Orders generated by character values

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Abstract

Let $K := \mathbb{Q}(G)$ be the number field generated by the complex character values of a finite group G. Let $\mathbb{Z}_K$ be the ring of integers of K. In this paper we investigate the suborder $\mathbb{Z}[G]$ of $\mathbb{Z}_K$ generated by the character values of G. We prove that every prime divisor of the order of the finite abelian group $\mathbb{Z}_K/\mathbb{Z}[G]$ divides $|G|$. Moreover, if G is nilpotent, we show that the exponent of $\mathbb{Z}_K/\mathbb{Z}[G]$ is a proper divisor of $|G|$ unless $G = 1$. We conjecture that this holds for arbitrary finite groups G.

Keywords: finite groups, field of character values, orders, algebraic integers
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1 Introduction

It is well-known that the complex character values of a finite group G are algebraic integers. We like to measure how "many" algebraic integers actually arise in this way. The field

$$K := \mathbb{Q}(G) := \mathbb{Q}(\chi(g) : \chi \in \mathrm{Irr}(G),\ g \in G) \subseteq \mathbb{C}$$

of character values of G is contained in $\mathbb{Q}_{\exp(G)}$ where $\exp(G)$ denotes the exponent of G and $\mathbb{Q}_n$ is the cyclotomic field generated by the complex n-th roots of unity. Let $\mathbb{Z}_K$ be the ring of integers of K. The character values of G also generate an order $\mathbb{Z}[G]$ contained in $\mathbb{Z}_K$ (here $\mathbb{Z}[G]$ is neither the group algebra nor the ring of generalized characters). The deviation of $\mathbb{Z}[G]$ from $\mathbb{Z}_K$ can be measured by the structure of the finite abelian group $\mathbb{Z}_K/\mathbb{Z}[G]$. If G is a rational group for instance, then $K = \mathbb{Q}$ and $\mathbb{Z}[G] = \mathbb{Z} = \mathbb{Z}_K$. If G is abelian, then $K = \mathbb{Q}_{\exp(G)}$ and $\mathbb{Z}_K = \mathbb{Z}[e^{2\pi\sqrt{-1}/\exp(G)}]$. In this case it is easy to see that $\mathbb{Z}[G] = \mathbb{Z}_K$ as well. On the other hand, we construct a group G of order 240 such that

$$\mathbb{Z}_K/\mathbb{Z}[G] \cong C_{120}^2 \times C_{60}^2 \times C_{12}^4 \times C_4^4 \times C_2^{14}$$

where C_n denotes a cyclic group of order n. Nevertheless, our main theorems show that the structure of $\mathbb{Z}_K/\mathbb{Z}[G]$ is restricted by the order of G.

Theorem A. *Let G be a finite group and $K := \mathbb{Q}(G)$. Then the prime divisors of $|\mathbb{Z}_K/\mathbb{Z}[G]|$ divide $|G|$.*

Theorem B. *Let $G \neq 1$ be a nilpotent group and $K := \mathbb{Q}(G)$. Then the exponent of $\mathbb{Z}_K/\mathbb{Z}[G]$ is a proper divisor of $|G|$. In particular, $|G|\mathbb{Z}_K \subseteq \mathbb{Z}[G]$.*

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In the final section we exhibit many examples which indicate that Theorem B might be true without the nilpotency hypothesis.

Conjecture C. *Let $G \neq 1$ be a finite group and $K := \mathbb{Q}(G)$. Then the exponent of $\mathbb{Z}_K/\mathbb{Z}[G]$ is a proper divisor of $|G|$.*

2 Preliminaries

In addition to the notation introduced above, we define

$$\begin{aligned}
\mathbb{Q}(g) &:= \mathbb{Q}(\chi(g) : \chi \in \mathrm{Irr}(G)) && (g \in G),\\
\mathbb{Z}[g] &:= \mathbb{Z}[\chi(g) : \chi \in \mathrm{Irr}(G)],\\
\mathbb{Q}(\chi) &:= \mathbb{Q}(\chi(g) : g \in G) && (\chi \in \mathrm{Irr}(G)),\\
\mathbb{Z}[\chi] &:= \mathbb{Z}[\chi(g) : g \in G].
\end{aligned}$$

For number fields $K \subseteq L$ we denote the relative discriminant of L with respect to K by $d_{L|K} \in \mathbb{Z}_K$. If $K = \mathbb{Q}$ we write $d_L := d_{L|\mathbb{Q}}$ as usual. We make use of the following tools from algebraic number theory.

Proposition 1. *The discriminant of any subfield of $\mathbb{Q}_n$ divides $n^{\varphi(n)}$.*

Proof. If $n = p^m$ is a power of a prime p, then by [9, Lemma I.10.1] the discriminant d_n of $\mathbb{Q}_n$ is $\pm p^{p^{m-1}(mp-m-1)}$, a divisor of $n^{\varphi(n)} = p^{mp^{m-1}(p-1)}$. For arbitrary n we obtain $d_n \mid n^n$ from [9, Proposition I.2.11]. Now if $K \subseteq \mathbb{Q}_n$ is any subfield, then by [9, Corollary III.2.10] even $d_K^{|\mathbb{Q}_n:K|}$ divides d_n. $\square$

Although we only need a weak version of the following result, it seems worth stating a strong form.

Proposition 2. *Let K and L be Galois number fields. Then*

$$\gcd(d_K, d_L)\mathbb{Z}_{KL} \subseteq \frac{\gcd(d_K, d_L)}{d_{K\cap L}^m}\mathbb{Z}_{KL} \subseteq \mathbb{Z}_K\mathbb{Z}_L$$

where $m := \min\{|KL : K|, |KL : L|\}$. In particular, $\mathbb{Z}_{KL} = \mathbb{Z}_K\mathbb{Z}_L$ if d_K and d_L are coprime.

Proof. Most textbooks only deal with the last claim. To prove the general case we follow [9, Proposition I.2.11]:

We consider the compositum KL as an extension over $M := K \cap L$. Note that $\mathbb{Z}_{KL}$ ($\mathbb{Z}_K$, $\mathbb{Z}_L$ respectively) is the integral closure of $\mathbb{Z}_M$ in KL (K, L respectively). Let $b_1, \ldots, b_n$ be a $\mathbb{Z}_M$-basis of $\mathbb{Z}_K$ and let $c_1, \ldots, c_m$ be a $\mathbb{Z}_M$-basis of $\mathbb{Z}_L$. Then $\{b_ic_j : i = 1, \ldots, n, j = 1, \ldots, m\}$ is an M-basis of KL as is well-known. Let $\alpha \in \mathbb{Z}_{KL}$ be arbitrary and write

$$\alpha = \sum_{i,j} a_{ij}b_ic_j$$

with $a_{ij} \in M$ for all i, j. Since KL is a Galois extension over $\mathbb{Q}$, it is also a Galois extension over K and over L. Thus, we may write $\mathrm{Gal}(KL|K) = \{\sigma_1, \ldots, \sigma_m\}$ and $\mathrm{Gal}(KL|L) = \{\tau_1, \ldots, \tau_n\}$. Then

$$\mathrm{Gal}(KL|M) = \{\sigma_i\tau_j : i = 1, \ldots, m, j = 1, \ldots, n\}$$

and restriction yields isomorphisms $\mathrm{Gal}(KL|K) \to \mathrm{Gal}(L|M)$ and $\mathrm{Gal}(KL|L) \to \mathrm{Gal}(K|M)$. Let

$$D = (\tau_i(b_j))_{i,j=1}^n \in \mathbb{Z}_K^{n\times n}, \qquad a = (\tau_1(\alpha), \ldots, \tau_n(\alpha)) \in \mathbb{Z}_M^n, \qquad b := \Big(\sum_{j=1}^m a_{ij}c_j\Big)_{i=1}^n \in L^n.$$

Then

$$\det(D)^2 = \det(D^{\mathrm{t}}D) = \det((\mathrm{Tr}_{K|M}(b_ib_j)_{i,j})) = d_{K|M}$$

(here D^{t} denotes the transpose of D and $\mathrm{Tr}_{K|M}$ is the trace map of K with respect to M). Moreover, $Db = a$. Denoting the adjoint matrix of D by $D^* \in \mathbb{Z}_K^{n\times n}$ we obtain $\det(D)b = D^*Db = D^*a$. The right hand side is an integral vector and so must be the left hand side. It follows that

$$d_{K|M}a_{ij} = \det(D)^2 a_{ij} \in \mathbb{Z}_M \subseteq \mathbb{Z}_K$$

for all i,j. Now by [9, Corollary III.2.10], we have

$$d_K = d_M^{|K:M|}\mathrm{N}_M(d_{K|M})$$

where N_M denotes the norm map of M with respect to $\mathbb{Q}$. Since M is a Galois extension, the norm of $d_{K|M}$ is the product of all Galois conjugates of $d_{K|M}$ in M. In particular, $d_{K|M}$ divides $\mathrm{N}_M(d_{K|M}) = d_K/d_M^{|K:M|}$ in $\mathbb{Z}_M$. Hence, $\frac{d_K}{d_M^{|K:M|}}a_{ij} \in \mathbb{Z}_M$ for all i,j. By a symmetric argument, $\frac{d_L}{d_M^{|L:M|}}a_{ij} \in \mathbb{Z}_M$ and therefore $\frac{\gcd(d_K,d_L)}{d_M^m}a_{ij} \in \mathbb{Z}_M$. Hence, we derive

$$\frac{\gcd(d_K,d_L)}{d_M^m}\alpha \in \mathbb{Z}_K\mathbb{Z}_L$$

as desired. □

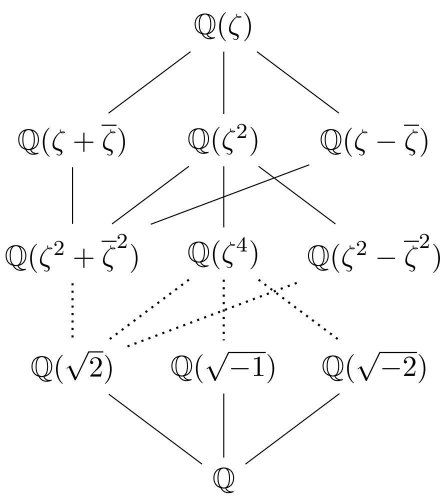
It is well-known that $\mathbb{Z}_{\mathbb{Q}_n} = \mathbb{Z}[\zeta]$ for every primitive n-th root of unity ζ. We also need the following refinements.

Proposition 3 (Leopoldt, see [12, Proposition 6.1])**.** *Let K be a number field contained in $\mathbb{Q}_n$. Then $\mathbb{Z}_K$ is generated as abelian group by the traces*

$$\sum_{\sigma\in\mathrm{Gal}(K(\zeta)|K)} \sigma(\zeta)$$

of n-th roots of unity ζ.

Lemma 4. *Every subfield of $\mathbb{Q}_{2^n}$ has the form $K = \mathbb{Q}(\xi)$ where $\xi \in \{\zeta, \zeta \pm \overline{\zeta}\}$ and ζ is a 2^n-th root of unity. The inclusion of subfields is given as follows*



If $\xi = \zeta \pm \overline{\zeta}$, then the elements 1 *and* $\zeta^k + (\pm\overline{\zeta})^k$ *with* $k = 1, \ldots, 2^{n-2} - 1$ *generate* $\mathbb{Z}_K$ *as abelian group.*

Proof. If $n \leq 2$, then $K \in \{\mathbb{Q}, \mathbb{Q}_4\}$ and the claim holds with $\xi = \zeta \in \{1, \sqrt{-1}\}$. Hence, let $n \geq 3$. By induction on n, we may assume that $K \nsubseteq \mathbb{Q}_{2^{n-1}}$ and ζ is a primitive 2^n-th root of unity. The subfields of $\mathbb{Q}_{2^n}$ correspond via Galois theory to the subgroups of the Galois group

$$\mathcal{G} := \mathrm{Gal}(\mathbb{Q}_{2^n}|\mathbb{Q}) \cong (\mathbb{Z}/2^n\mathbb{Z})^\times \cong C_2 \times C_{2^{n-2}}.$$

The involutions of $\mathcal{G}$ are $\alpha : \zeta \mapsto \zeta^{-1} = \overline{\zeta}$, $\beta : \zeta \mapsto \zeta^{-1+2^{n-1}} = -\overline{\zeta}$ and $\gamma : \zeta \mapsto \zeta^{1+2^{n-1}} = -\zeta$. Since $K \nsubseteq \mathbb{Q}_{2^{n-1}} = \mathbb{Q}_{2^n}^{\gamma}$, we must have $\mathrm{Gal}(\mathbb{Q}_{2^n}|K) \in \{\langle\alpha\rangle, \langle\beta\rangle\}$, i. e. $K = \mathbb{Q}(\zeta \pm \overline{\zeta})$.

As remarked above, $1, \zeta, \ldots, \zeta^{2^{n-1}-1}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_{\mathbb{Q}_{2^n}}$. Hence, every $x \in \mathbb{Z}_K$ can be written in the form

$$x = \sum_{k=0}^{2^{n-1}-1} a_k \zeta^k$$

with $a_0, \ldots, a_{2^{n-1}-1} \in \mathbb{Z}$. Since x is invariant under α or β, we obtain $a_k = -(\pm 1)^k a_{2^{n-1}-k}$ for $k = 1, \ldots, 2^{n-1} - 1$. Hence,

$$x = a_0 + \sum_{k=1}^{2^{n-2}-1} a_k(\zeta^k + (\pm\overline{\zeta})^k)$$

and the second claim follows. □

Proposition 5 ([8, Theorem 3.11])**.** *Let G be a finite group and $g \in G$. Then the natural map*

$$\mathrm{N}_G(\langle g\rangle)/\mathrm{C}_G(g) \to \mathrm{Gal}(\mathbb{Q}_{|\langle g\rangle|}|\mathbb{Q}(g))$$

is an isomorphism.

3 General results

We start our investigation with the "column fields" $\mathbb{Q}(g)$. Since products of characters are characters, we have $\mathbb{Z}[g] = \sum_{\chi \in \mathrm{Irr}(G)} \mathbb{Z}\chi(g)$.

Proposition 6. *For every finite group G and $g \in G$ we have*

$$|\mathrm{N}_G(\langle g\rangle)/\langle g\rangle|\mathbb{Z}_{\mathbb{Q}(g)} \subseteq \mathbb{Z}[g].$$

Proof. Let $n := |\langle g\rangle|$ and $K := \mathbb{Q}(g) \subseteq \mathbb{Q}_n$. By Proposition 3, $\mathbb{Z}_K$ is generated by the traces

$$\xi := \sum_{\sigma \in \mathrm{Gal}(K(\zeta)|K)} \sigma(\zeta)$$

of n-th roots of unity ζ. Let ψ be a character of $\langle g\rangle$ such that $\psi(g) = \xi \in K$. Then by Proposition 5 it follows that

$$\mathbb{Z}[g] \ni (\psi^G)(g) = \frac{1}{|\langle g\rangle|} \sum_{x \in \mathrm{N}_G(\langle g\rangle)} \psi(g^x) = |\mathrm{N}_G(\langle g\rangle)/\langle g\rangle|\xi.$$

This implies $|\mathrm{N}_G(\langle g\rangle)/\langle g\rangle|\mathbb{Z}_K \subseteq \mathbb{Z}[g]$. □

The following consequence implies Theorem A.

Corollary 7. *For every finite group G there exists $e \in \mathbb{N}$ such that*

$$|G|^e \mathbb{Z}_{\mathbb{Q}(G)} \subseteq \mathbb{Z}[G].$$

Proof. Clearly, $\mathbb{Q}(G) = \prod_{g \in G} \mathbb{Q}(g)$. By Proposition 1, the discriminants of the fields $\mathbb{Q}(g)$ for $g \in G$ divide $|G|^{|G|}$. Hence, Proposition 2 and Proposition 6 imply

$$|G|^e \mathbb{Z}_{\mathbb{Q}(G)} \subseteq |G|^{|G|} \prod_{g \in G} \mathbb{Z}_{\mathbb{Q}(g)} \subseteq \prod_{g \in G} \mathbb{Z}[g] \subseteq \mathbb{Z}[G]$$

for some (large) $e \in \mathbb{N}$. $\square$

For specific groups one can estimate the exponent e in Corollary 7 by using the full strength of Propositions 1 and 2. For nilpotent groups G we will prove next that e can be taken to be 1.

4 Nilpotent groups

Lemma 8. *Let G and H be finite groups of coprime order. Let $K := \mathbb{Q}(G)$ and $L := \mathbb{Q}(H)$. Then $\mathbb{Q}(G \times H) = KL$, $\mathbb{Z}_{KL} = \mathbb{Z}_K \mathbb{Z}_L$ and $\mathbb{Z}[G \times H] = \mathbb{Z}[G]\mathbb{Z}[H]$.*

Proof. Since $\mathrm{Irr}(G \times H) = \mathrm{Irr}(G) \times \mathrm{Irr}(H)$, it is clear that $\mathbb{Q}(G \times H) = KL$ and

$$\mathbb{Z}[G \times H] = \Big\{ \sum_{i=1}^{n} x_i y_i : n \in \mathbb{N}, x_1, \ldots, x_n \in \mathbb{Z}[G], y_1, \ldots, y_n \in \mathbb{Z}[H] \Big\} = \mathbb{Z}[G]\mathbb{Z}[H].$$

Since $K \subseteq \mathbb{Q}_{|G|}$ and $L \subseteq \mathbb{Q}_{|H|}$, the discriminants d_K and d_L are coprime according to Proposition 1. By Proposition 2, we obtain $\mathbb{Z}_{KL} = \mathbb{Z}_K \mathbb{Z}_L$ $\square$

In the situation of Lemma 8 it is easy to determine $\mathbb{Z}_{KL}/\mathbb{Z}[G \times H]$ from the elementary divisors of $\mathbb{Z}_K/\mathbb{Z}[G]$ and $\mathbb{Z}_L/\mathbb{Z}[H]$. For instance, if $\mathbb{Z}_K/\mathbb{Z}[G]$ has elementary divisors $1, 2, 4$ (in particular, $\mathbb{Z}_K$ has rank 3) and $\mathbb{Z}_L/\mathbb{Z}[L]$ has elementary divisors $1, 3$, then

$$\mathbb{Z}_{KL}/\mathbb{Z}[G \times H] \cong C_2 \times C_4 \times C_3 \times C_6 \times C_{12} \cong C_2 \times C_6 \times C_{12}^2.$$

The following is a special case of Theorem B.

Proposition 9. *Let G be a nilpotent group of odd order and let $p_1, \ldots, p_n$ be the prime divisors of $|G|$. Then*

$$|G|\mathbb{Z}_{\mathbb{Q}(G)} \subseteq q\mathbb{Z}[G]$$

where $q := \prod_{i=1}^{n} \min\{p_i^3, |G|_{p_i}\}$.

Proof. We may write $G = P_1 \times \ldots \times P_n$ with Sylow subgroups $P_1, \ldots, P_n$. By Lemma 8, it follows that

$$|G|\mathbb{Z}_{\mathbb{Q}(G)} = |P_1|\mathbb{Z}_{\mathbb{Q}(P_1)} \ldots |P_n|\mathbb{Z}_{\mathbb{Q}(P_n)}.$$

Thus, we may assume that G is a non-abelian p-group for some odd prime p. In particular, $|G| \geq p^3$. The Galois group of $\mathbb{Q}_{|G|}$ (and therefore of every subfield) is cyclic. By Proposition 5, $\mathrm{Gal}(\mathbb{Q}_{|\langle g \rangle|}|\mathbb{Q}(g))$ is a cyclic p-group for every $g \in G$. Hence, the fields $\mathbb{Q}(g)$ are all cyclotomic and therefore they are totally ordered. In particular, there exists $g \in G$ such that $K := \mathbb{Q}(G) = \mathbb{Q}(g)$. By Proposition 6, it follows that $N\mathbb{Z}_K \subseteq \mathbb{Z}[G]$ where $N := |\mathrm{N}_G(\langle g\rangle)/\langle g\rangle|$. If $N \le |G|/p^3$, then we are done. So we may assume that $N \ge |G|/p^2$. If $\mathbb{Q}(G) = \mathbb{Q}_p$, then $\mathbb{Z}_K = \mathbb{Z}[\lambda] \subseteq \mathbb{Z}[G]$ for any non-trivial linear character $\lambda \in \mathrm{Irr}(G)$. Therefore, we may assume that $|G| \ge p^4$, $|\langle g\rangle| = p^2$ and $\mathrm{N}_G(\langle g\rangle) = \mathrm{C}_G(g) = G$. By Proposition 5, $\mathbb{Q}(g) = \mathbb{Q}_{|\langle g\rangle|} = \mathbb{Q}(\zeta)$ for some root of unity ζ. Since the regular character of G is faithful, there exists $\chi \in \mathrm{Irr}(G)$ such that the restriction $\chi_{\langle g\rangle}$ is faithful. Since $g \in \mathrm{Z}(\chi)$, we have $\chi(g) = \chi(1)\zeta^k$ for some integer k coprime to p. Then for every $l \ge 0$ we also have $\chi(g^{p^l}) = \chi(1)\zeta^{kp^l}$. This implies $\chi(1)\mathbb{Z}_K \subseteq \mathbb{Z}[G]$. Since $|G| \ge p^4$ and $\chi(1)^2 < |G|$, we obtain $|G|\mathbb{Z}_K \subseteq p^3\mathbb{Z}[G]$. □

The analysis of 2-groups G is more delicate, since it may happen that $\mathbb{Q}(G) \ne \mathbb{Q}(g)$ for all $g \in G$.

Lemma 10. *Let G be a 2-group and $g \in G$ such that $\mathbb{Q}(g)$ is not a cyclotomic field. Then for every subfield K of $\mathbb{Q}(g)$ there exists $\chi \in \mathrm{Irr}(G)$ such that $K = \mathbb{Q}(\chi(g))$.*

Proof. We argue by induction on $|G|$. We may assume that $|\mathbb{Q}(g) : \mathbb{Q}| > 2$. In particular, $G \ne 1$. By Lemma 4, the subfields of $\mathbb{Q}(g)$ are totally ordered. In particular, there exists $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi(g)) = \mathbb{Q}(g)$. Let Z be a central subgroup of G of order 2. Then χ^2 is a character of G/Z and $|\mathbb{Q}(\chi(g)) : \mathbb{Q}(\chi(g)^2)| \le 2$. Since

$$\mathbb{Q}(gZ) = \mathbb{Q}(\psi(gZ) : \psi \in \mathrm{Irr}(G/Z)) \subseteq \mathbb{Q}(g),$$

we obtain $|\mathbb{Q}(g) : \mathbb{Q}(gZ)| \le 2$. Since $|\mathbb{Q}(g) : \mathbb{Q}| > 2$, also $\mathbb{Q}(gZ)$ is not a cyclotomic field. By induction, every proper subfield of $\mathbb{Q}(g)$ has the form $\mathbb{Q}(\psi(g))$ for some $\psi \in \mathrm{Irr}(G/Z)$. □

The cyclic group $G = \langle g\rangle \cong C_8$ shows the assumption on $\mathbb{Q}(g)$ in Lemma 10 is necessary.

Lemma 11. *Let G be a 2-group and $g \in G$ such that $K := \mathbb{Q}(g)$ is not a cyclotomic field. Then*

$$M\mathbb{Z}_K \subseteq 2\mathbb{Z}[G]$$

where $M := \max\{\chi(1) : \chi \in \mathrm{Irr}(G)\}$.

Proof. By Lemma 4, there exists a primitive 2^n-th root of unity ζ such that $K = \mathbb{Q}(\zeta \pm \overline{\zeta})$. Moreover, $\mathbb{Z}_K$ is generated by the elements 1 and $\xi_k := \zeta^k + (\pm\overline{\zeta})^k$ with $k = 1, \ldots, 2^{n-2} - 1$. For every such k there exists $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi(g)) = \mathbb{Q}(\xi_k)$ by Lemma 10. It suffices to show that $\chi(1)\xi_k$ is an integral linear combination of the Galois conjugates of $2\chi(g)$. To this end, we may assume that $k = 1$ and $\xi := \xi_1$.

Let $d := \chi(1)$ and note that $d > 1$ since $\mathbb{Q}(\chi(g)) = \mathbb{Q}(\xi) = K$ is not a cyclotomic field. There exist integers $a_0, \ldots, a_{2^{n-1}-1}$ such that

$$\chi(g) = \sum_{i=0}^{2^{n-1}-1} a_i\zeta^i = a_0 + \sum_{i=1}^{2^{n-2}-1} a_i\xi_i.$$

Since $\chi(g)$ is a sum of d roots of unity, $|a_0| + \ldots + |a_{2^{n-1}-1}| \le d$ (it may happen that other roots, even of higher order than 2^n, cancel each other out). The Galois group $\mathcal{G}$ of $\mathbb{Q}_{2^n}$ acts on K and on $\{\psi(g) : \psi \in \mathrm{Irr}(G)\}$. Let $\sigma \in \mathcal{G}$ such that $\sigma(\zeta) = \zeta^{1+2^{n-1}} = -\zeta$. Then

$$\omega := \sum_{i=0}^{s-1} b_i\xi_{2i+1} = \chi(g) - \sigma(\chi(g)) \in \mathbb{Z}[G]$$

where $s := 2^{n-3}$ and $b_i := 2a_{2i+1}$ for $i = 0, \ldots, s-1$. Let $\tau \in \mathcal{G}$ such that $\tau(\zeta) = \zeta^5$. Note that $\tau^s(\xi) = \sigma(\xi) = -\xi$. We may relabel the elements b_i in a suitable order such that

$$\omega = \sum_{i=0}^{s-1} b_i \tau^i(\xi).$$

Next we consider

$$\gamma := \sum_{i=0}^{s-1} b_i \zeta^{4i} \in \mathbb{Z}_{\mathbb{Q}_{2s}}.$$

It is known that the prime 2 is fully ramified in $\mathbb{Q}_{2s}$. More precisely, $(2) = (\zeta^4 - 1)^s$ and $(\zeta^4 - 1)$ is a prime ideal (see [9, Lemma I.10.1]). Let e be the 2-part of $\gcd(b_0, \ldots, b_{s-1})$. Then $\frac{1}{e}\gamma$ is an algebraic integer, but $\frac{1}{2e}\gamma$ is not. Hence, $(\frac{1}{e}\gamma) = (\zeta^4 - 1)^t \mathfrak{p}$ where $t < s$ and $\mathfrak{p}$ is an ideal of $\mathbb{Z}_{\mathbb{Q}_{2s}}$ coprime to $(\zeta^4 - 1)$. This implies the existence of some $\delta \in \mathbb{Z}_{\mathbb{Q}_{2s}}$ such that $\gamma\delta = 2em$ where m is an odd integer. We write $\delta = \sum_{i=0}^{s-1} c_i \zeta^{4i}$ with $c_0, \ldots, c_{s-1} \in \mathbb{Z}$. Then

$$2em = \gamma\delta = \sum_{i,j=0}^{s-1} b_i c_j \zeta^{4(i+j)}.$$

Comparing coefficients yields

$$\sum_{i+j=t} b_i c_j - \sum_{i+j=s-t} b_i c_j = \begin{cases} 2em & \text{if } t = 0, \\ 0 & \text{if } 1 \le t \le s-1. \end{cases}$$

Finally we compute

$$\sum_{j=0}^{s-1} c_j \tau^j(\omega) = \sum_{i,j=0}^{s-1} b_i c_j \tau^{i+j}(\xi) = \sum_{t=0}^{s-1} \Big(\sum_{i+j=t} b_i c_j - \sum_{i+j=s-t} b_i c_j \Big) \tau^t(\xi) = 2em\xi.$$

Hence, $2em\xi \in \mathbb{Z}[G]$. By Proposition 6 we also have $|G|\xi \in |G|\mathbb{Z}_{\mathbb{Q}(g)} \subseteq \mathbb{Z}[G]$. Therefore,

$$2e\xi = \gcd(2em, |G|)\xi \in \mathbb{Z}[G].$$

Note that

$$e \le \sum_{i=0}^{s-1} |b_i| = \sum_{i=0}^{2^{n-2}-1} |a_{2i+1}| \le \sum_{i=0}^{2^{n-1}-1} |a_i| \le d. \tag{4.1}$$

Suppose that $d\xi \notin 2\mathbb{Z}[G]$. Then $d \le 2e$ (keep in mind that d and e are 2-powers). If the first inequality in (4.1) is strict, then $2e \le \sum_{i=0}^{s-1} |b_i|$ since the right hand side is divisible by e. Thus, in any case one of the inequalities in (4.1) is an equality. If $e = \sum_{i=0}^{s-1} |b_i|$, then $e = |b_i|$ and $\omega = b_i \tau^i(\xi)$ for some $i \in \{0, \ldots, s-1\}$. Then we obtain $e\xi \in \mathbb{Z}[G]$. If, on the other hand, $\sum_{i=0}^{2^{n-2}-1} |a_{2i+1}| = \sum_{i=0}^{2^{n-1}-1} |a_i|$, then $\omega = 2\chi(g)$ and $e\xi \in \mathbb{Z}[G]$ by the computation above. Hence in any case we deduce that $d = e$. But now $\chi(g) = a_{2i+1}\tau^i(\xi)$ and $d = 2|a_{2i+1}|$. This implies $d\xi \in 2\mathbb{Z}[G]$ as desired. □

The next result is a restatement of Theorem B.

Theorem 12. *For every nilpotent group $G \neq 1$ the exponent of $\mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G]$ is a proper divisor of $|G|$.*

Proof. By Proposition 9 and its proof, we may assume that G is a 2-group. By Lemma 4, $\mathbb{Q}(G) = \mathbb{Q}(\xi)$ where $\xi \in \{\zeta, \zeta \pm \overline{\zeta}\}$ and ζ is a primitive 2^n-th root of unity. If there exists $g \in G$ such that $\mathbb{Q}(G) = \mathbb{Q}(g)$, then we obtain $|G|\mathbb{Z}_{\mathbb{Q}(G)} \subseteq \mathbb{Z}[G]$ from Proposition 6. Otherwise we have $n \geq 3$, $\mathbb{Q}(G) = \mathbb{Q}(\zeta)$ and there exists $g \in G$ such that $K := \mathbb{Q}(g) = \mathbb{Q}(\zeta \pm \overline{\zeta})$. Moreover, there exist $h \in G$ and $\psi \in \mathrm{Irr}(G)$ such that

$$\psi(h) = \sum_{i=0}^{2^{n-1}-1} a_i \zeta^i \notin K$$

where $a_0, \ldots, a_{2^{n-1}-1} \in \mathbb{Z}$. Lemma 11 shows that $M\mathbb{Z}_K \subseteq 2\mathbb{Z}[G]$ where $M := \max\{\chi(1) : \chi \in \mathrm{Irr}(G)\}$. It suffices to prove $|G|\zeta^k \in 2\mathbb{Z}[G]$ for every $k \in \mathbb{Z}$.

Let σ be the Galois automorphism of $\mathbb{Q}(\zeta)$ such that $\sigma(\zeta) = \pm\overline{\zeta}$. Since $\psi(h) \notin K$, we have $\psi(h) \neq \sigma(\psi(h))$. We consider

$$\omega := \psi(h) - \sigma(\psi(h)) = \sum_{i=1}^{2^{n-1}-1} b_i \zeta^i \in \mathbb{Z}[G]$$

where $b_i := a_i \pm a_{2^{n-1}-i}$ if i is odd and $b_i := a_i + a_{2^{n-1}-i}$ otherwise. Let e be the 2-part of $\gcd(b_0, \ldots, b_{2^{n-1}-1})$. As in the proof of Lemma 11 there exists an odd integer m such that $2em\omega^{-1}$ is an algebraic integer. Hence for every $k \in \mathbb{Z}$,

$$2em\frac{\zeta^k - \sigma(\zeta)^k}{\omega} \in \mathbb{Z}_{\mathbb{Q}(\zeta)} \cap \mathbb{Q}(\zeta)^\sigma = \mathbb{Z}_K.$$

We conclude that

$$2emM\zeta^k = emM(\zeta^k + \sigma(\zeta)^k) + emM\frac{\zeta^k - \sigma(\zeta)^k}{\omega}\omega \in \mathbb{Z}[G].$$

By Corollary 7, there exists $s \in \mathbb{N}$ such that $|G|^s\zeta^k \in \mathbb{Z}[G]$. Hence,

$$2eM\mathbb{Z}_{\mathbb{Q}(G)} \subseteq \gcd(2emM, |G|^s)\mathbb{Z}[\zeta] \subseteq \mathbb{Z}[G].$$

If $b_i \neq 0$ for some $i \neq 2^{n-2}$, then $e \leq |b_i| \leq |a_i| + |a_{2^{n-1}-1}| \leq \psi(1)$. Otherwise, $\omega = 2a_{2^{n-2}}\sqrt{-1}$. If, in this case, there exists some $a_i \neq 0$ with $i \neq 2^{n-2}$, then $e \leq |b_{2^{n-2}}| < 2|a_{2^{n-2}}| + |a_i| \leq 2\psi(1)$. Since e and $\psi(1)$ are 2-powers, we still have $e \leq \psi(1)$. Finally, let $\psi(h) = a_{2^{n-2}}\sqrt{-1} = \omega/2$. Then we may repeat the calculation above with $\psi(h)$ instead of ω in order to obtain $eM\mathbb{Z}_{\mathbb{Q}(G)} \subseteq \mathbb{Z}[G]$ where $e \leq 2\psi(1)$. In summary,

$$2M\psi(1)\mathbb{Z}_{\mathbb{Q}(G)} \subseteq \mathbb{Z}[G]$$

in every case. Since $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$, we have $2M\psi(1) \leq 2M^2 \leq |G|$. If $2M\psi(1) = |G|$, then $\psi(1) = M$ and ψ is the only irreducible character of degree M. But then ψ is rational and we derive the contradiction $\psi(h) \in K$. Therefore, $2M\psi(1) < |G|$ and the claim follows. □

5 Examples

We show first that Proposition 9 is sharp in the following sense.

Proposition 13. *For every prime p and every integer $n \geq 1$ there exists a group P of order p^{2n+2} and exponent p^2 such that $K := \mathbb{Q}(P) = \mathbb{Q}_{p^2}$ and $\mathbb{Z}_K/\mathbb{Z}[P] \cong C_{p^n}^{(p-1)^2}$.*

Proof. Let P be the central product of an extraspecial group E of order p^{2n+1} (it does not matter which one) and a cyclic group $C = \langle c \rangle$ of order p^2. The irreducible characters of P are those of $E \times C$ which agree on $\mathrm{Z}(E) = \langle z \rangle$ and $\langle c^p \rangle$. It is well-known that $\mathrm{Irr}(E)$ consists of p^{2n} linear character and $p-1$ faithful characters $\chi_1, \ldots, \chi_{p-1}$ of degree p^n (see [6, Example 7.6(b)] for instance). Since E/E' is elementary abelian, the linear character values of E and also of P generate $\mathbb{Q}_p$. Let ζ be a primitive p^2-th root of unity. After relabeling, we may assume that $\chi_i(z) = p^n \zeta^{ip}$ and $\chi_i(g) = 0$ for $g \in E \setminus \mathrm{Z}(E)$ and $i = 1, \ldots, p-1$. Hence, the non-linear character of P take the values 0 and $p^n \zeta^i$ for $i \in \mathbb{Z}$. This shows $K = \mathbb{Q}_{p^2}$ and

$$\mathbb{Z}[G] = \mathbb{Z}[\zeta^p, p^n \zeta^i : \gcd(i, p) = 1].$$

Since the elements $1, \zeta, \zeta^2, \ldots, \zeta^{p(p-1)-1}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}_K$, the claim follows easily. $\square$

Proposition 13 already shows that neither $|\langle g \rangle| \mathbb{Z}_{\mathbb{Q}(g)} \subseteq \mathbb{Z}[G]$ nor $\exp(G)\mathbb{Z}_{\mathbb{Q}(G)} \subseteq \mathbb{Z}[G]$ is true in general. Also the dual statements, motivated by Lemma 11, $\chi(1)\mathbb{Z}_{\mathbb{Q}(\chi)} \subseteq \mathbb{Z}[G]$ and

$$\mathrm{lcm}\{\chi(1) : \chi \in \mathrm{Irr}(G)\}\mathbb{Z}_{\mathbb{Q}(G)} \subseteq \mathbb{Z}[G]$$

do not always hold. Using GAP [5] and MAGMA [1] we computed the following example: The group

$$G = \texttt{SmallGroup}(48, 3) \cong C_4^2 \rtimes C_3$$

gives $K := \mathbb{Q}(G) = \mathbb{Q}_{12}$ and $\mathbb{Z}[G] = \mathbb{Z}[2\sqrt{-1}, \zeta]$ where ζ is a primitive third root of unity. Hence, $\mathbb{Z}_K/\mathbb{Z}[G] \cong C_2^2$, but $\mathrm{lcm}\{\chi(1) : \chi \in \mathrm{Irr}(G)\} = 3$.

For a single entry $\omega = \chi(g)$ of the character table of G the group $\mathbb{Z}_{\mathbb{Q}(\omega)}/\mathbb{Z}[\omega]$ usually has nothing to do with G. For instance, $G = D_{26} \times C_3$ has a character value ω such that $\mathbb{Z}_{\mathbb{Q}(\omega)}/\mathbb{Z}[\omega]$ is cyclic of order $5^2 \cdot 157 \cdot 547$. It is not hard to show that every algebraic integer of an abelian number field occurs in the character table of some finite group (see proof of [4, Theorem 6]).

For 2-groups the gap between G and $\mathbb{Z}_K/\mathbb{Z}[G]$ can get even bigger than in Proposition 13: The exponent and the largest character degree of $G = \texttt{SmallGroup}(2^9, 6480850)$ is 8, but

$$\mathbb{Z}_K/\mathbb{Z}[G] \cong C_{64} \times C_8 \times C_4.$$

Similarly, the group $G = \texttt{SmallGroup}(2^9, 60860)$ yields $|\mathbb{Z}_K/\mathbb{Z}[G]| = 2^{33}$.

For non-nilpotent groups, the arguments from the last section drastically fail as our next example shows. Let

$$G = \texttt{SmallGroup}(240, 13) \cong C_{15} \rtimes D_{16}$$

where the dihedral group D_{16} acts with kernel D_{16}' (commutator subgroup) on C_{15}. Then $K = \mathbb{Q}_{120}$ and $2\mathbb{Z}_{\mathbb{Q}(g)} \subseteq \mathbb{Z}[G]$ for all $g \in G$, but

$$\mathbb{Z}_K/\mathbb{Z}[G] \cong C_{120}^2 \times C_{60}^2 \times C_{12}^4 \times C_4^4 \times C_2^{14}.$$

Now we consider some simple groups which support Conjecture C.

Proposition 14.

(i) Let $G = \mathrm{PSL}(2, q)$ for some prime power $q \neq 1$. Then $\mathbb{Z}_{\mathbb{Q}(G)} = \mathbb{Z}[G]$.

(ii) Let $G = \mathrm{Sz}(q)$ for $q \geq 8$ an odd power of 2. Then $\mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G] \cong C_2^a$ where $a = \varphi((q^2+1)(q-1))/32$.

Proof.

(i) Assume first that $q \geq 5$ is odd. Then G has two irreducible characters taking only rational values and three families χ_i, θ_j, η_k taking (potentially) irrational values (see [3, Theorem 38.1] for instance). Let ζ_n be a primitive n-th root of unity and let $\epsilon := (-1)^{(q-1)/2}$. Set $r := (q-1)/2$ and $s := (q+1)/2$. Then the values of the χ_i lie in $K := \mathbb{Q}(\zeta_r + \overline{\zeta_r})$ and they contain the integral basis from Lemma 4. Similarly the values of the θ_j generate the ring of integers of $L := \mathbb{Q}(\zeta_s + \overline{\zeta_s})$. Finally, the values of the η_k generate the ring of integers of $M := \mathbb{Q}(\sqrt{\epsilon q})$. The discriminants of K, L and M are pairwise coprime by Proposition 1. Hence, by Proposition 2 we have

$$\mathbb{Z}[G] = \mathbb{Z}_K \mathbb{Z}_L \mathbb{Z}_M = \mathbb{Z}_{KLM} = \mathbb{Z}_{\mathbb{Q}(G)}.$$

For q a power of 2, the result follows for $\mathrm{PSL}(2, q) = \mathrm{SL}(2, q)$ with a similar argument from [3, Theorem 38.2].

(ii) The character table of the group $G = \mathrm{Sz}(q)$ was determined by Suzuki in [11, Theorem 13]. We use the names of characters in that theorem. Set $r := q-1$, $s := q+\sqrt{2q}+1$ and $t := q-\sqrt{2q}+1$ and note that these odd numbers are pairwise coprime. Observe that $\mathbb{Q}(G) = KLMN$, the composita of the fields $K = \mathbb{Q}(X_1) = \mathbb{Q}(\zeta_r + \bar{\zeta_r})$, $L = \mathbb{Q}(Y_1) = \mathbb{Q}(\zeta_s + \zeta_s^q + \zeta_s^{q^2} + \zeta_s^{q^3})$, $M = \mathbb{Q}(Z_1) = \mathbb{Q}(\zeta_t + \zeta_t^q + \zeta_t^{q^2} + \zeta_t^{q^3})$ and $N = \mathbb{Q}(W_1) = \mathbb{Q}(\sqrt{-1})$, which have pairwise coprime discriminant by Proposition 1. Now $\mathbb{Z}_K = \mathbb{Z}[\zeta_r + \bar{\zeta_r}] = \mathbb{Z}[X_1]$ and $\mathbb{Z}_L = \mathbb{Z}[\zeta_s + \zeta_s^q + \zeta_s^{q^2} + \zeta_s^{q^3}] = \mathbb{Z}[Y_1]$ and similarly for Z_1. Further $\mathbb{Z}[W_1] = \mathbb{Z}[W_2] = \mathbb{Z}[2\sqrt{-1}]$, hence $\mathbb{Z}_N/\mathbb{Z}[W_1]$ has elementary divisors 1 and 2. Similar to the remark following Lemma 8, we can conclude that $\mathbb{Z}_{KLMN}/\mathbb{Z}[G]$ has elementary divisors 1 and 2 each with multiplicity

$$[KLM : \mathbb{Q}] = \frac{\varphi(r)}{2}\frac{\varphi(s)}{4}\frac{\varphi(t)}{4} = \frac{\varphi((q^2+1)(q-1))}{32}. \qquad \square$$

A minimal simple group (i. e. a simple group with all proper subgroups solvable) is isomorphic to some $\mathrm{PSL}(2, q)$, to some $\mathrm{Sz}(2^{2f+1})$ or to $\mathrm{PSL}(3, 3)$. For the last group one can check easily that $\mathbb{Z}_{\mathbb{Q}(G)} = \mathbb{Z}[G]$. Hence, for minimal simple groups G, the exponent of $\mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G]$ is at most 2.

Finally we compute $\mathbb{Z}[G]$ for the alternating group $G = A_n$ of (small) degree n. Let $g \in G$ be non-rational. Then there exists a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n into pairwise distinct odd parts such that

$$\mathbb{Z}[g] = \mathbb{Z}[(1 + \sqrt{d})/2]$$

where $d = (-1)^{(n-k)/2}\lambda_1 \ldots \lambda_k \equiv 1 \pmod 4$ (see [7, Theorem 2.5.13] for instance). We may write $\sqrt{d} = e\sqrt{d'}$ such that d' is squarefree. Let $K := \mathbb{Q}(g) = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{d'})$. Then

$$\mathbb{Z}_K = \mathbb{Z}[(1 + \sqrt{d'})/2]$$

and we obtain $e\mathbb{Z}_K \subseteq \mathbb{Z}[g]$. Note that $e^2 \mid d \mid n! = 2|G|$. Since the discriminant of K is $d' \equiv 1 \pmod 2$, it follows that $|\mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G]|$ is odd by Proposition 2. It seems fairly difficult to determine the precise structure of $\mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G]$. For $n \geq 25$, a theorem by Robinson–Thompson [10] states that

$$\mathbb{Q}(G) = \mathbb{Q}(\sqrt{p^*} : p \text{ odd prime }, n-2 \neq p \leq n)$$

where $p^* := (-1)^{\frac{p-1}{2}} p$. By Proposition 2, $\mathbb{Z}_{\mathbb{Q}(G)}$ is generated as abelian group by all products of the elements $(1 + \sqrt{p^*})/2$ with p as above. The following table lists the (non-trivial) elementary divisors of $\mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G]$ for $n \leq 31$. In every case Conjecture C is fulfilled.

n	$\mathbb{Z}_{\mathbb{Q}(A_n)}/\mathbb{Z}[A_n]$
≤ 11	1
$12, 13, 14$	3^4
15	$3^4 \times 15^4 \times 45^4$
16	$3^4 \times 15^4$
17	$3^{12} \times 9^4 \times 45^4 \times 135^4$
18	$3^8 \times 15^8 \times 45^8$
19	$3^8 \times 15^8$
20	$3^{36} \times 9^{12} \times 45^{32} \times 10395^{28} \times 31185^4$
21	$3^{36} \times 105^4 \times 315^{12}$
22	$3^{52} \times 105^8 \times 315^{52} \times 945^4$
23	$3^{64} \times 4095^{32}$
24	1
25	$3^{32} \times 15^{32} \times 315^{32}$
26	$3^{38} \times 15^{40} \times 45^{40} \times 315^{56} \times 945^8$
27	$3^{112} \times 9^{112} \times 27^{16}$
28	$3^{96} \times 15^{80} \times 45^{48}$
29	$3^{224} \times 15^{128}$
30	$3^{128} \times 105^{128}$
31	3^{256}

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