On the projective height zero conjecture

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Abstract

Recently, Malle and Navarro put forward a projective version of Brauer's celebrated height zero conjecture on blocks of finite groups. In this short note we show that Brauer's original conjecture implies the projective version.

The following is a long-standing conjecture in representation theory of finite groups (see [2, Problem 23]):

Conjecture 1 (Brauer's height zero conjecture)**.** *Let B be a block of a finite group with defect group D. Then every irreducible character in B has height 0 if and only if D is abelian.*

Recently, Malle–Navarro [6] proposed the following generalization of Conjecture 1 (the case $Z = 1$ yields the original conjecture). An equivalent statement in terms of so-called θ-blocks was given by Rizo [9].

Conjecture 2 (Malle–Navarro's projective height zero conjecture)**.** *Let B be a p-block of a finite group G with defect group D. Let Z be a central p-subgroup of G and let $\lambda \in \mathrm{Irr}(Z)$. Then every irreducible character in B lying over λ has height 0 if and only if D/Z is abelian and λ extends to D.*

In their paper, Malle and Navarro already proved the "if direction" of Conjecture 2 by making use of the solution [5] of the "if direction" of Conjecture 1. Moreover, they showed that Conjecture 1 implies Conjecture 2 for blocks of maximal defect. Generalizing their argument, we prove that Conjecture 1 always implies Conjecture 2.

Theorem 3. *Suppose that Conjecture 1 holds for all blocks of finite groups. Then Conjecture 2 holds for all blocks of finite groups.*

A direct consequence of Theorem 3 and [6, Theorem 2.2] gives the following.

Corollary 4. *Suppose that Conjecture 1 holds for all blocks of finite groups. Let B be a p-block of a finite group G with defect group D. Let N be a normal p-subgroup of G and $\theta \in \mathrm{Irr}(N)$ be G-invariant. If all irreducible characters in B lying over θ have the same height as θ, then D/N is abelian.*

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Our proof uses the notation from [6] and the language of fusion systems. Recall that every block B with defect group D induces a (saturated) fusion system $\mathcal{F}$ on D (see [1, Theorem IV.3.2] for instance). The *focal subgroup* and the *center* of $\mathcal{F}$ are given by

$$\begin{aligned} \mathfrak{foc}(\mathcal{F}) &:= \langle x^{-1}x^f : x \in Q \le D,\ f \in \mathrm{Aut}_{\mathcal{F}}(Q)\rangle \trianglelefteq D, \\ \mathrm{Z}(\mathcal{F}) &:= \{x \in D : x \text{ is fixed by every morphism in } \mathcal{F}\} \le \mathrm{Z}(D) \end{aligned}$$

respectively.

Proof of Theorem 3. Let B be as in Conjecture 2. Since the "if direction" of Conjecture 2 holds, we may assume that $\mathrm{Irr}(B|\lambda) = \mathrm{Irr}_0(B|\lambda)$. By [8, Theorem 9.4], the set $\mathrm{Irr}(B|\lambda)$ is not empty and a result of Murai [7, Theorem 4.4] implies that λ extends to D. We show by induction on $|G|$ that D/Z is abelian.

Let $K \trianglelefteq G$ be the kernel of λ. Suppose first that $K \ne 1$. Then B dominates a unique block $\overline{B}$ of G/K with defect group D/K (see [8, Theorem 9.10]). Since the kernel of every $\chi \in \mathrm{Irr}(B|\lambda)$ contains K, we have $\mathrm{Irr}(\overline{B}|\lambda) = \mathrm{Irr}(B|\lambda) = \mathrm{Irr}_0(B|\lambda) = \mathrm{Irr}_0(\overline{B}|\lambda)$. By induction, it follows that $(D/K)/(Z/K) \cong D/Z$ is abelian.

Therefore, we may assume that λ is faithful. This implies $D' \cap Z = 1$, since λ extends to D. Let $\mathcal{F}$ be the fusion system of B. Then $Z \le \mathrm{Z}(\mathcal{F})$ and it follows from [4, Lemma 4.3] that $\mathfrak{foc}(\mathcal{F}) \cap Z = 1$.

Let $\chi \in \mathrm{Irr}(B)$ and $\mu \in \mathrm{Irr}(Z|\chi)$. Since $\mathfrak{foc}(\mathcal{F}) \cap Z = 1$, there exists an extension $\theta \in \mathrm{Irr}(D)$ of $\lambda\mu^{-1}$ with $\mathfrak{foc}(\mathcal{F}) \le \mathrm{Ker}(\theta)$. By Broué–Puig [3, Corollary], we obtain a character

$$\psi := \theta * \chi \in \mathrm{Irr}(B|\lambda)$$

(cf. [10]). By hypothesis, ψ has height 0 and the same holds for χ, since $\chi(1) = \psi(1)$. Consequently, $\mathrm{Irr}(B) = \mathrm{Irr}_0(B)$ and Conjecture 1 shows that D is abelian and so is D/Z. □

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