Exponent and p-rank of finite p-groups and applications

Benjamin Sambale

September 23, 2018

Abstract

We bound the order of a finite p-group in terms of its exponent and p-rank. Here the p-rank is the maximal rank of an abelian subgroup. These results are applied to defect groups of p-blocks of finite groups with given Loewy length. Doing so, we improve results in a recent paper by Koshitani, Külshammer and Sambale. In particular, we determine possible defect groups for blocks with Loewy length 4.

Keywords: exponent, p-rank, Loewy length
AMS classification: 20D15, 20C20

1 Exponent and p-rank

Let P be a finite p-group for a prime p. Then the *exponent* of P is the smallest positive integer e such that $x^e = 1$ for all $x \in P$. Moreover, the *p-rank* of P is the maximal rank of an abelian subgroup of P. It is often useful to bound the order of P if its exponent and p-rank are given. Most of our notation is standard (see e. g. [7]). We denote a cyclic group of order $n \geq 1$ by C_n. Moreover, define $P^m = P \times \ldots \times P$ (m copies). We use the abbreviations $\Omega(P) := \Omega_1(P)$ and $\mho(P) := \mho_1(P)$ for a finite p-group P.

Theorem 1.1 (Laffey [11])**.** *Let P be a finite p-group with exponent p^e, and let r be the rank of a maximal elementary abelian normal subgroup of P. Then $|P| \leq p^k$ where*

$$k := \begin{cases} re + \binom{r}{2} + r^2 & \text{if } p = 2, \\ re + \binom{r}{2} & \text{if } p > 2. \end{cases}$$

Corollary 1.2. *Let P be a finite p-group with exponent p^e and p-rank r. If $p > 2$, then $|P| \leq p^{re+\binom{r}{2}}$.*

For $p = 2$ we improve Theorem 1.1 as follows.

Theorem 1.3. *Let P be a 2-group with exponent 2^e, and let r be the rank of a maximal elementary abelian normal subgroup of P. Then $|P| \leq 2^k$ where*

$$k := r(e+1) + \binom{r}{2} - \frac{1}{2}\Big(\big|\lfloor \log_2(r) \rfloor + 1 - e\big| + \lfloor \log_2(r) \rfloor + 1 - e\Big). \tag{1}$$

Proof. Let E be a maximal elementary abelian normal subgroup of P of rank r. We consider $C := \mathrm{C}_P(E) \trianglelefteq P$. Choose a maximal abelian normal subgroup A of exponent at most 4 of P which contains E. Then obviously, $\mathrm{C}_P(A) \subseteq C$. Moreover, $\Omega(A) = E$. By a result of Alperin (see Satz III.12.1 in [7]) we have $\Omega_2(\mathrm{C}_P(A)) = A \subseteq \mathrm{Z}(\mathrm{C}_P(A))$. Lemma 1 in [11] implies $|\mathrm{C}_P(A)| \leq 2^{re}$. Let $x \in C$. Then x acts trivially on E and thus also on A/E. It follows that $x^2 \in \mathrm{C}_P(A)$ and $C/\mathrm{C}_P(A)$ is elementary abelian. In particular, $\Phi(C) \subseteq \mathrm{C}_P(A)$. Since $\Phi(C) = \mho(C)$, we also have $\Phi(C) \subseteq \Omega_{e-1}(\mathrm{C}_P(A))$. Corollary 1 in [10] shows that $\Omega_{e-1}(\mathrm{C}_P(A))$ has exponent at most 2^{e-1}. Hence, again by Lemma 1 in [11] we obtain $|\Phi(C)| \leq 2^{r(e-1)}$.

Now we count the involutions in C. Let $\mathcal{M}$ be the set of all elementary abelian subgroups of C of rank $r+1$ which contain E (possibly $\mathcal{M}=\varnothing$). For an involution $x\in C\setminus E$ we have $\langle E,x\rangle\in\mathcal{M}$. Moreover, two distinct elements of $\mathcal{M}$ intersect in E. Since E is maximal, the action of G on $\mathcal{M}$ by conjugation has no fixed points. In particular, $|\mathcal{M}|$ is even. We conclude that the number γ of involutions in C satisfies $\gamma\equiv 2^r-1 \pmod{2^{r+1}}$. Now a result of MacWilliams (see Theorem 37.1 in [2]) shows that $|C:\Phi(C)|\le 2^{2r}$. Hence,

$$|C|=|\Phi(C)||C:\Phi(C)|\le 2^{r(e-1)+2r}=2^{r(e+1)}.$$

Now we consider $P/C\le \operatorname{Aut}(E)\cong \operatorname{GL}(r,2)$. Let $S\le \operatorname{GL}(r,2)$ be the group of upper unitriangular matrices. Then $|S|=2^{\binom{r}{2}}$ and $S\in\operatorname{Syl}_2(\operatorname{GL}(r,2))$. In particular, $P/C\cong S_0\le S$. By Satz III.16.5 in [7],

$$2^{\lceil\log_2(r)\rceil}=\exp(S)\le\exp(S_0)|S:S_0|\le 2^e|S:S_0|.$$

This gives $|S_0|\le 2^{\binom{r}{2}+e-\lceil\log_2(r)\rceil}$ whenever $\lceil\log_2(r)\rceil\ge e$. Now assume $r=2^e$. Let $\alpha\in S$ be a Jordan block of size r. Suppose that there is $x\in P$ such that xC corresponds to α. Then $|\langle x\rangle|=2^e$ and $\langle x\rangle\cap C=1$. Moreover, α has minimal polynomial $(X+1)^r$. In particular, $1+\alpha+\alpha^2+\ldots+\alpha^{r-1}\ne 0$. Choose $a\in E$ such that $(1+\alpha+\alpha^2+\ldots+\alpha^{r-1})(a)\ne 1$. Then

$$(ax)^{2^e}=a\cdot xax^{-1}\cdot x^2ax^{-2}\cdot\ldots\cdot x^{r-1}ax^{1-r}\cdot x^{2^e}\ne 1.$$

This contradiction shows that $|S_0|\le 2^{\binom{s}{2}+e-\lfloor\log_2(r)\rfloor-1}$ whenever $\lfloor\log_2(r)\rfloor+1\ge e$. The result follows. □

The last summand in Eq. (1) is only relevant if r is large compared to e. Since this will not happen in the applications in the next section, we note the following consequence.

Corollary 1.4. *Let P be a finite 2-group with exponent 2^e and 2-rank r. Then $|P|\le 2^{r(e+1)+\binom{r}{2}}$.*

The analysis of the subgroup $S\in\operatorname{Syl}_p(\operatorname{GL}(r,p))$ in the proof of Theorem 1.3 also applies to odd primes p. In fact one may count the matrices $\alpha\in S$ such that $(\alpha-1)^{p^e-1}=0$. Unfortunately, these matrices do not form a subgroup. However, the Jordan form of such a matrix consists only of blocks of size $\le p^e-1$. In the proof of Theorem 1.3, $|S_0|$ can be bounded by the order of the largest subgroup $T\le S$ such that $(\alpha-1)^{p^e-1}=0$ for all $\alpha\in T$. Computer calculations suggest that this is a better bound than $p^{\binom{r}{2}+e-\lfloor\log_p(r)\rfloor-1}$. For example if $p=e=2$ and $r=6$, one gets $|S_0|\le 2^{12}$ instead of $|S_0|\le 2^{14}$.

In the following we improve the corollaries above for special cases which will play an important role in the second part of the paper.

Let P be a 2-group of 2-rank r and exponent 2^e. If $r=1$, then Corollary 1.4 shows that $|P|\le 2^{e+1}$ (as is well-known), and this bound is assumed by the quaternion group. If $e=1$, then P is elementary abelian and satisfies $|P|\le 2^r$. In case $r=2$, Corollary 1.4 implies $|P|\le 2^{2e+3}$. This can be slightly improved.

Proposition 1.5. *Let P be a 2-group with exponent 2^e and 2-rank $r\le 2$. Then $|P|\le 2^{r(e+1)}$.*

Proof. By the remark above, we may assume that $r=2$ and $e\ge 2$. Obviously, a metacyclic group of exponent 2^e has order at most 2^{2e}. Hence, we may assume that P is not metacyclic. By Theorem 50.1 in [3] there exists a metacyclic normal subgroup $N\trianglelefteq P$ such that $\mathrm{C}_P(\Omega_2(N))\le N$. If $|P:N|\le 4$, then we are done. Thus, by way of contradiction we may assume that $P/N\cong D_8$ and

$$N=\langle a,b\mid a^{2^e}=b^{2^e}=1,\ bab^{-1}=a^{1+2^i}\rangle\cong C_{2^e}\rtimes C_{2^e}$$

where $i\in\{2^{e-1},2^e\}$ (see Theorem 50.1 in [3]). Observe that $C_{2^{e-1}}^2\cong\Omega_{e-1}(N)=\Phi(N)\subseteq \mathrm{Z}(N)$. Let $x\in P$ such that $x^2\notin N$. Suppose that x acts trivially on $\Omega(N)\cong C_2^2$ by conjugation. Then it is easy to see that x must also act trivially on $\Omega_2(N)/\Omega(N)\cong C_2^2$. Then however, $x^2\in\mathrm{C}_P(\Omega_2(N))\le N$. This contradiction shows that $\mathrm{C}_P(\Omega(N))<P$. We can find an element $y\in P\setminus\mathrm{C}_P(\Omega(N))$ such that $y^2\in N$. Since $\Omega(N)=\mho_{e-1}(N)$, y acts non-trivially on $N/\Phi(N)$. In particular, $\langle N,y\rangle/\Phi(N)\cong D_8$. Hence, let us choose an element $z\in\langle N,y\rangle$ such that $z^2\in N\setminus\Phi(N)$. Since all elements in $N\setminus\Phi(N)$ have order 2^e, we derive the contradiction $|\langle z\rangle|=2^{e+1}$. □

Now we turn to the case $r = 3$. Here Corollary 1.4 yields $|P| \le 2^{3e+6}$. This can be improved for $e = 2$ as follows.

Proposition 1.6. *Let P be a 2-group with exponent 4 and 2-rank 3. Then $|P| \le 2^9$.*

Proof. Let E be a maximal elementary abelian normal subgroup of P. If E has rank at most 2, then the claim follows from Theorem 1.3. Hence, we may assume that E has rank 3. Suppose that $|P| \ge 2^9$. Let $x \in P \setminus E$ be an involution. Then $\langle E, x\rangle$ is a group of order 2^4 with more than 7 involutions. Obviously, $\langle E, x\rangle$ lies in a subgroup of P of order 2^9. However, using GAP [5] one can show that all groups of order 2^9 with exponent 4 and 2-rank 3 have precisely 7 involutions. This contradiction shows that all involutions of P lie in E. In particular $\Phi(P) = \mho(P) \subseteq E$. Now Theorem 37.1 in [2] implies $|P| = 2^9$. □

By the results above we raise the following question:

Question: Let P be a 2-group with exponent 2^e and 2-rank r. Is it true that $|P| \le 2^{r(e+1)}$?[1],

A direct product of quaternion groups shows that the bound would be sharp. Moreover, a counterexample must have at least 2^{13} elements.

Next we turn to odd primes. Here a group of p-rank 1 is cyclic and therefore, Corollary 1.2 is optimal in this case. By Lemma 3.2 in [9], Corollary 1.2 is also optimal for $e = 1$, $p \ge 7$ and $r \le 3$. Now let $p = 3$ and consider the group

$$P := \langle x, y, a \mid x^{3^e} = y^{3^e} = a^3 = [x, y] = 1,\ axa^{-1} = xy^{-3},\ aya^{-1} = xy^{-2}\rangle$$

of order 3^{2e+1}. Since a acts non-trivially on $\langle x^{3^{e-1}}, y^{3^{e-1}}\rangle$, it follows that P has 3-rank 2. Moreover, $P/\langle x^3, y^3\rangle$ has exponent 3 and P has exponent 3^e. Hence, Corollary 1.2 is optimal for $p = 3$ and $r = 2$. On the other hand, Blackburn's classification of the p-groups with p-rank 2 (see Satz III.12.4 in [7]) implies that Corollary 1.2 is not optimal for $r = 2 \le e$ and $p \ge 5$. For the 3-groups of 3-rank 3 we give another improvement.

Lemma 1.7. *Let P be a group of order 3^6, exponent 9 and 3-rank 3 such that $\mathrm{Z}(P)$ has rank 3. Then $\Omega(P) = \mho(P) \cong P/\mho(P) \cong C_3^3$. Let $C_3^2 \cong A \le \mathrm{Aut}(P)$. Then A does not act faithfully on $\Omega(P)$.*

Proof. The result can of course be achieved by computer, but we prefer to give a theoretical argument. Since P has 3-rank 3 and $\mathrm{Z}(P)$ has rank 3, we conclude that $C_3^3 \cong \Omega(P) \subseteq \mathrm{Z}(P)$. Obviously, $P/\Omega(P)$ has exponent 3. By Lemma C in [13], we have $P/\Omega(P) \cong C_3^3$. In particular, P has class at most 2. By Satz III.10.2 in [7], P is regular and thus $\Omega(P) = \mho(P)$. Write $P = \langle x, y, z\rangle$ such that $\Omega(P) = \langle x^3, y^3, z^3\rangle$. Let $A = \langle a, b\rangle$. Assume that A acts faithfully on $\Omega(P)$. Since $\mathrm{Aut}(\Omega(P)) \cong \mathrm{GL}(3, 3)$, we may regard A as a subgroup of upper unitriangular matrices. In particular ${}^a(x^3) = {}^b(x^3) = x^3$. Moreover, we may assume that ${}^a(y^3) = y^3$ and ${}^a(z^3) = x^3z^3$ (i. e. $\langle a\rangle$ represents the center of the group of upper unitriangular matrices). Hence there are elements $\alpha_x, \alpha_y, \alpha_z \in \Omega(P)$ such that ${}^ax = x\alpha_x$, ${}^ay = y\alpha_y$ and ${}^az = xz\alpha_z$. Since $a^3 = 1$, it follows that

$$z = {}^{a^3}z = {}^{a^2}(xz\alpha_z) = {}^a(x\alpha_x xz\alpha_z {}^a\alpha_z) = x\alpha_x {}^a\alpha_x x\alpha_x xz\alpha_z {}^a\alpha_z {}^{a^2}\alpha_z = x^3\alpha_x^2 {}^a\alpha_x z$$

This shows that ${}^a\alpha_x = \alpha_x x^{-3}$. Therefore, $\alpha_x \equiv z^{-3} \pmod{\langle x^3, y^3\rangle}$. Now we consider b.

Suppose first that ${}^bx = x\beta_x$ and ${}^by = xy\beta_y$ for some $\beta_x, \beta_y \in \Omega(P)$. Then

$$xy\beta_y {}^b\alpha_y = {}^b(y\alpha_y) = {}^{ba}y = {}^{ab}y = {}^a(xy\beta_y) = x\alpha_x y\alpha_y {}^a\beta_y.$$

This gives the contradiction

$$\langle x^3, y^3\rangle \ni {}^b\alpha_y\alpha_y^{-1} = \alpha_x {}^a\beta_y\beta_y^{-1} \in z^{-3}\langle x^3\rangle.$$

Hence, we may assume that the action of b on $P/\Omega(P)$ is given by ${}^bx = x\beta_x$, ${}^by = y\beta_y$ and ${}^bz = yz\beta_z$ for some $\beta_x, \beta_y, \beta_z \in \Omega(P)$. This yields

$$x\beta_x {}^b\alpha_x = {}^b(x\alpha_x) = {}^{ba}x = {}^{ab}x = {}^a(x\beta_x) = x\alpha_x {}^a\beta_x$$

[1]Shortly after this question appeared in the 19th edition of the Kourovka Notebook, Avinoam Mann pointed out that the answer is "no" according to [A. Y. Ol'shanskii, *The number of generators and orders of Abelian subgroups of finite p-groups*, Math. Notes **23** (1978), 183–185]

and

$$\langle y^3\rangle \ni {}^b\alpha_x\alpha_x^{-1} = {}^a\beta_x\beta_x^{-1} \in \langle x^3\rangle.$$

Since $\alpha_x \equiv z^{-3} \pmod{\langle x^3, y^3\rangle}$, we derive the contradiction ${}^b\alpha_x\alpha_x^{-1} \neq 1$. □

Proposition 1.8. *Let P be a 3-group with exponent 9 and 3-rank 3. Then $|P| \le 3^7$.*

Proof. Let E be a maximal elementary abelian normal subgroup of P. By Theorem 1.1, we may assume that E has rank 3. Consider $C := \mathrm{C}_P(E)$. By a result of Alperin (see Satz III.12.1 in [7]), $\Omega(C) = E \subseteq \mathrm{Z}(C)$. Thus, by Lemma 1 in [11] we have $|C| \le 3^6$. Since P/C acts faithfully on E, we obtain $|P/C| \le 3^3$.

Suppose first that $|C| = 3^6$. By way of contradiction, let us assume that there is a subgroup $Q \le P$ such that $C \le Q$ and $|Q : C| = 9$. Since Q/C acts faithfully on E, we obtain $Q/C \cong C_3^2$. By Lemma 1.7 we have $E = \Omega(C) = \mho(C)$. Hence, $\overline{Q} := Q/E$ also acts faithfully on $\overline{C} := C/E$. Assume first that $\overline{Q}$ has 3-rank 4. Then there exists an elementary abelian normal subgroup $\overline{K} = K/E$ of $\overline{Q}$ such that $|\overline{K} \cap \overline{C}| = 9$. Hence, we find elements $a, b \in K \setminus C$ such that $Q = \langle a, b, C\rangle$ and $[a, b] \in \Omega(C) \subseteq \mathrm{Z}(C)$. Since $a^3, b^3 \in \Omega(C) \subseteq \mathrm{Z}(C)$, it follows that $\langle a, b\rangle$ induces an elementary abelian subgroup $A \le \mathrm{Aut}(C)$ of order 9. However, this contradicts Lemma 1.7.

Therefore, we may assume that $\overline{Q}$ has 3-rank 3. Since $\overline{Q}$ has exponent 3, one can show by computer that there is only one possible isomorphism type for $\overline{Q}$. One can show further that $\overline{Q}$ is a semidirect product of $\overline{C}$ and a subgroup of type C_3^2. Thus, we find elements $a, b \in Q \setminus C$ such that $a^3, b^3, [a, b] \in \Omega(C)$, and we get a contradiction as above.

For the remainder of the proof we can assume that $|C| = 3^5$ and $|P| = 3^8$. A computer calculations shows that there are only three possible isomorphism types for C, namely $C_9^2 \times C_3$, $(C_9 \rtimes C_9) \times C_3$ and a group of type $(C_9 \times C_3) \rtimes C_9$. Let us consider the last case. Let $A \in \mathrm{Syl}_3(\mathrm{Aut}(C))$. Then one can show that the kernel of the canonical map $A \to \mathrm{Aut}(\Omega(C))$ has index 3. However, this is impossible, since $|P/C| = 27$. Hence, there are two remaining isomorphism types for C. We may choose a maximal subgroup $Q \le P$ such that $C \le Q$ and $\mathrm{Z}(Q)$ is cyclic (choose a suitable action on $\Omega(C)$). Suppose that Q contains a subgroup Q_1 of order 3^6 such that the rank of $\mathrm{Z}(Q_1)$ is 3. Then there must be another subgroup $Q_1 \neq Q_2 \le Q$ such that $Q_1 \cong Q_2$, because otherwise Q_1 would be characteristic in Q and normal in P and we were back to the first case of the proof by setting $E := \Omega(Q_1)$. Therefore, Q satisfies the following properties:

- Q has order 3^7, exponent 9, 3-rank 3 and $\mathrm{Z}(Q)$ is cyclic,
- there exists a normal subgroup $C \trianglelefteq Q$ such that $\mathrm{C}_Q(C) \subseteq C$ and C is isomorphic to $C_9^2 \times C_3$ or to $(C_9 \rtimes C_9) \times C_3$,
- for every subgroup $Q_1 \le Q$ of order 3^6 such that the rank of $\mathrm{Z}(Q_1)$ is 3, there exists a subgroup $Q_1 \neq Q_2 \le Q$ such that $Q_2 \cong Q_1$.

A computation yields that there are precisely 68 isomorphism types of groups with these three properties. Using the `GrpConst` package in GAP we can determine all extensions of these groups by C_3. It follows that P is not among them and thus cannot exist. □

There are in fact 3-groups of order 3^7, exponent 9 and 3-rank 3. The results of the present section give the impression that there is no uniform bound on the order of a p-group in terms of exponents and p-ranks which is optimal for all odd primes.

2 Applications

In this section we consider p-blocks of finite groups over algebraically closed fields of characteristic p. The following result improves Theorem 2.3 in [9].

Theorem 2.1. *Let B be a p-block of a finite group G with defect d and Loewy length $\lambda > 1$. Then*

$$d \le \begin{cases} (\lambda-1)(1+\lfloor \log_p(\lambda-1)\rfloor) + \binom{\lambda}{2} & \text{if } p = 2,\\ (\lambda-1)\lfloor \log_p(\lambda-1)\rfloor + \binom{\lambda}{2} & \text{if } p > 2. \end{cases}$$

Proof. The result follows from Lemma 2.2 in [9] and Corollaries 1.2 and 1.4 above. □

The p-blocks with Loewy length at most 3 are determined in [14] (see also Proposition 3.1 in [9]). In [9] we started the investigation of p-blocks with Loewy length 4. Using the results above we give more precise information now.

Proposition 2.2. *Let B be a p-block of a finite group with Loewy length 4 and defect d. Then*

$$d \leq \begin{cases} 9 & \text{if } p = 2, \\ 7 & \text{if } p = 3, \\ 5 & \text{if } p = 5, \\ 6 & \text{if } p \geq 7. \end{cases}$$

Proof. For $p \in \{2, 3\}$ apply Lemma 2.2 in [9] in combination with Propositions 1.6 and 1.8 above. For $p \geq 5$ the claim was already shown in [9] (see remark after Proposition 3.3). □

In case $p \geq 5$ we have given a short list of possible defect groups in the situation of Proposition 2.2 (see Proposition 3.3 and Corollary 3.5 in [9]). For $p = 5$ (respectively $p \equiv 1 \pmod 5$, $p \equiv 1 \pmod 7$) there are at most 10 (respectively 11, 11) isomorphism types, and for the remaining primes $p \geq 7$ we have at most 12 possible isomorphism types. Now, using the "Small Groups Library" we can do the same for the remaining primes $p = 2, 3$. In order to reduce the number of 2-groups we apply the theory of fusion systems (see e.g. [1]).

Lemma 2.3. *Let $f(n)$ be the number of 2-groups of order 2^n, exponent 4 and 2-rank at most 3 which admit only trivial fusion systems. Then $f(6) = 30$, $f(7) = 104$, $f(8) = 496$ and $f(9) = 933$.*

Proof. Let P be a 2-group of order 2^n which admits only the trivial fusion system. Then by Alperin's Fusion Theorem (see Theorem I.3.5 in [1]), $\mathrm{Aut}(P)$ is a 2-group and there are no candidates for essential subgroups. We list some necessary condition on an essential subgroup $Q \leq P$ for a fusion system $\mathcal{F}$. Since Q is $\mathcal{F}$-centric we have $\mathrm{C}_P(Q) \subseteq Q$. Since $\mathrm{O}_2(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$, it follows that $\mathrm{N}_P(Q)/Q \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(Q))$ acts faithfully on $Q/\Phi(Q)$. If Q is generated by at most 5 elements, we have $|\mathrm{N}_P(Q) : Q| \leq 4$ and the possible isomorphism types of $\mathrm{Out}_{\mathcal{F}}(Q)$ are described in Corollary 6.12 and Lemma 6.13 in [16]. If Q happens to be normal in P, we find a non-trivial constrained fusion system $\mathrm{N}_{\mathcal{F}}(Q)$ on P. By Theorem III.5.10 in [1], $\mathrm{N}_{\mathcal{F}}(Q)$ is the fusion system of a group of order $|Q||\mathrm{Out}_{\mathcal{F}}(Q)|$. Usually we can check by computer if there are groups with the desired properties.

These properties suffice to determine 30 groups of order 2^6 which admit only the trivial fusion system. On the other hand, we can construct non-trivial fusion systems on the remaining groups of order 2^6, exponent 4 and 2-rank at most 3.

For $n = 7, 8$ we find 104 respectively 496 groups with the given constraints. It turns out that there are no non-trivial fusion systems if $\mathrm{Aut}(P)$ is a 2-group. Now we will show that this is also true for $n = 9$. A computer calculation (as in Proposition 1.6) shows that all groups P of order 2^9, exponent 4 and 2-rank 3 satisfy $C_3^3 \cong \Omega(P) \subseteq \mathrm{Z}(P)$. Let $Q \leq P$ be a candidate for an essential subgroup. Then

$$P' \subseteq \Phi(P) = \mho(P) \subseteq \Omega(P) \subseteq \mathrm{Z}(P) \subseteq \mathrm{C}_P(Q) \subseteq Q$$

and $Q \trianglelefteq P$. Hence, Q cannot be generated by three or less elements (otherwise $|\mathrm{N}_P(Q) : Q| = |P : Q| > 2$). Now suppose that Q is generated by four elements. Then $\Phi(Q) < \Omega(P)$, since otherwise P/Q acts trivially on $Q/\Phi(Q)$. Therefore, $|Q| \leq 2^6$ and $|P : Q| \geq 2^3$. However, this contradicts Lemma 6.13 in [16]. Thus we have shown that Q cannot be generated by four elements. Suppose next that Q is generated by five elements. Then again $\Phi(Q) < \Omega(P)$ and $|Q| \in \{2^6, 2^7\}$. Since $\Phi(Q) < \mathrm{Z}(P) \subseteq \mathrm{Z}(Q) < Q$, we have a characteristic subgroup lying between $\Phi(Q)$ and Q. In case $\mathrm{Z}(Q) = \Omega(P)$ we have $|\mathrm{Z}(Q) : \Phi(Q)| \leq 4$. Hence, $\mathrm{Out}_{\mathcal{F}}(Q)$ must act non-trivially on $Q/\mathrm{Z}(Q) = Q/\Omega(P)$. However, $P/Q \leq \mathrm{Out}_{\mathcal{F}}(Q)$ acts trivially on $Q/\Omega(P)$. It follows that $\Omega(P) < \mathrm{Z}(Q)$. However, one can show that there are no groups Q with the given properties such that $|\mathrm{Z}(Q)| \geq 2^4$. We conclude that Q is not generated by five or less elements. If Q is generated by more elements, one can show with GAP that $|\mathrm{Z}(Q)| \leq 4$. This contradicts $\Omega(P) \subseteq \mathrm{Z}(P) \subseteq \mathrm{Z}(Q)$. In summary, we proved that there are only trivial fusion systems on a group P of order 2^9, exponent 4 and 2-rank 3, if and only if $\mathrm{Aut}(P)$ is a 2-group. □

Lemma 2.4. *Let B be a 2-block of a finite group with defect group D such that $\mathrm{Z}(D)$ is isomorphic either to $C_4 \times C_2$, $C_2^2 \times C_4$ or to $C_4^2 \times C_2$. Then B does not have Loewy length 4.*

Proof. Let (D, b) be a maximal Brauer pair of B. Let $\overline{T} := \mathrm{N}_G(D, b)/\mathrm{C}_G(\mathrm{Z}(D))$. Then we have $|\overline{T}| \in \{1, 3\}$. In case $\overline{T} = 1$ the result follows from Corollary 2.7 in [9]. Now let $|\overline{T}| = 3$. Then $\mathrm{Z}(D) \cong C_2^2 \times C_4$ or $\mathrm{Z}(D) \cong C_4^2 \times C_2$. One can show by computer that the Loewy length of the centralizer algebra $F\,\mathrm{Z}(D)^{\overline{T}}$ is at least 5 where F is an algebraically closed field of characteristic 2. Hence, again the claim follows from Corollary 2.7 in [9]. □

Proposition 2.5. *Let B be a 2-block of a finite group with Loewy length 4 and defect group D. Then there are at most 196 possible isomorphism types for D and three of them are known to occur, namely C_4, C_2^3 and D_8.*

Proof. There are 1799 2-groups of exponent at most 4 and 2-rank at most 3. It is known that the groups C_4, C_2^3 and D_8 do actually occur as defect groups of 2-blocks with Loewy length 4 (see [8]). Now let D be metacyclic, but not isomorphic to C_4 or D_8. Then $|D| \le 16$ and the remark after Corollary 3.9 in [9] shows that D is not a defect group of a block with Loewy length 4. This excludes 14 groups from our list. The abelian groups $C_2^2 \times C_4$ and $C_4^2 \times C_2$ are impossible by Lemma 2.4. Another group (minimal non-abelian of order 32) can be excluded by using [4]. Using the list of defect groups of order 32 in [18], we can further exclude 7 groups which can only correspond to nilpotent blocks (cf. Corollary 3.8 in [9]).

Now let $|D| \ge 2^6$. By Lemma 2.3, $30+104+496+933 = 1563$ of these groups lead to nilpotent blocks. Two more groups can be excluded by Lemma 2.4. Moreover, there is one group whose center is isomorphic to C_4^2. Here one can show that the image of the restriction map $\mathrm{Aut}(D) \to \mathrm{Aut}(\mathrm{Z}(D))$ is a 2-group. Hence, by Corollary 2.7 in [9], D is not the defect group of a block with Loewy length 4. Using the same technique we eliminate 13 more groups of higher order.

Thus, altogether we have $1799 - 14 - 3 - 7 - 1563 - 16 = 196$ possible defect groups for 2-blocks with Loewy length 4 where three of them are known to occur. □

Lemma 2.6. *Let B be a 3-block of a finite group with defect group D and Loewy length 4. Then $\mathrm{Z}(D)$ is elementary abelian.*

Proof. The same argument as in Lemma 2.4 works. □

Lemma 2.7. *There are (at least) 2 (respectively 13) groups of order 3^6 (respectively 3^7) with exponent 9 and 3-rank at most 3 which admit only trivial fusion systems.*

Proof. Let P be a group of order 3^n with exponent 9 and 3-rank at most 3. Assume that P admits only the trivial fusion system. Then $n \ge 6$, since otherwise $\mathrm{Aut}(P)$ is not a 3-group (see [12]). We may assume that $\mathrm{Aut}(P)$ is a 3-group. We use the following algorithm in order to find possible groups P:

(1) Make a list $\mathcal{L}$ of all candidates of essential subgroups of P by using Lemma 6.15 in [16].

(2) We may assume that $\mathcal{F}$ is a *sparse* fusion system on P in the sense of [6].

(3) By Theorem 3.5 in [6], $\mathcal{F}$ is constrained, i. e. there is a self-centralizing subgroup $N \trianglelefteq P$ such that $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(N)$. Moreover, N lies in at least one member of $\mathcal{L}$.

(4) Theorem III.5.10 in [1] shows that there is a finite group G such that $P \in \mathrm{Syl}_3(G)$, $N \trianglelefteq G$, $G/\mathrm{Z}(N) \cong \mathrm{Aut}_{\mathcal{F}}(N)$ and $\mathcal{F} = \mathcal{F}_P(G)$. In particular, $\mathrm{Aut}(N)$ has no normal Sylow 3-subgroup (otherwise $\mathcal{F}$ would be controlled and thus trivial).

(5) It follows that $|N| \ge 3^3$ and in case $|N| = 3^3$ we have $N \cong C_3^3$ and $n = 6$.

This gives 2 groups of order 3^6 and 13 groups of order 3^7. □

Proposition 2.8. *Let B be a 3-block of a finite group with Loewy length 4 and defect group D. Then there are at most 386 possible isomorphism types for D and none of them is known to occur.*

Proof. There are 820 3-groups of exponent at most 9 and 3-rank at most 3. However, the three cyclic groups C_1, C_3 and C_9 cannot occur by Corollary 3.9 in [9]. Also the abelian groups $C_9 \times C_3$, C_9^2, $C_3^2 \times C_9$, $C_9^2 \times C_3$ and C_9^3 cannot occur by Lemma 2.6. Using Corollary 2.7 in [9] we can exclude 411 more groups. Among the remaining groups, 15 lead to nilpotent blocks by Lemma 2.7. Hence, there are $820 - 8 - 411 - 15 = 386$ groups left. □

We remark that the proof of Proposition 2.8 exhausts the known methods, i. e. in the remaining case there are always non-trivial fusion systems and neither Corollary 2.7 nor Corollary 3.9 in [9] applies. We add a result for principal blocks which was suggested by Koshitani with a different proof.

Proposition 2.9. *Let B be a principal 3-block with defect group D and Loewy length 4. Then D is not metacyclic. Moreover, $|D| \ge 3^4$.*

Proof. By Proposition 4.13 in [9] we may assume that D is non-abelian. Since D has exponent at most 9, it follows that $D \cong C_9 \rtimes C_3$ or $C_9 \rtimes C_9$. In the first case, Theorem 4.5 in [17] implies that all Cartan invariants of B are divisible by 3. The same holds in case $|D| = 3^4$ by Corollary 5 in [15] (cf. [17, Section 2]). Now Proposition 4.6 in [9] gives a contradiction. The last statement follows from Propositions 4.13 and 4.14 in [9]. □

Acknowledgment

This work is supported by the Carl Zeiss Foundation and the Daimler and Benz Foundation. The author thanks Heiko Dietrich, Bettina Eick, Max Horn and Eamonn O'Brien for the assistance with computations of automorphism groups.

References

[1] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, Vol. 391, Cambridge University Press, Cambridge, 2011.

[2] Y. Berkovich, *Groups of prime power order. Vol. 1*, de Gruyter Expositions in Mathematics, Vol. 46, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[3] Y. Berkovich and Z. Janko, *Groups of prime power order. Vol. 2*, de Gruyter Expositions in Mathematics, Vol. 47, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[4] C. W. Eaton, B. Külshammer and B. Sambale, 2-*Blocks with minimal nonabelian defect groups II*, J. Group Theory **15** (2012), 311–321.

[5] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.4*; 2014, (`http://www.gap-system.org`).

[6] A. Glesser, *Sparse fusion systems*, Proc. Edinb. Math. Soc. (2) **56** (2013), 135–150.

[7] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

[8] S. Koshitani, *On the projective cover of the trivial module over a group algebra of a finite group*, Comm. Algebra (to appear).

[9] S. Koshitani, B. Külshammer and B. Sambale, *On Loewy lengths of blocks*, Math. Proc. Cambridge Philos. Soc. **156** (2014), 555–570.

[10] T. J. Laffey, *A lemma on finite p-groups and some consequences*, Proc. Cambridge Philos. Soc. **75** (1974), 133–137.

[11] T. J. Laffey, *Bounding the order of a finite p-group*, Proc. Roy. Irish Acad. Sect. A **80** (1980), 131–134.

[12] D. MacHale and R. Sheehy, *Finite groups with odd order automorphism groups*, Proc. Roy. Irish Acad. Sect. A **95** (1995), 113–116.

[13] A. Mann, *The power structure of p-groups. II*, J. Algebra **318** (2007), 953–956.

[14] T. Okuyama, *On blocks of finite groups with radical cube zero*, Osaka J. Math. **23** (1986), 461–465.

[15] G. R. Robinson, *On the focal defect group of a block, characters of height zero, and lower defect group multiplicities*, J. Algebra **320** (2008), 2624–2628.

[16] B. Sambale, *Blocks of finite groups and their invariants*, Habilitationsschrift, Jena, 2013.

[17] B. Sambale, *Brauer's Height Zero Conjecture for metacyclic defect groups*, Pacific J. Math. **262** (2013), 481–507.

[18] B. Sambale, *Further evidence for conjectures in block theory*, Algebra Number Theory **7** (2013), 2241–2273.

Benjamin Sambale
Institut für Mathematik
Friedrich-Schiller-Universität
07743 Jena
Germany
benjamin.sambale@uni-jena.de