Cartan matrices and Brauer’s $k(B)$-conjecture

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Let $G$ be a finite group and $p$ be a prime.

Let $B$ be a $p$-block of $G$ with respect to a sufficiently large $p$-modular system.

We denote the number of ordinary irreducible characters of $B$ by $k(B)$, and the number of modular irreducible characters by $l(B)$.

It is well known that the Cartan matrix $C$ of $B$ cannot be arranged in the form $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$, i.e. $C$ is indecomposable.

However, in practice $C$ is often only known up to basic sets, i.e. up to a matrix $S \in \text{GL}(l(B), \mathbb{Z})$ with $S^TCS$. 

A question

We call two matrices $M_1, M_2 \in \mathbb{Z}^{l \times l}$ equivalent if there exists a matrix $S \in \text{GL}(l, \mathbb{Z})$ such that $M_1 = S^T M_2 S$.

Open question

Is the Cartan matrix $C$ equivalent to a decomposable matrix?

This can certainly happen for arbitrary matrices. For example $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is indecomposable, but $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^T A \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not.
Proposition

Let $d$ be the defect of $B$. If $\det C = p^d$, then the matrices $S^TCS$ are indecomposable for every $S \in \text{GL}(l(B), \mathbb{Z})$.

- In general, $p^d$ divides $\det C$. Thus, in the proposition $\det C$ is minimal.
- This holds for blocks with cyclic defect groups for instance.
- Moreover, $\det C$ can be determined locally with the notion of lower defect groups.
Some results

**Proposition**

If $G$ is $p$-solvable and $l := l(B) \geq 2$, then $C$ is not equivalent to a matrix of the form $\begin{pmatrix} p^d & 0 \\ 0 & C_1 \end{pmatrix}$, where $C_1 \in \mathbb{Z}^{(l-1) \times (l-1)}$. In particular $C$ is not equivalent to a diagonal matrix.

- The proof of this proposition uses a result by Fong that the Cartan invariants are bounded by $p^d$ for $p$-solvable groups.
- This is known to be false for arbitrary groups.
Proposition (Külshammer-Wada)

Let $B$ be a block with Cartan matrix $C = (c_{ij})$ up to equivalence. Then for every positive definite, integral quadratic form $q := \sum_{1 \leq i \leq j \leq l(B)} q_{ij} X_i X_j$ we have

$$k(B) \leq \sum_{1 \leq i \leq j \leq l(B)} q_{ij} c_{ij}.$$ 

In particular

$$k(B) \leq \sum_{i=1}^{l(B)} c_{ii} - \sum_{i=1}^{l(B)-1} c_{i,i+1}. \quad (KW)$$
These bounds are usually sharper for indecomposable Cartan matrices.

Example

Assume that $l(B) = p = 2$ and $C$ has elementary divisors 2 and 16. Then $C$ is equivalent to

$$\begin{pmatrix} 2 & 0 \\ 0 & 16 \end{pmatrix} \text{ or } \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}.$$ 

Inequality (KW) gives $k(B) \leq 18$ in the first case and $k(B) \leq 10$ in the second.
One can view $C = (c_{ij})$ as a quadratic form $q(x) := xC x^T$ for $x \in \mathbb{Z}^l$ with $l := l(B)$.

The reduction theory of quadratic forms allows to replace $C$ by an equivalent matrix with “small” entries.

More precisely we may assume that

$$c_{ii} \leq q(x) \text{ for } x = (x_1, \ldots, x_l) \in \mathbb{Z}^l \text{ with } \gcd(x_1, \ldots, x_l) = 1.$$  

It follows that $c_{11} \leq c_{22} \leq \ldots \leq c_{ll}$ and $2|c_{ij}| \leq \min\{c_{ii}, c_{jj}\}$ for $i \neq j$. 
Moreover, the “fundamental inequality”

\[ c_{11} c_{22} \ldots c_{ll} \leq \lambda_l \det C \]

holds for a constant \( \lambda_l \) which only depends on \( l \).

In particular there are only finitely many equivalence classes of Cartan matrices for a given block.

However, \( \lambda_l \) increases rapidly with \( l \).
A bound for $k(B)$

**Theorem**

Let $B$ be a block with defect $d$ and Cartan matrix $C$. If $\det C = p^d$ and $l(B) \leq 4$, then

$$k(B) \leq \frac{p^d - 1}{l(B)} + l(B).$$

Moreover, this bound is sharp.
Let $u$ be a $p$-element of $G$, and let $b$ be a Brauer correspondent of $B$ in $C_G(u)$.

Then the pair $(u, b)$ is called $B$-subsection.

If $b$ and $B$ have the same defect, then $(u, b)$ is called major.

If $u$ lies in the center of a defect group of $B$, then $(u, b)$ is major.

For the rest of this talk we assume $p = 2$. 
Proposition

Let \((u, b)\) be a major \(B\)-subsection. Then for every positive definite, integral quadratic form \(q(x_1, \ldots, x_{l(b)}) = \sum_{1 \leq i \leq j \leq l(b)} q_{ij} x_i x_j\) we have

\[
k(B) \leq \sum_{1 \leq i \leq j \leq l(b)} q_{ij} c_{ij},
\]

where \(C = (c_{ij})\) is the Cartan matrix of \(b\). In particular

\[
k(B) \leq \sum_{i=1}^{l(b)} c_{ii} - \sum_{i=1}^{l(b)-1} c_{i,i+1}.
\]
The Cartan invariants of $b$ can often be determined easier than the Cartan invariants of $B$.

Brauer's $k(B)$-conjecture asserts that $k(B) \leq p^d$ holds for every block $B$ of defect $d$.

**Theorem**

Brauer's $k(B)$-conjecture holds for defect groups which are central extensions of metacyclic 2-groups by cyclic groups. In particular the $k(B)$-conjecture holds for abelian defect 2-groups of rank at most 3.
2-Blocks of defect at most 4

**Theorem**

*Brauer’s $k(B)$-conjecture holds for defect groups which contain a central cyclic subgroup of index 8.*

**Corollary**

*Brauer’s $k(B)$-conjecture holds for 2-blocks of defect at most 4.*
A group is called **minimal nonabelian** if every proper subgroup is abelian.

**Proposition (Rédei)**

A minimal nonabelian 2-group is metacyclic or of type

\[ D(r, s) := \langle x, y \mid x^{2r} = y^{2s} = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle \]

with \( r \geq s \geq 1 \), \([x, y] := xyx^{-1}y^{-1} \) and \([x, x, y] := [x, [x, y]]\).
2-Blocks with minimal nonabelian defect groups

**Theorem**

Brauer’s $k(B)$-conjecture holds for 2-blocks with minimal nonabelian defect groups. Moreover, let $Q$ be a minimal nonabelian 2-group, but not of type $D(r, r)$ with $r \geq 3$ (these groups have order $2^{2r+1} \geq 128$). Then Brauer’s $k(B)$-conjecture holds for defect groups with are central extensions of $Q$ by a cyclic group.
Proposition

Brauer’s $k(B)$-conjecture holds for defect groups which are central extensions of $C_4 \wr C_2$ by a cyclic group.

This follows from the PhD thesis of Külshammer about defect groups of type $C_{2n} \wr C_2$. 
A question

- For a block $B$ with Cartan matrix $C = (c_{ij})$ there is not always a positive definite quadratic form $q$ such that

$$k(B) = \sum_{1 \leq i \leq j \leq l(B)} q_{ij} c_{ij}.$$ 

- One may ask if there is always a positive definite quadratic form $q$ such that

$$\sum_{1 \leq i \leq j \leq l(B)} q_{ij} c_{ij} \leq p^d,$$

where $d$ is the defect of $B$.

- A positive answer would imply Brauer’s $k(B)$-conjecture in general.
A counterexample

Let $D \cong C_2^4$, $A \in \text{Syl}_3(\text{Aut}(D))$, $G = D \rtimes A$ and $B = B_0(G)$.

Then $k(B) = |D| = 16$, $l(B) = 9$, and the Cartan matrix $C$ of $B$ is given by

$$C = \begin{pmatrix}
4 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\
2 & 4 & 2 & 1 & 2 & 1 & 1 & 2 & 1 \\
2 & 2 & 4 & 2 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 2 & 4 & 1 & 2 & 1 & 2 & 2 \\
1 & 2 & 1 & 1 & 4 & 1 & 2 & 2 & 2 \\
2 & 1 & 1 & 2 & 1 & 4 & 2 & 2 & 1 \\
2 & 1 & 1 & 1 & 2 & 2 & 4 & 1 & 2 \\
1 & 2 & 1 & 2 & 2 & 1 & 4 & 1 & 1 \\
1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 4
\end{pmatrix}.$$
A counterexample

- Assume
  \[ \sum_{1 \leq i \leq j \leq 9} q_{ij} c_{ij} \leq 16. \]

- Then it is easy to see that \( q_{ii} = 1 \) for \( i = 1, \ldots, 9 \) and \( q_{ij} \in \{-1, 0, 1\} \) for \( i \neq j \).

- Using GAP, we showed that \( q \) does not exist.
Recently Kessar, Koshitani and Linckelmann obtained the invariants of blocks with defect group $C_2^3$ (using the classification).

We use their result (and thus the classification) to extend the previous results.

For this, let $e(B)$ be the inertial index of a block $B$.

**Theorem**

Let $B$ be a block with a defect group which is a central extension of a group $Q$ of order 16 by a cyclic group. If $Q \not\cong C_2^4$ or $9 \nmid e(B)$, then Brauer’s $k(B)$-conjecture holds for $B$. 
2-Blocks of defect 5

The exception in this theorem is due to the counterexample shown above.

**Corollary**

Let $B$ be a block with defect group $D$ of order 32. If $D$ is not extraspecial of type $D_8 \rtimes D_8$ or if $9 \nmid e(B)$, then Brauer’s $k(B)$-conjecture holds for $B$.

In particular the $k(B)$-conjecture holds for $D \cong C_2^5$. In this case it is possible to choose a major subsection $(u, b)$ such that $9 \nmid e(b)$. 
A group is called **minimal nonmetacyclic** if every proper subgroup is metacyclic.

**Proposition (Blackburn)**

*There are just four minimal nonmetacyclic 2-groups:*

(i) $C_2^3$,

(ii) $Q_8 \times C_2$,

(iii) $D_8 \ast C_4 \cong Q_8 \ast C_4$ (*central product*),

(iv) $D := \langle x, y, z \mid x^4 = y^4 = [x, y] = 1, \ z^2 = x^2, \ zxz^{-1} = xy^2, \ zyz^{-1} = x^2y \rangle$. 
One can show that $\mathcal{D}$ has order 32.

**Proposition**

*Every fusion system on $\mathcal{D}$ is nilpotent. In particular every block with defect group $\mathcal{D}$ is nilpotent.*

- As a byproduct of the former results, we obtain the block invariants of 2-blocks with minimal nonmetacyclic defect groups.
- For this, let $k_i(B)$ be the number of irreducible characters of height $i \in \mathbb{N}_0$. 
Theorem

Let $B$ be a 2-block with minimal nonmetacyclic defect group $D$. Then one of the following holds:

(i) $B$ is nilpotent. Then $k_i(B)$ is the number of ordinary characters of $D$ of degree $2^i$. In particular $k(B)$ is the number of conjugacy classes of $D$ and $k_0(B) = |D : D'|$. Moreover, $l(B) = 1$.

(ii) $D \cong C_2^3$. Then $k(B) = k_0(B) = 8$ and $l(B) \in \{3, 5, 7\}$ (all cases occur).

(iii) $D \cong Q_8 \times C_2$ or $D \cong D_8 \rtimes C_4$. Then $k(B) = 14$, $k_0(B) = 8$, $k_1(B) = 6$ and $l(B) = 3$. 