^k)} \sum_{\tau\in\mathcal{A}} \sigma_l(t_i\tau(\overline{t_j})) \\ &= \sum_{\tau\in\mathcal{A}} \sum_{l=1}^{\varphi(p^k)} \sigma_l((\zeta^{-i} - \zeta^{i'})\tau(\zeta^j - \zeta^{-j'})) \\ &= \sum_{\tau\in\mathcal{A}} \sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j\tau-i} + \zeta^{i'-j'\tau} - \zeta^{-i-j'\tau} - \zeta^{i'+j\tau}). \end{aligned} \tag{4.3}$$

As in the proof of Lemma 4.3 we have

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j\tau-i}) = \begin{cases} \varphi(p^k) & \text{if } p^k \mid j\tau - i, \\ 0 & \text{if } p^{k-1} \nmid j\tau - i, \\ -p^{k-1} & \text{otherwise.} \end{cases}$$

This can be combined to

$$\sum_{\tau\in\mathcal{A}} \sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j\tau-i}) = p^k|\{\tau \in \mathcal{A} : p^k \mid j\tau - i\}| - p^{k-1}|\{\tau \in \mathcal{A} : p^{k-1} \mid j\tau - i\}|.$$

We get similar expressions for the other numbers $i' - j'\tau$, $-i - j'\tau$ and $i' + j\tau$. Since $i + i' \equiv j + j' \equiv 0 \pmod{p^{k-1}}$, we have $j\tau - i \equiv i' - j'\tau \equiv -i - j'\tau \equiv i' + j\tau \pmod{p^{k-1}}$. Thus, the terms of the form $p^{k-1}|\{\ldots\}|$ in (4.3) cancel out each other. This proves the proposition. $\square$

Since the group $\mathrm{Aut}(\langle u\rangle)$ is cyclic, $\mathcal{A}$ is uniquely determined by its order. We introduce a notation.

Definition 4.5. *Let $\mathcal{A}$ be as in Proposition 4.4. Then we define $\Gamma(d, k, |\mathcal{A}|)$ as the minimum of the expressions*

$$\sum_{1\le i\le j\le\varphi(p^k)} q_{ij}(a^u_{i-1}, a^u_{j-1}),$$

where q ranges over all positive definite, integral quadratic forms. By Proposition 4.2 we have $k_0(B) \le \Gamma(d, k, |\mathcal{A}|)$, and $\sum_{i=0}^{\infty} p^{2i}k_i(B) \le \Gamma(d, k, |\mathcal{A}|)$ if (u, b_u) is major.

We will calculate $\Gamma(d, k, |\mathcal{A}|)$ by induction on k. First we collect some easy facts.

Lemma 4.6. *Let $\mathcal{H} \le (\mathbb{Z}/p^k\mathbb{Z})^\times$ where we regard $\mathcal{H}$ as a subset of $\{1, \ldots, p^k\}$. Then $|\{\sigma \in \mathcal{H} : \sigma \equiv 1 \pmod{p^j}\}| = \gcd(|\mathcal{H}|, p^{k-j})$ for $1 \le j \le k$.*

Proof. The canonical epimorphism $(\mathbb{Z}/p^k\mathbb{Z})^\times \to (\mathbb{Z}/p^j\mathbb{Z})^\times$ has kernel $\mathcal{K}$ of order p^{k-j}. Hence, $|\{\sigma \in \mathcal{H} : \sigma \equiv 1 \pmod{p^j}\}| = |\mathcal{H} \cap \mathcal{K}| = \gcd(|\mathcal{H}|, p^{k-j})$, since the p-subgroups of the cyclic group $(\mathbb{Z}/p^k\mathbb{Z})^\times$ are totally ordered by inclusion. $\square$

Lemma 4.7. *Let $|\mathcal{A}|_p$ be the order of a Sylow p-subgroup of $\mathcal{A}$. Then we have*

$$(a^u_0, a^u_0) = \big(|\mathcal{A}| + |\mathcal{A}|_p\big)p^{d-k}$$

and

$$\frac{p^{k-d}}{\gcd(|\mathcal{A}|_p, j)}(a^u_i, a^u_j) \in \{0, \pm1, \pm2\}$$

for $i + j > 0$. If $a^u_i \ne 0$ for some $i \ge 1$, then $(a^u_i, a^u_i) = 2p^{d-k}\gcd(|\mathcal{A}|_p, i)$. Moreover, $(a^u_i, a^u_j) = 0$ whenever $\gcd(i, p^{k-1}) \ne \gcd(j, p^{k-1})$.

Proof. For $i = j = 0$ we have $i + j'\tau = p^{k-1}\tau \not\equiv 0 \pmod{p^k}$ and $i' + j\tau = p^{k-1} \not\equiv 0 \pmod{p^k}$ for all $\tau \in \mathcal{A}$. Moreover, by Lemma 4.6 there are precisely $|\mathcal{A}|_p$ elements $\tau \in \mathcal{A}$ such that $i' - j'\tau = p^{k-1}(1-\tau) \equiv 0 \pmod{p^k}$. The first claim follows from Proposition 4.4.

Now let $i + j > 0$ and $\tau \in \mathcal{A}$ such that $i \equiv j\tau \pmod{p^k}$. Then we have $j \ne 0$. Assume that also $\tau_1 \in \mathcal{A}$ satisfies $i \equiv j\tau_1 \pmod{p^k}$. Then $j(\tau - \tau_1) \equiv 0 \pmod{p^k}$ and $\tau^{-1}\tau_1 \equiv 1 \pmod{p^k/\gcd(p^k, j)}$. Thus, Lemma 4.6 implies

$$|\{\tau \in \mathcal{A} : p^k \mid i - j\tau\}| \in \{0, \gcd(|\mathcal{A}|_p, j)\}.$$

The same argument also works for the other summands in (4.2), since $\gcd(|\mathcal{A}|_p, j) = \gcd(|\mathcal{A}|_p, j')$. This gives

$$p^{k-d}(a^u_i, a^u_j) \in \{0, \pm\gcd(|\mathcal{A}|_p, j), \pm2\gcd(|\mathcal{A}|_p, j)\}$$

whenever $i + j > 0$.

Suppose $i \geq 1$ and $i \equiv i\tau \pmod{p^k}$ for some $\tau \in \mathcal{A}$. Then $\tau \equiv 1 \pmod p$ and thus $i \equiv i\tau - (i+i')(\tau-1) \equiv -i'\tau + i + i' \pmod{p^k}$. Hence $i' \equiv i'\tau \pmod{p^k}$. This shows $|\{\tau \in \mathcal{A} : p^k \mid i - i\tau\}| = |\{\tau \in \mathcal{A} : p^k \mid i' - i'\tau\}|$. Moreover, we have $|\{\tau \in \mathcal{A} : p^k \mid i + i'\tau\}| = |\{\tau \in \mathcal{A} : p^k \mid i\tau^{-1} + i'\}| = |\{\tau \in \mathcal{A} : p^k \mid i' + i\tau\}|$. This shows $a_i^u = 0$ or $(a_i^u, a_i^u) = 2p^d \gcd(|\mathcal{A}|_p, i)/p^k$.

Finally suppose that $\gcd(i, p^{k-1}) \neq \gcd(j, p^{k-1})$. Then $i \not\equiv j\tau \pmod{p^{k-1}}$ and thus $p^k \nmid i - j\tau$ for all $\tau \in \mathcal{A}$. The same holds for the other terms in (4.2), since $i + i' \equiv j + j' \equiv 0 \pmod{p^{k-1}}$. The last claim follows. ☐

Proposition 4.8. *We have*

$$\Gamma(d, 1, |\mathcal{A}|) = \big(|\mathcal{A}| + (p-1)/|\mathcal{A}|\big)p^{d-1}.$$

Proof. Since $|\mathcal{A}| \mid p-1$, we have $|\mathcal{A}|_p = 1$. Hence, $(a_0^u, a_0^u) = (|\mathcal{A}|+1)p^{d-1}$ and $(a_i^u, a_j^u) \in \{0, \pm p^{d-1}, \pm 2p^{d-1}\}$ for $i + j > 0$ by Lemma 4.7. First we determine the indices i such that $a_i^u = 0$. For this we use Proposition 4.4. Observe that we always have $i' = 1$. In particular for all i, j we have $p \mid i' - j'\tau$ for $\tau = 1$. It follows that $a_i^u = 0$ if and only if $-i \equiv \tau \pmod p$ for some $\tau \in \mathcal{A}$. We write this condition in the form $-i \in \mathcal{A}$. This gives exactly $|\mathcal{A}| - 1$ vanishing rows and columns. Thus, all the scalar products (a_i^u, a_j^u) with $-i \in \mathcal{A}$ or $-j \in \mathcal{A}$ vanish. Hence, assume that $-i \notin \mathcal{A}$ and $-j \notin \mathcal{A}$. Then $(a_i^u, a_j^u) \in \{p^{d-1}, 2p^{d-1}\}$ for $i + j > 0$. In case $(a_i^u, a_j^u) = 2p^{d-1}$ we have $a_i^u = a_j^u$. This happens exactly when $j \neq 0$ and $ij^{-1} \in \mathcal{A}$. Since $-i \notin \mathcal{A}$, the coset $i\mathcal{A}$ in $\mathcal{G}$ does not contain -1. Hence, there are precisely $|\mathcal{A}|$ choices for j such that $ij^{-1} \in \mathcal{A}$.

Hence, we have shown that the rows a_i^u for $i = 1, \ldots, p-2$ split in $|\mathcal{A}| - 1$ zero rows and $(p-1)/|\mathcal{A}| - 1$ groups consisting of $|\mathcal{A}|$ equal rows each. If we replace the matrix A by UA for a suitable matrix $U \in \mathrm{GL}(p-1, \mathbb{Z})$, we get a new matrix with exactly $(p-1)/|\mathcal{A}|$ non-vanishing rows (this is essentially the same as taking another (positive definite) quadratic form in (4.1), see [32]). After leaving out the zero rows we get a $(p-1)/|\mathcal{A}| \times (p-1)/|\mathcal{A}|$ matrix

$$AA^{\mathrm{T}} = p^{d-1} \begin{pmatrix} |\mathcal{A}|+1 & 1 & \ldots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \ldots & 1 & 2 \end{pmatrix}.$$

Now we can apply the quadratic form q corresponding to the Dynkin diagram $A_{(p-1)/|\mathcal{A}|}$ in Eq. (4.1). This gives

$$\Gamma(d, 1, |\mathcal{A}|) \leq \big(|\mathcal{A}| + (p-1)/|\mathcal{A}|\big)p^{d-1}.$$

On the other hand $p^{1-d}AA^{\mathrm{T}}$ is the square of the matrix

$$\begin{pmatrix} 1 & \cdots & 1 & & & \\ 1 & & & 1 & & \\ \vdots & & & & \ddots & \\ 1 & & & & & 1 \end{pmatrix}$$

which has exactly $|\mathcal{A}| + (p-1)/|\mathcal{A}|$ columns. This shows that $\Gamma(d, 1, |\mathcal{A}|)$ cannot be smaller. ☐

The next proposition gives an induction step.

Proposition 4.9. *If* $|\mathcal{A}|_p \neq 1$*, then*

$$\Gamma(d, k, |\mathcal{A}|) = \Gamma(d, k-1, |\mathcal{A}|/p).$$

Proof. Since $|\mathcal{A}|_p \neq 1$, we have $k \geq 2$. Let $i \in \{1, \ldots, \varphi(p^k) - 1\}$ such that $\gcd(i, p) = 1$. We will see that $(a_i^u, a_i^u) = 0$ and thus $a_i^u = 0$. By Lemma 4.7 and Eq. (4.2) it suffices to show that there is some $\tau \in \mathcal{A}$ such that $p^k \mid i' + i\tau$. We can write this in the form $-i^{-1}i' \in \mathcal{A}$, since i represents an element of $(\mathbb{Z}/p^k\mathbb{Z})^\times$. Now let $-i' = i + \alpha p^{k-1}$ for some $\alpha \in \mathbb{Z}$. Then $-i^{-1}i' = 1 + i^{-1}\alpha p^{k-1}$ is an element of order p in $\mathcal{G}$. Since $\mathcal{G}$ has only one subgroup of order p, it follows that $-i^{-1}i' \in \mathcal{A}$.

Hence, in order to apply Proposition 4.2 it remains to consider the indices which are divisible by p. Let $\overline{\mathcal{A}}$ be the image of the canonical map $(\mathbb{Z}/p^k\mathbb{Z})^\times \to (\mathbb{Z}/p^{k-1}\mathbb{Z})^\times$ under $\mathcal{A}$. Then $|\overline{\mathcal{A}}| = |\mathcal{A}|/p$ (cf. Lemma 4.6). If i and j are divisible by p, we have

$$|\{\tau \in \mathcal{A} : p^k \mid i + j\tau\}| = p \cdot |\{\tau \in \overline{\mathcal{A}} : p^{k-1} \mid (i/p) + (j/p)\tau\}|.$$

A similar equality holds for the other summands in (4.2). Here observe that $(i/p)' = i'/p$, where the dash on the left refers to the case p^{k-1}. Thus, the remaining matrix is just the matrix in case p^{k-1}. Hence $\Gamma(d, k, |\mathcal{A}|) = \Gamma(d, k-1, |\overline{\mathcal{A}}|) = \Gamma(d, k-1, |\mathcal{A}|/p)$. □

Now we are in a position to prove the main theorem of this section.

Theorem 4.10. *Let B be a p-block of a finite group G where p is an odd prime, and let (u, b_u) be a B-subsection such that $l(b_u) = 1$ and b_u has defect d. Moreover, let $\mathcal{F}$ be the fusion system of B and $|\mathrm{Aut}_{\mathcal{F}}(\langle u\rangle)| = p^s r$, where $p \nmid r$ and $s \ge 0$. Then we have*

$$k_0(B) \le \frac{|\langle u\rangle| + p^s(r^2-1)}{|\langle u\rangle| \cdot r} p^d. \tag{4.4}$$

If (in addition) (u, b_u) is major, we can replace $k_0(B)$ by $\sum_{i=0}^{\infty} p^{2i} k_i(B)$ in (4.4).

Proof. As before let $|\langle u\rangle| = p^k$. We will prove by induction on k that

$$\Gamma(d, k, p^s r) = \frac{p^k + p^s(r^2-1)}{p^k r} p^d.$$

By Proposition 4.8 we may assume $k \ge 2$. By Proposition 4.9 we can also assume that $s = 0$. As before we consider the matrix A. Like in the proof of Proposition 4.9 it is easy to see that the indices divisible by p form a block of the matrix AA^{T} which contributes $\Gamma(d, k-1, r)/p$ to $\Gamma(d, k, r)$. It remains to deal with the matrix $\widetilde{A} := \big(a_i^u : \gcd(i,p) = 1\big)$. By Lemma 4.7 the entries of $p^{k-d}\widetilde{A}\widetilde{A}^{\mathrm{T}}$ lie in $\{0, \pm1, \pm2\}$. Moreover, if $\gcd(i,p) = 1$ we have $(a_i^u, a_i^u) = 2p^{d-k}$ (see proof of Proposition 4.9).

With the notation of the proof of Proposition 4.4 we have $VA = W$. In particular $\mathrm{rk}\, AA^{\mathrm{T}} = \mathrm{rk}\, A = \mathrm{rk}\, W = |\mathcal{G} : \mathcal{A}|$. If we set $A_1 := \big(a_i^u : \gcd(i,p) > 1\big)$, it also follows that $\mathrm{rk}\, A_1A_1^{\mathrm{T}} = \mathrm{rk}\, A_1 = \varphi(p^{k-1})/r$. Since the rows of $\widetilde{A}$ are orthogonal to the rows of A_1 (see Lemma 4.7), we see that $\mathrm{rk}\, \widetilde{A} = (\varphi(p^k) - \varphi(p^{k-1}))/r = p^{k-2}(p-1)^2/r$.

Now we will find $p^{k-2}(p-1)^2/r$ linearly independent rows of $\widetilde{A}$. For this observe that $\mathcal{A}$ acts on $\Omega := \{i : 1 \le i \le p^{k-1},\ \gcd(i,p) = 1\}$ by ${}^\tau i := \tau \cdot i \pmod{p^{k-1}}$ for $\tau \in \mathcal{A}$. Since $p \nmid r$, every orbit has length r (see Lemma 4.6). We choose a set of representatives Δ for these orbits. Then $|\Delta| = p^{k-2}(p-1)/r$. Finally for $i \in \Delta$ we set $\Delta_i := \{i + jp^{k-1} : j = 0, \ldots, p-2\}$. We claim that the rows a_i^u with $i \in \bigcup_{j\in\Delta} \Delta_j$ are linearly independent. We do this in two steps.

Step 1: $(a_i^u, a_j^u) = 0$ for $i, j \in \Delta$, $i \ne j$.
We will show that all summands in (4.2) vanish. First assume that $i \equiv j\tau \pmod{p^k}$ for some $\tau \in \mathcal{A}$. Then of course we also have $i \equiv j\tau \pmod{p^{k-1}}$ which contradicts the choice of Δ. Exactly the same argument works for the other summands. For the next step we fix some $i \in \Delta$.

Step 2: a_j^u for $j \in \Delta_i$ are linearly independent.
It suffices to show that the matrix $A' := p^{k-d}(a_l^u, a_m^u)_{l,m\in\Delta_i}$ is invertible. We already know that the diagonal entries of A' equal 2. Now write $m = l + jp^{k-1}$ for some $j \ne 0$. We consider the summands in (4.2). Assume that there is some $\tau \in \mathcal{A}$ such that $l \equiv m\tau \equiv (l + jp^{k-1})\tau \pmod{p^k}$. Then we have $\tau \equiv 1 \pmod{p^{k-1}}$ which implies $\tau = 1$. However, this contradicts $j \ne 0$. On the other hand we have $l' \equiv m'\tau \equiv l'\tau \pmod{p^k}$ for $\tau = 1 \in \mathcal{A}$. Now assume $-l \equiv m'\tau \pmod{p^k}$. Then the argument above implies $\tau = 1$ and $l + l' \equiv 0 \pmod{p^k}$ which is false. Similarly the last summand in (4.2) equals 0. Thus, we have shown that $A' = (1 + \delta_{lm})_{l,m\in\Delta_i}$ is invertible.

This implies that the rank of $\widetilde{A}$ is $p^{k-2}(p-1)^2/r$. Hence, there exists an integral matrix $U \in \mathrm{GL}(p^{k-2}(p-1)^2, \mathbb{Q})$ such that the only non-zero rows of $U\widetilde{A}$ are a_i^u for $i \in \bigcup_{j\in\Delta} \Delta_j$. Then we can leave out the zero rows and obtain a matrix (still denoted by $\widetilde{A}$) of dimension $p^{k-2}(p-1)^2/r$. Moreover, the two steps above show that $p^{k-d}\widetilde{A}\widetilde{A}^{\mathrm{T}}$

consists of $p^{k-2}(p-1)/r$ blocks of the form $(1+\delta_{ij})_{1\le i,j\le p-1}$. Thus, an application of the quadratic form q corresponding to the Dynkin diagram $A_{p^{k-2}(p-1)^2/r}$ in Eq. (4.1) gives

$$\Gamma(d,k,r) \le \frac{\Gamma(d,k-1,r)}{p} + \frac{p^{k-1}(p-1)}{p^k r}p^d = \frac{p^k+r^2-1}{p^k r}p^d.$$

The minimality of $\Gamma(d,k,r)$ is not so clear as in the proof of Proposition 4.8, since here we do not know if $\det U \in \{\pm 1\}$. However, it suffices to give an example where $k_0(B) = \Gamma(d,k,r)$. By Proposition 4.4 we already know that $\Gamma(d,k,r) = p^{d-k}\Gamma(k,k,r)$. Hence, we may assume $d = k$. Let $G = \langle u\rangle \rtimes C_r$ and B be the principal block of G. Then it is easy to see that the hypothesis of the theorem is satisfied. Moreover,

$$k_0(B) = k(B) = \frac{|D|-1}{r} + r = \Gamma(d,k,r).$$

Hence, the proof is complete. □

We add some remarks. It is easy to see that the right hand side of (4.4) is always an integer. Moreover, if $\mathcal{A} = \mathcal{G}$ (i. e. $s = k-1$ and $r = p-1$) or $\mathcal{A}$ is a p-group (i. e. $r = 1$), we get the same bound as in Proposition 2.1 and Proposition 2.2. In all other cases Theorem 4.10 really improves Proposition 2.1 and Proposition 2.2. For $k \ge 2$ the case $s = 0$ and $r = p-1$ gives the best bound for $k_0(B)$. If k tends to infinity, $\Gamma(d,k,p-1)$ goes to $p^d/(p-1)$.

Coming back to our intended task, i. e. to prove Olsson's Conjecture (in some cases), we have to say (in contrast to the case $p = 2$) that Olsson's Conjecture does not follow from Theorem 4.10 if it does not already follow from Proposition 2.1, since the right hand side of (4.4) is always larger than p^{d-1}.

In the proof we already saw that Inequality (4.4) is sharp for blocks with cyclic defect groups. Perhaps it is possible that this can provide a more elementary proof of Dade's Theorem. For this it would be sufficient to bound $l(B)$ from below, since the difference $k(B) - l(B)$ is locally determined.

As an application of Theorem 4.10 we give a concrete example. Let B be an 11-block with defect group $D \cong C_{11} \times C_{11}$ (for smaller primes results by Usami and Puig give more complete informations, e. g. [59, 45]). Then it is known that Brauer's $k(B)$-Conjecture and thus Olsson's Conjecture hold. However, the precise values for $k(B)$ and $l(B)$ are unknown. Since D is abelian, the fusion system $\mathcal{F}$ is controlled by $\mathrm{Aut}_{\mathcal{F}}(D)$. Assume that $\mathrm{Aut}_{\mathcal{F}}(D)$ has order 5 and acts only on one factor C_{11} non-trivially. Then there are two non-trivial (major) subsections (u, b_u) and (v, b_v) such that $l(b_u) = 1$ and $l(b_v) = 5$. Moreover, $|\mathcal{A}| = 5$ and Theorem 4.10 implies $k(B) \le 77$ which is better than Brauer's $k(B)$-Conjecture. On the other hand the block b_v dominates a block of $\mathrm{C}_G(v)/\langle v\rangle$ with cyclic defect group $D/\langle v\rangle \cong C_{11}$. Hence, the Cartan matrix of b_v has the form $11(2+\delta_{ij})_{1\le i,j\le 5}$ up to basic sets (see [14, 48]). Now Theorem 2.4 applied to (v, b_v) gives exactly the same bound on $k(B)$. Under the action of $\mathrm{Aut}_{\mathcal{F}}(D)$ the group D splits in 11 orbits of length 1 and 22 orbits of length 5. Hence, by Theorem 5.9.4 in [37] we get $k(B) - l(B) = 72$. Since B is centrally controlled, Theorem 1.1 in [31] implies $l(B) \ge l(b_v) = 5$. This shows $k(B) = 77$ and $l(B) = 5$ (this can also be obtained from Corollary 2 in [60]). By Theorem IV.4.18 in [18] we also have $k_0(B) = k(B)$, because B has defect 2.

Now assume that $\mathrm{Aut}_{\mathcal{F}}(D)$ acts diagonally (and thus fixed point freely) on both factors C_{11}. Then we have $l(b_u) = 1$ for all non-trivial subsections (u, b_u). Thus, Theorem 2.4 and Theorem 1 in [60] become useless in this situation, but Theorem 4.10 still implies $k(B) \le 77$. However, for the principal block B of $G = D \rtimes \mathrm{Aut}_{\mathcal{F}}(D)$ we have $k(B) = k_0(B) = 29$ and $l(B) = 5$.

As was pointed out earlier, for odd primes p and $l(b_u) > 1$ there is not always a stable character in $\mathrm{IBr}(b_u)$ under $\mathrm{N}_G(\langle u\rangle, b_u)$, even for $l(b_u) = 2$ (see Proposition (2E)(ii) and the example following it in [28]). However, the situation is better if we consider the principal block.

Proposition 4.11. *Let B be the principal p-block of G for an odd prime p, and let (u, b_u) be a B-subsection such that $l(b_u) = 2$, and b_u has defect d and Cartan matrix $C_u = (c_{ij})$. Then we may replace C_u by an equivalent matrix such that $p^d c_{11}/\det C_u$ is divisible by p. Moreover, let $\mathcal{F}$ be the fusion system of B and $|\mathrm{Aut}_{\mathcal{F}}(\langle u\rangle)| = p^s r$, where $p \nmid r$ and $s \ge 0$. Then we have*

$$k_0(B) \le \frac{|\langle u\rangle| + p^s(r^2-1)}{|\langle u\rangle| \cdot r} c_{11}.$$

Proof. By Brauer's third main theorem b_u is the principal block of $\mathrm{C}_G(u)$ and so $\mathrm{IBr}(b_u)$ contains the trivial Brauer character. Hence, both characters of $\mathrm{IBr}(b_u)$ are stable under $\mathrm{N}_G(\langle u\rangle)$. As in the proof of Theorem 3.1, $\frac{p^d}{\det C_u}C_u \pmod p$ has rank 1. Hence, we can replace C_u by an equivalent matrix (still denoted by $C_u = (c_{ij})$) such that $p^d c_{11}/\det C_u$ and $p^d c_{12}/\det C_u$ are divisible by p. As in the proof of Theorem 3.1, the rows d_i^u and a_j^i become $\widehat{d_i^u}$ and $\widehat{a}_j^i$ for $i = 1, 2$ and $j = 0, \ldots, \varphi(|\langle u\rangle|) - 1$. Write $p^d C_u^{-1} = (\widetilde{c}_{ij})$. For $\chi \in \mathrm{Irr}_0(B)$ we have

$$0 \not\equiv p^d m_{\chi\chi}^{(u,b_u)} \equiv \widetilde{c}_{11}\big(\widehat{d}^u_{\chi\varphi_1}\big)^2 \pmod{(\pi)};$$

in particular $\widehat{a}_j^1(\chi) \neq 0$ for some $j \in \{0, \ldots, \varphi(p^k) - 1\}$. Now since

$$(\widehat{d_1^u}, \gamma(\widehat{d_1^u})) = \begin{cases} c_{11} & \text{if } \gamma \in \mathcal{A}, \\ 0 & \text{if } \gamma \in \mathcal{G} \setminus \mathcal{A}, \end{cases}$$

the proof works as in case $l(b_u) = 1$. □

5 Controlled blocks

In this section we will use Proposition 2.1 to show that Olsson's Conjecture is satisfied for controlled blocks with certain defect groups. Here a block B of G with defect group D is *controlled* if $\mathrm{N}_G(D, b_D)$ controls the fusion system $\mathcal{F}$ of B (see Section 2 for notations). Recall that in this situation all subgroups of D are fully $\mathcal{F}$-normalized. In particular for a subsection (u, b_u) the block b_u has defect group $\mathrm{C}_D(u)$ (cf. Proposition 2.3). Our strategy will be to find a subsection (u, b_u) such that $l(b_u) = 1$ and $|\mathrm{C}_D(u)| = |D : D'|$. Then Olsson's Conjecture follows from Proposition 2.1. Observe that the inequality $|D : \mathrm{C}_D(u)| \le |D'|$ always holds by elementary group theory. The next proposition gives a general criterion for this situation.

Proposition 5.1. *Let B be a controlled block of G with defect group D. Suppose that there exists an element $u \in D$ such that $|D : \mathrm{C}_D(u)| = |D'|$ and $\mathrm{N}_G(D, b_D) \cap \mathrm{C}_G(u) \subseteq \mathrm{C}_D(u)\,\mathrm{C}_G(\mathrm{C}_D(u))$. Then Olsson's Conjecture holds for B.*

Proof. By Proposition 2.1(b) in [3], also b_u is a controlled block and it suffices to show that b_u has inertial index 1, since then b_u is nilpotent and $l(b_u) = 1$. Observe that $(\mathrm{C}_D(u), b_{\mathrm{C}_D(u)})$ is a maximal b_u-subpair. Hence, Proposition 2.2 in [3] implies

$$\mathrm{N}_G(\mathrm{C}_D(u), b_{\mathrm{C}_D(u)}) = \big[\mathrm{N}_G(D, b_D) \cap \mathrm{N}_G(\mathrm{C}_D(u), b_{\mathrm{C}_D(u)})\big]\,\mathrm{C}_G(\mathrm{C}_D(u)) = \big[\mathrm{N}_G(D, b_D) \cap \mathrm{N}_G(\mathrm{C}_D(u))\big]\,\mathrm{C}_G(\mathrm{C}_D(u)).$$

Thus,

$$\mathrm{N}_{\mathrm{C}_G(u)}(\mathrm{C}_D(u), b_{\mathrm{C}_D(u)}) = \big[\mathrm{N}_G(D, b_D) \cap \mathrm{N}_G(\mathrm{C}_D(u)) \cap \mathrm{C}_G(u)\big]\,\mathrm{C}_G(\mathrm{C}_D(u)) = \mathrm{C}_D(u)\,\mathrm{C}_G(\mathrm{C}_D(u)),$$

and the claim follows. □

Recall that the inertial quotient $\mathrm{N}_G(D, b_D)/D\,\mathrm{C}_G(D)$ is always a p'-group. Thus, we can formulate Proposition 5.1 in the following abstract setting. Let P be a finite p-group and let A be a p'-group of automorphisms on P. Then we can form the semidirect product $G := P \rtimes A$. The conclusion of Proposition 5.1 applies if we find an element $u \in P$ such that $|P : \mathrm{C}_P(u)| = |P'|$ and $\mathrm{C}_G(u) \le P$. Observe that the requirement $\mathrm{C}_A(u) = 1$ alone is not sufficient, since for a P-conjugate v of u we might have $\mathrm{C}_A(v) \neq 1$. In the following results we verify this condition for several families of 2-generator p-groups. We start with a useful lemma.

Lemma 5.2. *Let P be a p-group such that $|P : \Phi(P)| \le p^2$. Let $A \le \mathrm{Aut}(P)$ be a p'-group and $G = P \rtimes A$. If P contains an A-invariant maximal subgroup C, then there is an element $u \in P \setminus C$ such that $\mathrm{C}_G(u) \le P$.*

Proof. In case $|P : \Phi(P)| = p$ the claim is trivial. Hence, assume $|P : \Phi(P)| = p^2$. By Maschke's Theorem there is another A-invariant maximal subgroup C_1 of P. Let $u \in P \setminus (C \cup C_1)$. Then $\mathrm{C}_A(u)$ acts trivially on $\langle u\rangle\Phi(P)/\Phi(P)$. Since $P/\Phi(P) = C/\Phi(P) \times C_1/\Phi(P)$, it follows that $\mathrm{C}_A(u)$ acts trivially on $C/\Phi(P)$ and on P/C. This shows $\mathrm{C}_A(u) = 1$, because A is a p'-group. By way of contradiction assume that $\mathrm{C}_G(u)$ is not a p-group. Let $\alpha \in \mathrm{C}_G(u)$ be a non-trivial p'-element. By Schur-Zassenhaus α is P-conjugate to an element of A. In particular $\mathrm{C}_A(v) \neq 1$ for some P-conjugate v of u. However, this contradicts the first part of the proof, since $v \in P \setminus (C \cup C_1)$. □

Proposition 5.3. *Let p be an odd prime, and let P be a p-group of maximal class with $|P| \geq p^4$. If $A \leq \operatorname{Aut}(P)$ is a p'-group and $G = P \rtimes A$, then there exists an element $u \in P$ such that $|P : \mathrm{C}_P(u)| = |P'|$ and $\mathrm{C}_G(u) \leq P$.*

Proof. Let $|P| = p^n$. We denote the terms of the lower central series of P by $P_2 = P'$, $P_3 = [P_2, P]$, etc. Then $P_1 := \mathrm{C}_P(P_2/P_4)$ is a characteristic maximal subgroup of P by Hilfssatz III.14.4 in [25]. Moreover, Hauptsatz III.14.6(a) tells us that the set $\{\mathrm{C}_P(P_i/P_{i+2}) : 2 \leq i \leq n-2\}$ contains at most one subgroup $C := \mathrm{C}_P(P_{n-2}) < P$ different from P_1. By (the proof of) Lemma 5.2 there exists an element $u \in P \setminus (P_1 \cup C)$ such that $\mathrm{C}_G(u) \leq P$. By Hilfssatz III.14.13 in [25] we also have $|P : \mathrm{C}_P(u)| = |P'|$. $\square$

Proposition 5.4. *Let p be an odd prime, and let P be a p-group such that P' is cyclic, $|P : \Phi(P)| = p^2$ and $|P| \geq p^4$. If $A \leq \operatorname{Aut}(P)$ is a p'-group and $G = P \rtimes A$, then there exists an element $u \in P$ such that $|P : \mathrm{C}_P(u)| = |P'|$ and $\mathrm{C}_G(u) \leq P$.*

Proof. Assume first that P is abelian. By Lemma 5.2 we may assume $P \cong C_{p^s} \times C_{p^s}$ for some $s \geq 2$. Since $\mathrm{C}_G(u) = P\,\mathrm{C}_A(u)$ for all $u \in P$, it suffices to show $\mathrm{C}_A(u) = 1$ for some $u \in P$. After replacing P by $\Omega_2(P)$, we may also assume that $s = 2$. Let $x \in P \setminus \Phi(P)$. Suppose that $A_1 := \mathrm{C}_A(x) \neq 1$. Since A_1 acts faithfully on $\Omega_1(P)$, we have $\mathrm{C}_P(A_1) = \langle x \rangle$. The group $A_2 := \mathrm{C}_A(x^p)$ must be cyclic, since it acts faithfully on $\Omega_1(P)/\langle x^p \rangle$. Thus, it follows from $A_1 \leq A_2$ that A_2 acts on $\langle x \rangle = \mathrm{C}_P(A_1)$. But since A_2 fixes $x^p \in \Omega_1(\langle x \rangle)$, we derive $A_1 = A_2$. Now choose an element $u \in P$ such that $\Omega_1(P) \subseteq \langle x, u \rangle$ and $\langle u^p \rangle = \langle x^p \rangle$. Then $\mathrm{C}_A(u) = \mathrm{C}_A(u) \cap \mathrm{C}_A(u^p) = \mathrm{C}_{A_2}(u) = \mathrm{C}_{A_1}(u) \subseteq \mathrm{C}_A(\Omega_1(P)) = 1$.

Now suppose that P has class 2. Then for $P = \langle a, b \rangle$ we have $P' = \langle [a, b] \rangle = \{[a, b^n] : n \in \mathbb{Z}\} = \{[a, x] : x \in P\}$. In particular $|P : \mathrm{C}_P(u)| = |P'|$ for all $u \in P \setminus \Phi(P)$. Hence, it suffices to show $\mathrm{C}_A(u) = 1$ for all u in a certain P-conjugacy class lying in $P \setminus \Phi(P)$ (compare with proof of Lemma 5.2). For this we may replace P by P/P'. In case $|P : P'| > p^2$ the claim follows from the arguments above. Thus, assume $|P : P'| = p^2$. Then $P' = \mathrm{Z}(P)$ and $|P'| = p$. This contradicts $|P| \geq p^4$.

Finally let P be a group of class at least 3. Then $P' \nsubseteq \mathrm{Z}(P)$ and $1 \neq P/\mathrm{C}_P(P') \leq \operatorname{Aut}(P')$ is cyclic. Hence, $C := \mathrm{C}_P(P')\Phi(P)$ is a characteristic maximal subgroup of P. By Lemma 5.2 there is an element $u \in P \setminus C$ such that $\mathrm{C}_G(u) \leq P$. Choose $x \in \mathrm{C}_P(P')$ such that $P = \langle u, x \rangle$. Now $N := \langle x \rangle P'$ is an abelian normal subgroup of P, and $P/N = \langle uN \rangle$ is cyclic. Thus, Aufgabe 2 on page 259 of [25] implies that $P' = \{[y, u] : y \in N\} = \{[y, u] : y \in P\}$; in particular, we have $|P'| = |P : \mathrm{C}_P(u)|$. $\square$

We observe that $\mathrm{GL}(2, p)$ contains a p'-subgroup A of order $2(p-1)^2$ which is bigger than p^2 for $p > 3$. Thus, when P is elementary abelian of order p^2, then there is no regular orbit of A on P.

Proposition 5.5. *Let p be an odd prime, and let P be a p-group of p-rank 2 with $|P| \geq p^4$. If $A \leq \operatorname{Aut}(P)$ is a p'-group and $G = P \rtimes A$, then there exists an element $u \in P$ such that $|P : \mathrm{C}_P(u)| = |P'|$ and $\mathrm{C}_G(u) \leq P$.*

Proof. By Theorem A.1 in [15], a result of Blackburn, there are four cases to consider. The metacyclic case follows from Proposition 5.4. In the next case P is a 3-group of maximal class and the result holds by Proposition 5.3.

Now suppose that P is presented as

$$P = \langle a, b, c \mid a^p = b^p = c^{p^{n-2}} = [a, c] = [b, c] = 1,\ [a, b] = c^{p^{n-3}} \rangle$$

for some $n \geq 4$. Then it is easy to see that $P = \Omega_1(P) * \mathrm{Z}(P)$, where $\Omega_1(P) = \langle a, b \rangle$ is a non-abelian group of order p^3 and exponent p, and $\mathrm{Z}(P) = \langle c \rangle$ is cyclic of order p^{n-2}. Thus, $|P'| = p$. Then

$$\mathcal{U} := \{x \in P \setminus \mathrm{Z}(P) : |\langle x \rangle| = p^{n-2}\} \neq \varnothing.$$

For $u \in \mathcal{U}$ we have $\mathrm{C}_A(u) \leq \mathrm{C}_A(u^p) = \mathrm{C}_A(c^p)$. Hence, $\mathrm{C}_A(u)$ acts trivially on $\mathrm{Z}(P) = \langle c \rangle$ and on $\langle u, c \rangle$. Now Problem 4D.1 in [26] implies $\mathrm{C}_A(u) = 1$ for all $u \in \mathcal{U}$. Since $\mathcal{U}$ is closed under conjugation in P, we obtain $\mathrm{C}_G(u) \leq P$ easily (compare with proof of Lemma 5.2). Obviously, we also have $|P : \mathrm{C}_P(u)| = p = |P'|$ for all $u \in \mathcal{U}$.

Finally, it remains to handle the case

$$P = \langle a, b, c \mid a^p = b^p = c^{p^{n-2}} = [b, c] = 1,\ [a, b^{-1}] = c^{\epsilon p^{n-3}},\ [a, c] = b \rangle,$$

where $n \geq 4$ and ϵ is 1 or a fixed quadratic non-residue modulo p. Obviously, $P = \langle a, c\rangle$ and $P' = \langle b, c^{p^{n-3}}\rangle \cong C_p \times C_p$. Moreover, $\mathrm{C}_P(P') = \langle b, c\rangle$ is abelian and maximal in P. Hence, by Lemma 5.2 we find an element $u \in P \setminus \mathrm{C}_P(P')$ such that $\mathrm{C}_G(u) \leq P$. It remains to show $|P : \mathrm{C}_P(u)| = |P'|$. By way of contradiction suppose that $\mathrm{C}_P(u)$ is maximal in P. Then $\Phi(P) = \mathrm{C}_P(P') \cap \mathrm{C}_P(u) \subseteq \mathrm{C}_P(\langle \mathrm{C}_P(P'), u\rangle) = \mathrm{Z}(P)$. Thus, P is minimal non-abelian and we get the contradiction $|P'| = p$. This completes the proof. □

Theorem 5.6. *Let D be a finite p-group, where p is an odd prime, and suppose that one of the following holds:*

(i) D has p-rank 2,

(ii) D has maximal class,

(iii) D' is cyclic and $|D : \Phi(D)| = p^2$.

Then Olsson's Conjecture holds for all controlled blocks with defect group D.

Proof. In case $|D| \leq p^3$ the claim follows easily from Proposition 2.5(i) and Theorem VII.10.14 in [18] (observe that D is not elementary abelian of order p^3). The other cases are consequences of the previous propositions. □

As mentioned earlier in this paper, Olsson's Conjecture holds also for 2-blocks with maximal class defect groups. We also like to point out that Olsson's Conjecture for controlled blocks with maximal class defect groups follows easily from Proposition 2.5(i) (without considering the action of an automorphism group). In connection with (iii) in Theorem 5.6 we mention that by a result of Burnside, D' is already cyclic if $\mathrm{Z}(D')$ is (see Satz III.7.8 in [25]).

If u is an element of D such that $|D : \mathrm{C}_D(u)| = |D'|$, then $D' = \{[u, v] : v \in D\}$; in particular, every element in D' is a commutator. Thus, one cannot expect to prove Olsson's Conjecture for all possible defect groups in this way (see for example [22]).

6 Defect groups of p-rank 2

In this section we discuss Olsson's Conjecture for blocks which are not necessarily controlled. We begin with a special case for which the method of the previous section does not suffice. For this reason we use the classification of finite simple groups.

Proposition 6.1. *Let B be a block of a finite group G with a non-abelian defect group D of order 5^3 and exponent 5. Suppose that the fusion system $\mathcal{F}$ of B is the same as the fusion system of the sporadic simple Thompson group Th for the prime 5. Then B is Morita equivalent to the principal 5-block of Th; in particular, Olsson's Conjecture holds for B.*

Proof. By Fong reduction, we may assume that $\mathrm{O}_{5'}(G)$ is central and cyclic (cf. Section IV.6 in [7]). The ATLAS [13] shows that Th has a unique conjugacy class of elements of order 5. Thus, by our hypothesis, all non-trivial B-subsections are conjugate in G; in particular, all B-subsections are major. Since $\mathrm{O}_5(G) \leq D$, this implies that $\mathrm{O}_5(G) = 1$. Thus $\mathrm{F}(G) = \mathrm{Z}(G) = \mathrm{O}_{5'}(G)$.

Let $N/\mathrm{Z}(G)$ be a minimal normal subgroup of $G/\mathrm{Z}(G)$. By Fong reduction, we may assume that B covers a unique block b of N. Then $D \cap N$ is a defect group of b. By Fong reduction, we may also assume that $D \cap N \neq 1$. Since all non-trivial B-subsections are conjugate in G this implies that $D \cap N = D$, i.e. $D \subseteq N$. In particular, $N/\mathrm{Z}(G)$ is the only minimal normal subgroup of $G/\mathrm{Z}(G)$. Hence $N = \mathrm{F}^*(G)$, and $\mathrm{E}(G)$ is a central product of the components $L_1, \ldots, L_n$ of G.

For $i = 1, \ldots, n$, b covers a unique block b_i of L_i. Let D_i be a defect group of b_i. Then $D_1 \times \ldots \times D_n$ is a defect group of b (since $\mathrm{O}_5(G) = 1$). Thus $D_1 \times \ldots \times D_n \cong D$. This shows that we must have $n = 1$. Hence $\mathrm{E}(G)$ is quasisimple, and $S := \mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G))$ is simple. Since $\mathrm{F}^*(G) = \mathrm{E}(G)\,\mathrm{F}(G) = \mathrm{E}(G)\,\mathrm{Z}(G)$, we conclude that $\mathrm{C}_G(\mathrm{E}(G)) = \mathrm{C}_G(\mathrm{F}^*(G)) = \mathrm{Z}(\mathrm{F}(G)) = \mathrm{Z}(G)$, so that $G/\mathrm{Z}(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{E}(G))$.

Now we discuss the various possibilities for S, by making use of the classification of finite simple groups. In each case we apply [4].

If S is an alternating group then, by Section 2 in [4], the block b cannot exist. Similarly, if S is exceptional group of Lie type then, by Theorem 5.1 in [4], the block b cannot exist.

Now suppose that S is a classical group. Then, by Theorem 4.5 in [4], $p = 5$ must be the defining characteristic of S. Moreover, S has to be isomorphic to $\mathrm{PSL}(3, 5)$ or $\mathrm{PSU}(3, 5)$. Also, D is a Sylow 5-subgroup of $\mathrm{E}(G)$. But now the ATLAS shows that S contains non-conjugate elements $\overline{x}$ and $\overline{y}$ of order 5 such that $|\mathrm{C}_S(\overline{x})| \neq |\mathrm{C}_S(\overline{y})|$. Thus there are elements x and y of order 5 in $\mathrm{E}(G)$ which are not conjugate in G. This contradicts the fact that all non-trivial B-subsections are conjugate in G.

The only remaining possibility is that S is a sporadic simple group. Then Table 1 in [4] implies that $S \in \{HS, McL, Ru, Co_2, Co_3, Th\}$. In all cases D is a Sylow 5-subgroup of S. In the first five cases we derive a contradiction as above, using the ATLAS. So we may assume that $S = Th$. Since Th has trivial Schur multiplier and trivial outer automorphism group, we must have $G = S \times \mathrm{Z}(G)$. Thus $B \cong b \otimes_{\mathbf{R}} \mathbf{R} \cong b$, and b is the principal 5-block of Th, by [58]. Moreover, we have $k_0(B) = k_0(b) = 20 \leq |D : D'|$. This completes the proof. □

Theorem 6.2. *Let $p > 3$. Then Olsson's Conjecture holds for all p-blocks with defect groups of p-rank 2.*

Proof. Let B be a p-block with defect group D of p-rank 2 for $p > 3$. Then, by the Theorems 4.1, 4.2 and 4.3 in [15], B is controlled unless D is non-abelian of order p^3 and exponent p (see also [57]). Hence, by Theorem 5.6 we may assume that D is non-abelian of order p^3 and exponent p.

If in addition $p > 7$, Hendren has shown that there is at least one non-major B-subsection. In this case the result follows easily from Proposition 2.5(i). Now let $p = 7$. Then the fusion system $\mathcal{F}$ of B is one of the systems given in [49]. Kessar and Stancu showed using the classification of finite simple groups that three of them cannot occur for blocks (see [27]). In the remaining cases the number of $\mathcal{F}$-radical and $\mathcal{F}$-centric subgroups of D is always less than $p + 1 = 8$. In particular, there is an element $u \in D \setminus \mathrm{Z}(D)$ such that $\langle u\rangle \mathrm{Z}(D)$ is not $\mathcal{F}$-radical, $\mathcal{F}$-centric. Then by Alperin's fusion theorem $\langle u\rangle$ is not $\mathcal{F}$-conjugate to $\mathrm{Z}(D)$. Hence, the subsection (u, b_u) is non-major, and Olsson's Conjecture follows from Proposition 2.5(i).

In case $p = 5$ the same argument shows that we can assume that $\mathcal{F}$ is the fusion system of the principal 5-block of Th. However, in this case Olsson's Conjecture holds by Proposition 6.1. □

For $p = 3$, there are two fusion systems on the non-abelian group of order 27 and exponent 3 in [49], such that all subsections are major. These correspond to the simple groups ${}^2F_4(2)'$ and J_4. However, Olsson's Conjecture holds for the 3-blocks of ${}^2F_4(2)'$, ${}^2F_4(2)$, J_4, Ru and $2.Ru$ (see [1, 2, 6, 5]; cf. Remark 1.3 in [49]). More generally, Olsson's Conjecture is known to hold for all principal blocks with a non-abelian defect group of order 27 and exponent 3, by Remark 64 in [38]. In addition to 3-blocks of defect 3, there are also non-controlled 3-blocks whose defect groups have maximal class and 3-rank 2. We plan to come back to this situation in a separate paper. On the other hand Brauer's $k(B)$-Conjecture is satisfied for all 3-blocks of defect 3 (see [54]).

We finish this paper with a similar result about minimal non-abelian defect groups.

Theorem 6.3. *Let $p \neq 3$. Then Olsson's Conjecture holds for all p-blocks with minimal non-abelian defect groups.*

Proof. By [52] we may assume $p > 3$. Let B be a block with minimal non-abelian defect group D. Then by Rédei's classification of minimal non-abelian groups (see [50]), we may assume that

$$D := \langle x, y \mid x^{p^r} = y^{p^s} = [x, y]^p = [x, x, y] = [y, x, y] = 1\rangle$$

for $r \geq s \geq 1$. We set $z := [x, y] \in \mathrm{Z}(D)$. Observe that $\Phi(D) = \mathrm{Z}(D) = \langle x^p, y^p, z\rangle$ and $D' = \langle z\rangle$. Let $\mathcal{F}$ be the fusion system of B.

First assume $s \geq 2$. Then we show that B is controlled. By Alperin's Fusion Theorem it suffices to show that D does not contain $\mathcal{F}$-essential subgroups. By way of contradiction, assume that $Q < D$ is $\mathcal{F}$-essential. Since $\mathrm{C}_D(Q) \subseteq Q$, Q is a maximal subgroup of D. Let $a \in D$ be an element of order p. Then also $aD' \in D/D' \cong C_{p^r} \times C_{p^s}$ has order p. Since $r \geq s \geq 2$, we see that $a \in \mathrm{Z}(D)$ and $\Omega_1(D) \subseteq \mathrm{Z}(D)$. This shows that $1 \neq D/Q = \mathrm{Aut}_D(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q)$ acts trivially on $\Omega_1(Q)$. On the other hand every p'-automorphism of $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts non-trivially on $\Omega_1(Q)$ (see Theorem 5.2.4 in [20]). Hence, $\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \neq 1$ which contradicts the choice of Q. Thus, we have proved that B is a controlled block. Now the claim follows from Theorem 5.6(iii).

Now assume that $s = 1$. If also $r = 1$, then D is non-abelian of order p^3 and exponent p. In this case we have seen in the proof of Theorem 6.2 that Olsson's Conjecture holds for B, since $p > 3$. Thus, let $r \geq 2$. Since $\mathrm{Z}(D)$ has exponent p^{r-1}, we see that x is not $\mathcal{F}$-conjugate to an element in $\mathrm{Z}(D)$. In particular (x, b_x) is a non-major B-subsection. Moreover, $\langle x \rangle$ is fully $\mathcal{F}$-centralized, since $\mathrm{C}_D(x)$ is a maximal subgroup of D. Hence, $\mathrm{C}_D(x)$ is a defect group of b_x by Theorem 2.4(ii) in [33]. Now the claim follows from Proposition 2.5(i). $\square$

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References

[1] J. An, *Dade's conjecture for the Tits group*, New Zealand J. Math. **25** (1996), 107–131.

[2] J. An, *The Alperin and Dade conjectures for Ree groups* ${}^2F_4(q^2)$ *in non-defining characteristics*, J. Algebra **203** (1998), 30–49.

[3] J. An, *Controlled blocks of the finite quasisimple groups for odd primes*, Adv. Math. **227** (2011), 1165–1194.

[4] J. An and C. W. Eaton, *Blocks with extraspecial defect groups of finite quasisimple groups*, J. Algebra **328** (2011), 301–321.

[5] J. An and E. A. O'Brien, *The Alperin and Dade conjectures for the O'Nan and Rudvalis simple groups*, Comm. Algebra **30** (2002), 1305–1348.

[6] J. An, E. A. O'Brien and R. A. Wilson, *The Alperin weight conjecture and Dade's conjecture for the simple group* J_4, LMS J. Comput. Math. **6** (2003), 119–140.

[7] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, Vol. 391, Cambridge University Press, Cambridge, 2011.

[8] Y. Berkovich, *Groups of prime power order. Vol. 1*, de Gruyter Expositions in Mathematics, Vol. 46, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[9] R. Brauer, *On blocks and sections in finite groups. II*, Amer. J. Math. **90** (1968), 895–925.

[10] R. Brauer, *On 2-blocks with dihedral defect groups*, in Symposia Mathematica, Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), 367–393, Academic Press, London, 1974.

[11] M. Broué, *On characters of height zero*, in The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), 393–396, Proc. Sympos. Pure Math., Vol. 37, Amer. Math. Soc., Providence, R.I., 1980.

[12] D. A. Buell, *Binary quadratic forms*, Springer-Verlag, New York, 1989.

[13] J. H. Conway, R. T. Curtis et al., *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.

[14] E. C. Dade, *Blocks with cyclic defect groups*, Ann. of Math. (2) **84** (1966), 20–48.

[15] A. Díaz, A. Ruiz and A. Viruel, *All p-local finite groups of rank two for odd prime p*, Trans. Amer. Math. Soc. **359** (2007), 1725–1764.

[16] C. W. Eaton, B. Külshammer and B. Sambale, *2-Blocks with minimal nonabelian defect groups II*, J. Group Theory **15** (2012), 311–321.

[17] S. Enßlen and B. Külshammer, *A note on blocks with dihedral defect groups*, Arch. Math. (Basel) **91** (2008), 205–211.

[18] W. Feit, *The representation theory of finite groups*, North-Holland Mathematical Library, Vol. 25, North-Holland Publishing Co., Amsterdam, 1982.

[19] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*; 2008, (`http://www.gap-system.org`).

[20] D. Gorenstein, *Finite groups*, Harper & Row Publishers, New York, 1968.

[21] P. G. Gres', *On conjectures of Olsson, Brauer and Alperin*, Mat. Zametki **52** (1992), 32–35, 155.

[22] R. M. Guralnick, *Commutators and commutator subgroups*, Adv. in Math. **45** (1982), 319–330.

[23] S. Hendren, *Extra special defect groups of order p^3 and exponent p^2*, J. Algebra **291** (2005), 457–491.

[24] S. Hendren, *Extra special defect groups of order p^3 and exponent p*, J. Algebra **313** (2007), 724–760.

[25] B. Huppert, *Endliche Gruppen I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

[26] I. M. Isaacs, *Finite group theory*, Graduate Studies in Mathematics, Vol. 92, American Mathematical Society, Providence, RI, 2008.

[27] R. Kessar and R. Stancu, *A reduction theorem for fusion systems of blocks*, J. Algebra **319** (2008), 806–823.

[28] M. Kiyota, *On 3-blocks with an elementary abelian defect group of order* 9, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 33–58.

[29] B. Külshammer, *On 2-blocks with wreathed defect groups*, J. Algebra **64** (1980), 529–555.

[30] B. Külshammer, *A remark on conjectures in modular representation theory*, Arch. Math. (Basel) **49** (1987), 396–399.

[31] B. Külshammer and T. Okuyama, *On centrally controlled blocks of finite groups*, unpublished.

[32] B. Külshammer and T. Wada, *Some inequalities between invariants of blocks*, Arch. Math. (Basel) **79** (2002), 81–86.

[33] M. Linckelmann, *Simple fusion systems and the Solomon 2-local groups*, J. Algebra **296** (2006), 385–401.

[34] M. Linckelmann, *Introduction to fusion systems*, in Group representation theory, 79–113, EPFL Press, Lausanne, 2007.

[35] V. D. Mazurov, *Finite groups with metacyclic Sylow 2-subgroups*, Sibirsk. Mat. Ž. **8** (1967), 966–982.

[36] G. O. Michler and J. B. Olsson, *The Alperin-McKay conjecture holds in the covering groups of symmetric and alternating groups, $p \neq 2$*, J. Reine Angew. Math. **405** (1990), 78–111.

[37] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press Inc., Boston, MA, 1989.

[38] R. Narasaki and K. Uno, *Isometries and extra special Sylow groups of order p^3*, J. Algebra **322** (2009), 2027–2068.

[39] J. Neukirch, *Algebraische Zahlentheorie*, Springer-Verlag, Berlin, 1992.

[40] Y. Ninomiya, *Finite p-groups with cyclic subgroups of index p^2*, Math. J. Okayama Univ. **36** (1994), 1–21 (1995).

[41] T. Okuyama and M. Wajima, *Irreducible characters of p-solvable groups*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), 309–312.

[42] J. B. Olsson, *On 2-blocks with quaternion and quasidihedral defect groups*, J. Algebra **36** (1975), 212–241.

[43] J. B. Olsson, *On subpairs and modular representation theory*, J. Algebra **76** (1982), 261–279.

[44] J. B. Olsson, *On the number of characters in blocks of finite general linear, unitary and symmetric groups*, Math. Z. **186** (1984), 41–47.

[45] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and cyclic inertial quotients of order* 4, J. Algebra **172** (1995), 205–213.

[46] G. R. Robinson, *On the number of characters in a block*, J. Algebra **138** (1991), 515–521.

[47] G. R. Robinson, *On Brauer's $k(B)$ problem*, J. Algebra **147** (1992), 450–455.

[48] R. Rouquier, *The derived category of blocks with cyclic defect groups*, in Derived equivalences for group rings, 199–220, Lecture Notes in Math., Vol. 1685, Springer-Verlag, Berlin, 1998.

[49] A. Ruiz and A. Viruel, *The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p*, Math. Z. **248** (2004), 45–65.

[50] L. Rédei, *Das „schiefe Produkt" in der Gruppentheorie*, Comment. Math. Helv. **20** (1947), 225–264.

[51] B. Sambale, *Blocks with central product defect group $D_{2^n} * C_{2^m}$*, Proc. Amer. Math. Soc. (to appear), arXiv:1105.4977v1.

[52] B. Sambale, 2-*Blocks with minimal nonabelian defect groups*, J. Algebra **337** (2011), 261–284.

[53] B. Sambale, *Cartan matrices and Brauer's $k(B)$-conjecture*, J. Algebra **331** (2011), 416–427.

[54] B. Sambale, *Cartan matrices and Brauer's $k(B)$-conjecture II*, J. Algebra **337** (2011), 345–362.

[55] B. Sambale, *Fusion systems on metacyclic* 2-*groups*, Osaka J. Math. **49** (2012), 325–329.

[56] P. Schmid, *The solution of the $k(GV)$ problem*, ICP Advanced Texts in Mathematics, Vol. 4, Imperial College Press, London, 2007.

[57] R. Stancu, *Control of fusion in fusion systems*, J. Algebra Appl. **5** (2006), 817–837.

[58] K. Uno, *Conjectures on character degrees for the simple Thompson group*, Osaka J. Math. **41** (2004), 11–36.

[59] Y. Usami, *On p-blocks with abelian defect groups and inertial index* 2 *or* 3. *I*, J. Algebra **119** (1988), 123–146.

[60] A. Watanabe, *Notes on p-blocks of characters of finite groups*, J. Algebra **136** (1991), 109–116.

[61] S. Yang, *On Olsson's conjecture for blocks with metacyclic defect groups of odd order*, Arch. Math. (Basel) **96** (2011), 401–408.

Lászlo Héthelyi
Department of Algebra
University of Technology and Economics
H-1521 Budapest Műegyetem Rkp. 3-9
Hungary
hethelyi@math.bme.hu

Burkhard Külshammer
Mathematisches Institut
Friedrich-Schiller-Universität
D-07737 Jena
Germany
kuelshammer@uni-jena.de

Benjamin Sambale
Mathematisches Institut
Friedrich-Schiller-Universität
D-07737 Jena
Germany
benjamin.sambale@uni-jena.de