Cartan matrices and Brauer's $k(B)$-conjecture

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Introduction

- Let G be a finite group and p be a prime.
- Let B be a p-block of G with respect to a sufficiently large p-modular system.
- We denote the number of ordinary irreducible characters of B by $k(B)$, and the number of modular irreducible characters by $l(B)$.
- It is well known that the Cartan matrix C of B cannot be arranged in the form $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$, i. e. C is indecomposable.
- However, in practice C is often only known up to basic sets, i. e. up to a matrix $S \in \mathrm{GL}(l(B), \mathbb{Z})$ with $S^{\mathrm{T}}CS$.

A question

We call two matrices $M_1, M_2 \in \mathbb{Z}^{l\times l}$ equivalent if there exists a matrix $S \in \mathrm{GL}(l,\mathbb{Z})$ such that $M_1 = S^{\mathsf{T}} M_2 S$.

Open question

Is the Cartan matrix C equivalent to a decomposable matrix?

This can certainly happen for arbitrary matrices. For example $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ is indecomposable, but $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)^{\mathsf{T}} A \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not.

Some results

Proposition

Let d be the defect of B. If $\det C = p^d$, then the matrices $S^T C S$ are indecomposable for every $S \in \mathrm{GL}(l(B), \mathbb{Z})$.

- In general p^d divides $\det C$. Thus, in the proposition $\det C$ is minimal.
- This holds for blocks with cyclic defect groups for instance.
- Moreover, $\det C$ can be determined locally with the notion of lower defect groups.

Some results

Proposition

If G is p-solvable and $l := l(B) \geq 2$, then C is not equivalent to a matrix of the form $\begin{pmatrix} p^d & 0 \\ 0 & C_1 \end{pmatrix}$, where $C_1 \in \mathbb{Z}^{(l-1)\times(l-1)}$. In particular C is not equivalent to a diagonal matrix.

- The proof of this proposition uses a result by Fong that the Cartan invariants are bounded by p^d for p-solvable groups.
- This is known to be false for arbitrary groups.

Motivation

Proposition (Külshammer-Wada)

Let B be a block with Cartan matrix $C = (c_{ij})$ up to equivalence. Then for every positive definite, integral quadratic form $q := \sum_{1\le i\le j\le l(B)} q_{ij}X_iX_j$ we have

$$k(B) \le \sum_{1\le i\le j\le l(B)} q_{ij}c_{ij}.$$

In particular

$$k(B) \le \sum_{i=1}^{l(B)} c_{ii} - \sum_{i=1}^{l(B)-1} c_{i,i+1}. \qquad \text{(KW)}$$

An example

These bounds are usually sharper for indecomposable Cartan matrices.

Example

Assume that $l(B) = p = 2$ and C has elementary divisors 2 and 16. Then C is equivalent to

$$\begin{pmatrix} 2 & 0 \\ 0 & 16 \end{pmatrix} \text{ or } \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}.$$

Inequality (KW) gives $k(B) \leq 18$ in the first case and $k(B) \leq 10$ in the second.

Quadratic forms

- One can view $C = (c_{ij})$ as a quadratic form $q(x) := xCx^{\mathrm{T}}$ for $x \in \mathbb{Z}^l$ with $l := l(B)$.
- The reduction theory of quadratic forms allows to replace C by an equivalent matrix with "small" entries.
- More precisely we may assume that

$$c_{ii} \leq q(x) \text{ for } x = (x_1, \ldots, x_l) \in \mathbb{Z}^l \text{ with } \gcd(x_i, \ldots, x_l) = 1.$$

- It follows that $c_{11} \leq c_{22} \leq \ldots \leq c_{ll}$ and $2|c_{ij}| \leq \min\{c_{ii}, c_{jj}\}$ for $i \neq j$

Quadratic forms

- Moreover, the "fundamental inequality"

$$c_{11}c_{22}\ldots c_{ll} \le \lambda_l \det C$$

holds for a constant λ_l which only depends on l.

- In particular there are only finitely many equivalence classes of Cartan matrices for a given block.
- However, λ_l increases rapidly with l.

A bound for $k(B)$

Theorem

Let B be a block with defect d and Cartan matrix C. If $\det C = p^d$ *and* $l(B) \leq 4$*, then*

$$k(B) \leq \frac{p^d - 1}{l(B)} + l(B).$$

Moreover, this bound is sharp.

Subsections

- Let u be a p-element of G, and let b be a Brauer correspondent of B in $C_G(u)$.
- Then the pair (u, b) is called *B-subsection*.
- If b and B have the same defect, then (u, b) is called major.
- If u lies in the center of a defect group of B, then (u, b) is major.
- For the rest of this talk we assume $p = 2$.

A generalization for $p = 2$

Proposition

Let (u, b) be a major B-subsection. Then for every positive definite, integral quadratic form $q(x_1, \ldots, x_{l(b)}) = \sum_{1\le i\le j\le l(b)} q_{ij}x_ix_j$ we have

$$k(B) \le \sum_{1\le i\le j\le l(b)} q_{ij}c_{ij},$$

where $C = (c_{ij})$ is the Cartan matrix of b. In particular

$$k(B) \le \sum_{i=1}^{l(b)} c_{ii} - \sum_{i=1}^{l(b)-1} c_{i,i+1}.$$

Central extensions

- The Cartan invariants of b can often be determined easier than the Cartan invariants of B.
- Brauer's $k(B)$-conjecture asserts that $k(B) \le p^d$ holds for every block B of defect d.

Theorem

Brauer's $k(B)$-conjecture holds for defect groups which are central extensions of metacyclic 2-groups by cyclic groups. In particular the $k(B)$-conjecture holds for abelian defect 2-groups of rank at most 3.

2-Blocks of defect at most 4

Theorem

Brauer's $k(B)$-conjecture holds for defect groups which contain a central cyclic subgroup of index 8.

Corollary

Brauer's $k(B)$-conjecture holds for 2-blocks of defect at most 4.

Minimal nonabelian groups

A group is called minimal nonabelian if every proper subgroup is abelian.

Proposition (Rédei)

A minimal nonabelian 2-group is metacyclic or of type

$$\mathcal{D}(r,s) := \langle x, y \mid x^{2^r} = y^{2^s} = [x,y]^2 = [x,x,y] = [y,x,y] = 1\rangle$$

with $r \geq s \geq 1$, $[x,y] := xyx^{-1}y^{-1}$ *and* $[x,x,y] := [x,[x,y]]$.

2-Blocks with minimal nonabelian defect groups

Theorem

Brauer's $k(B)$-conjecture holds for 2-blocks with minimal nonabelian defect groups. Moreover, let Q be a minimal nonabelian 2-group, but not of type $\mathcal{D}(r,r)$ with $r \geq 3$ (these groups have order $2^{2r+1} \geq 128$). Then Brauer's $k(B)$-conjecture holds for defect groups with are central extensions of Q by a cyclic group.

Wreath products

Proposition

Brauer's $k(B)$-conjecture holds for defect groups which are central extensions of $C_4 \wr C_2$ by a cyclic group.

This follows from the PhD thesis of Külshammer about defect groups of type $C_{2^n} \wr C_2$.

A question

- For a block B with Cartan matrix $C = (c_{ij})$ there is not always a positive definite quadratic form q such that

$$k(B) = \sum_{1 \le i \le j \le l(B)} q_{ij} c_{ij}.$$

- One may ask if there is always a positive definite quadratic form q such that

$$\sum_{1 \le i \le j \le l(B)} q_{ij} c_{ij} \le p^d,$$

  where d is the defect of B.
- A positive answer would imply Brauer's $k(B)$-conjecture in general.

A counterexample

- Let $D \cong C_2^4$, $A \in \mathrm{Syl}_3(\mathrm{Aut}(D))$, $G = D \rtimes A$ and $B = B_0(G)$.
- Then $k(B) = |D| = 16$, $l(B) = 9$, and the Cartan matrix C of B is given by

$$C = \begin{pmatrix} 4 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 2 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 & 1 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 & 4 & 1 & 2 & 2 & 2 \\ 2 & 1 & 1 & 2 & 1 & 4 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 & 2 & 2 & 4 & 1 & 2 \\ 1 & 2 & 1 & 2 & 2 & 2 & 1 & 4 & 1 \\ 1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 4 \end{pmatrix}.$$

A counterexample

- Assume
$$\sum_{1\le i\le j\le 9} q_{ij}c_{ij} \le 16.$$
- Then it is easy to see that $q_{ii} = 1$ for $i = 1, \ldots, 9$ and $q_{ij} \in \{-1, 0, 1\}$ for $i \neq j$.
- Using GAP, we showed that q does not exist.

2-Blocks of defect 5

- Recently Kessar, Koshitani and Linckelmann obtained the invariants of blocks with defect group C_2^3 (using the classification).
- We use their result (and thus the classification) to extend the previous results.
- For this, let $e(B)$ be the inertial index of a block B.

Theorem

Let B be a block with a defect group which is a central extension of a group Q of order 16 by a cyclic group. If $Q \not\cong C_2^4$ or $9 \nmid e(B)$, then Brauer's $k(B)$-conjecture holds for B.

2-Blocks of defect 5

The exception in this theorem is due to the counterexample shown above.

Corollary

Let B be a block with defect group D of order 32. If D is not extraspecial of type $D_8 * D_8$ or if $9 \nmid e(B)$, then Brauer's $k(B)$-conjecture holds for B.

In particular the $k(B)$-conjecture holds for $D \cong C_2^5$. In this case it is possible to choose a major subsection (u, b) such that $9 \nmid e(b)$.

Minimal nonmetacyclic groups

A group is called minimal nonmetacyclic if every proper subgroup is metacyclic.

Proposition (Blackburn)

There are just four minimal nonmetacyclic 2-groups:

(i) C_2^3,

(ii) $Q_8 \times C_2$,

(iii) $D_8 * C_4 \cong Q_8 * C_4$ *(central product),*

(iv) $\mathcal{D} := \langle x, y, z \mid x^4 = y^4 = [x, y] = 1,\ z^2 = x^2,$
$zxz^{-1} = xy^2,\ zyz^{-1} = x^2y \rangle$.

Fusion systems

One can show that $\mathcal{D}$ has order 32.

Proposition

Every fusion system on $\mathcal{D}$ is nilpotent. In particular every block with defect group $\mathcal{D}$ is nilpotent.

- As a byproduct of the former results, we obtain the block invariants of 2-blocks with minimal nonmetacyclic defect groups.
- For this, let $k_i(B)$ be the number of irreducible characters of height $i \in \mathbb{N}_0$.

2-Blocks with minimal nonmetacyclic defect groups

Theorem

Let B be a 2-block with minimal nonmetacyclic defect group D. Then one of the following holds:

(i) *B is nilpotent. Then $k_i(B)$ is the number of ordinary characters of D of degree 2^i. In particular $k(B)$ is the number of conjugacy classes of D and $k_0(B) = |D : D'|$. Moreover, $l(B) = 1$.*

(ii) *$D \cong C_2^3$. Then $k(B) = k_0(B) = 8$ and $l(B) \in \{3, 5, 7\}$ (all cases occur).*

(iii) *$D \cong Q_8 \times C_2$ or $D \cong D_8 * C_4$. Then $k(B) = 14$, $k_0(B) = 8$, $k_1(B) = 6$ and $l(B) = 3$.*