Donovan's Conjecture for certain defect groups

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Donovan's Conjecture

Let D be a finite p-group for a prime number p.

Donovan's Conjecture

There are only finitely many Morita equivalence classes of blocks of finite groups with defect group D.

A weak version is the following.

Conjecture

There is a bound on the Cartan invariants of blocks of finite groups with defect group D which only depends on D.

Known results

- If D is a cyclic group, then Donovan's Conjecture is true.
- If $p = 2$ and D has maximal class, all blocks with defect group D have tame representation type.
- Then, by Erdmann's work the Morita equivalence class of a block with defect group D is known up to certain parameters.
- Donovan's Conjecture also holds if one restricts to blocks of p-solvable, symmetric or alternating groups.
- Hiss and Kessar showed Donovan's Conjecture for some blocks of classical groups.

Rédei's classification

- In the following we consider blocks with respect to a splitting p-modular system $(\mathbb{K}, \mathcal{O}, \mathbb{F})$.
- Assume that $p = 2$ and D is a minimal nonabelian 2-group.
- This means all proper subgroups of D are abelian, but D is not.
- Then by a result of Rédei D is isomorphic to one of the following groups:
  - (a) $\langle x, y \mid x^{2^r} = y^{2^s} = 1,\ xyx^{-1} = y^{1+2^{s-1}} \rangle$ with $r \geq 1$ and $s \geq 2$,
  - (b) $\langle x, y \mid x^{2^r} = y^{2^s} = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle$ with $2 \leq r \geq s \geq 1$,
  - (c) Q_8.
- In case (a) or (c) D is metacyclic.

Rédei's classification

- Let B be a block of a finite group with defect group D as given above.
- If D is metacyclic, then B is nilpotent unless $D \cong D_8$ or $D \cong Q_8$.
- In the nilpotent case B is Morita equivalent to the group algebra $\mathcal{O}D$ by Puig's Theorem.
- For $D \cong D_8$ or $D \cong Q_8$ we can apply Erdmann's work.
- Hence, we may assume that case (b) in Rédei's classification occurs, i. e.

$$D := \langle x, y \mid x^{2^r} = y^{2^s} = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle,$$

where $2 \leq r \geq s \geq 1$, $[x, y] := xyx^{-1}y^{-1}$ and $[x, x, y] := [x, [x, y]]$.

The case $s = 1$

- Assume that B is not nilpotent.
- Then it turns out that $s = 1$ or $r = s$.
- Let $k_i(B)$ be the number of ordinary irreducible characters of height $i \geq 0$ of B, and let $k(B) = \sum_{i=0}^{\infty} k_i(B)$.
- Similarly $l(B)$ is the number of irreducible Brauer characters of B.
- For $s = 1$ these block invariants are given by the following theorem.

The case $s = 1$

Theorem (S., 2010)

Let B be a non-nilpotent block of a finite group with defect group

$$D = \langle x, y \mid x^{2^r} = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle$$

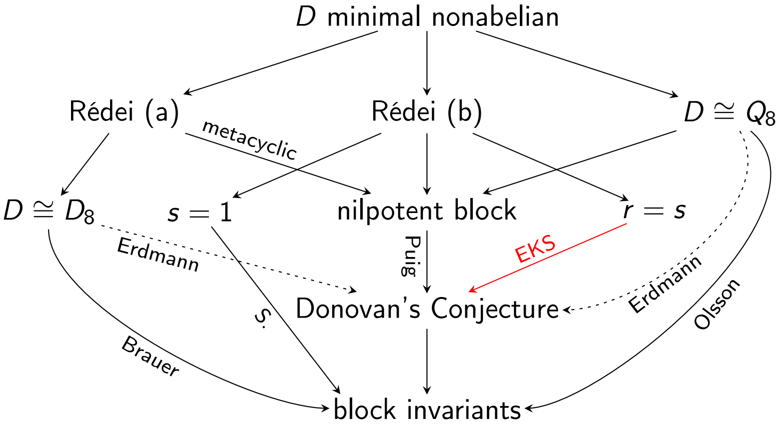
for some $r \geq 2$. Then

$$k(B) = 5 \cdot 2^{r-1}, \quad k_0(B) = 2^{r+1}, \quad k_1(B) = 2^{r-1}, \quad l(B) = 2$$

and the Cartan matrix of B is given by

$$2^{r-1} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

up to basic sets.


D minimal nonabelian
Rédei (a)
Rédei (b)
$D \cong Q_8$
metacyclic
$D \cong D_8$
$s = 1$
nilpotent block
$r = s$
Erdmann
Puig
EKS
Erdmann
Olsson
S.
Donovan's Conjecture
Brauer
block invariants

The case $r = s$

In the $r = s$ case we were able to prove Donovan's Conjecture:

Theorem

Let B be a non-nilpotent block of a finite group with defect group

$$D = \langle x, y \mid x^{2^r} = y^{2^r} = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$$

for some $r \geq 2$. Then B is Morita equivalent to $\mathcal{O}[D \rtimes E]$ where E is a subgroup of $\mathrm{Aut}(D)$ of order 3. In particular, we have

$$k(B) = \frac{5 \cdot 2^{2r-2} + 16}{3}, \quad k_0(B) = \frac{2^{2r} + 8}{3}, \quad k_1(B) = \frac{2^{2r-2} + 8}{3}$$

and $l(B) = 3$.

Sketch of the proof (1)

It turns out that the claim holds for solvable groups. Now the idea is to reduce the situation to quasisimple groups.

Lemma

Let G be a finite group with Sylow 2-subgroup D as given above. Then G is solvable.

Proof.

- By Feit-Thompson we may assume $O_{2'}(G) = 1$.
- Then the Z^*-Theorem implies $z := [x, y] \in Z(G)$. Thus, $G/\langle z\rangle$ has Sylow 2-subgroup $D/\langle z\rangle \cong C_{2^r} \times C_{2^r}$.
- By a result of Brauer, $G/\langle z\rangle$ and thus also G is solvable. □

Sketch of the proof (2)

- Let G be a finite group with a non-nilpotent block B as above.
- Then by Fong reduction we may assume that $O_{2'}(G)$ is cyclic and central.
- An application of the Külshammer-Puig Theorem gives

$$O_2(G) \subseteq D' = \langle z \rangle$$

and $Z(G) = F(G)$.

- It turns out that B covers a non-nilpotent block b of the layer $E(G)$ of G with defect group D.
- Moreover, b covers a non-nilpotent block of a component of G also with defect group D.

Sketch of the proof (3)

- Hence, by way of contradiction we may assume that G is quasi-simple, i. e. $G' = G$ and $G/\mathrm{Z}(G)$ is simple.
- Now we apply the classification of the finite simple groups.
- By the lemma above, $64\,(\le 2|D|)$ divides $|G|$.
- If $G/\mathrm{Z}(G)$ is an alternating group, the situation is very easy, since one can use the representation theory of symmetric groups.
- For the non-principal 2-blocks of the (covering groups of the) sporadic groups results of Landrock and An-Eaton can be used.

Sketch of the proof (4)

- Thus, it remains to deal with the simple groups of Lie type.
- By a result of Humphreys it suffices to consider Lie groups in odd characteristic.
- Here one can use methods going back to Deligne, Lusztig and others.
- We illustrate these for the case $G/\mathrm{Z}(G) \cong \mathrm{PSL}(n, q)$.
- Here one can go over to $H := \mathrm{GL}(n, q)$ (ignoring exceptional covers).

Sketch of the proof (5)

- Then there is a semisimple element $s \in H$ such that a Sylow 2-subgroup of $C_H(s)$ is related to D.
- In particular it can be shown that $C_H(s)$ is solvable.
- On the other hand $C_H(s)$ has the form

$$C_H(s) \cong \bigtimes_{i=1}^{t} GL(n_i, q^{m_i}).$$

- This leads to $q = 3$ and eventually to a contradiction. □

Proposition

In the situation of the last theorem, all simple B-modules have vertex D.

Proof.

- The result holds for $G = D \rtimes E$, since the irreducible Brauer characters are restrictions of ordinary characters of height 0 in this case.
- Since Morita equivalence preserves decomposition matrices, the result follows for arbitrary G. □

Corollary

For a 2-block B of a finite group with minimal nonabelian defect group the following conjectures are satisfied:

- *Alperin's Weight Conjecture*
- *Brauer's $k(B)$-Conjecture*
- *Brauer's Height-Zero Conjecture*
- *Dade's Ordinary Conjecture*
- *Alperin-McKay Conjecture*
- *Olsson's Conjecture*
- *Eaton's Conjecture*
- *Eaton-Moretó Conjecture*
- *Malle-Navarro Conjecture*

Background

- For primes $p > 3$, Héthelyi, Külshammer and myself proved Olsson's Conjecture for p-blocks with defect groups of p-rank 2.
- In the process the extraspecial defect group D of order 5^3 and exponent 5 turns out to be very difficult.
- In particular blocks with defect group D and a specific fusion system are complicated.
- We end up by determining the Morita equivalence class of such blocks.

A result

Proposition

Let B be a block of a finite group G with an extraspecial defect group D of order 5^3 and exponent 5. Suppose that the fusion system of B is the same as the fusion system of the sporadic simple Thompson group Th for the prime 5. Then B is Morita equivalent to the principal 5-block of Th.

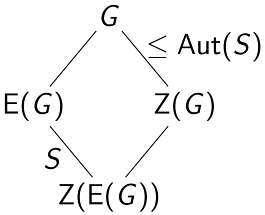
Sketch of the proof (1)

- We observe that all nontrivial elements of D are conjugate in the fusion system.
- By Fong reduction we may assume $\mathrm{F}(G) = \mathrm{Z}(G) = \mathrm{O}_{5'}(G)$.
- It turns out that G has only one component, i. e. $\mathrm{E}(G)$ is quasi-simple.
- So $S := \mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G))$ is simple and

$$G/\mathrm{Z}(G) \le \mathrm{Aut}(\mathrm{E}(G)) \le \mathrm{Aut}(S).$$

- B covers a block b of $\mathrm{E}(G)$ with defect group D.

Sketch of the proof (2)



- For the block b we use An and Eaton's classification of blocks of quasisimple groups with extraspecial defect groups.
- This shows that D must be a Sylow 5-subgroup of $\mathrm{E}(G)$.
- But then in most cases there are elements $x, y \in S$ of order 5 such that $|\mathrm{C}_S(x)| \neq |\mathrm{C}_S(y)|$.

Sketch of the proof (3)

- In particular, x and y cannot be conjugate in G. This contradicts the structure of the fusion system.
- The only remaining case is $S = Th$ and $b = B_0(\mathrm{E}(G))$.
- Fortunately here we have $\mathrm{Out}(Th) = \mathrm{M}(Th) = 1$, so that $G = S \times \mathrm{Z}(G)$.
- Hence, $B \cong b \otimes_{\mathcal{O}} \mathcal{O} \cong B_0(Th)$. □