The Alperin-McKay Conjecture for a special class of defect groups

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Let $G$ be a finite group and $p$ be a prime.

Let $B$ be a $p$-block of $G$, i.e. an ideal of $OG$ where $O$ is a complete discrete valuation ring of characteristic 0.

Let $k_i(B)$ be the number of irreducible characters of height $i \geq 0$ in $B$. Then $k(B) := \sum k_i(B) = |\text{Irr}(B)|$.

Let $\text{Irr}_0(B)$ be the subset of $\text{Irr}(B)$ of characters of height 0.

Let $l(B)$ be the number of irreducible Brauer characters of $B$.

Suppose that $B$ has metacyclic defect group $D$, i.e. $D$ has a cyclic normal subgroup such that the corresponding quotient is also cyclic.
What is known?

The case $p = 2$:

- $D$ is dihedral, semidihedral or quaternion (tame case):
  - $k(B)$, $k_i(B)$, $l(B)$ computed by Brauer and Olsson
  - perfect isometries constructed by Cabanes-Picaronny
  - Dade’s Invariant Conjecture verified by Uno
  - Donovan’s Conjecture almost settled by Erdmann and Holm (up to certain scalars)
- $D \cong C_{2^n} \times C_{2^n}$ is homocyclic for some $n \geq 1$:
  - case $n = 1$ known to Brauer (also tame case)
  - perfect isometries constructed by Usami-Puig
  - Donovan’s Conjecture and Broué’s Conjecture recently checked as follows:
What is known?

**Theorem (Eaton-Kessar-Külshammer-S., 2013)**

Suppose that $B$ is a 2-block with homocyclic defect group $D$. Then one of the following holds:

1. $B$ is nilpotent and thus Morita equivalent to $O_D$.
2. $B$ is Morita equivalent to $O[D \rtimes C_3]$.
3. $D \cong C_2 \times C_2$ and $B$ is Morita equivalent to $B_0(OA_5)$.

- remaining metacyclic 2-groups:
  - $B$ is nilpotent (Craven-Glesser, Robinson, S. independently)
  - algebra structure of $B$ known by a result of Puig

**Conclusion:** The case $p = 2$ is well-understood.
What is known?

The case $p > 2$:
- Brauer’s $k(B)$-Conjecture is true (Gao)
- Olsson’s Conjecture is true (Yang)
- Brauer’s Height Zero Conjecture is true (S.)
- Fusion system $\mathcal{F}$ on the subpairs of $B$ is controlled (Stancu)
- $D$ is cyclic:
  - case $|D| = p$ known to Brauer
  - in general, $k(B)$, $k_i(B)$, $l(B)$ computed by Dade
  - Donovan’s Conjecture verified (Brauer trees)
  - Broué’s Conjecture verified by Rickard and Linckelmann
What is known?

- **$D$ is abelian but non-cyclic:**
  - smallest case $D \cong C_3 \times C_3$ still open! Partial results by Kiyota and Watanabe.
  - Broué’s Conjecture and Donovan’s Conjecture checked for principal blocks in case $D \cong C_3 \times C_3$ by Koshitani, Kunugi and Miyachi.
  - perfect isometries known for $D \cong C_{3^m} \times C_{3^n}$ if $n \neq m$ (Usami-Puig)
  - More partial results for $D \cong C_p \times C_p$ by Kessar-Linckelmann

- **$D$ is non-abelian and non-split:**
  - $\text{Aut}(D)$ is a $p$-group (Dietz)
  - all blocks are nilpotent
\textbf{What is known?}

- $D$ is non-abelian and split:
  - $k(B)$, $k_i(B)$, $l(B)$ known if $B$ has \textit{maximal} defect (Gao)
  - perfect isometries constructed if $B$ is \textit{principal} by Horimoto and Watanabe
  - the $p$-\textit{solvable} case follows from a result by Külshammer
  - $G$ is not \textit{quasisimple} by work of An
  - $D \cong C_p^m \rtimes C_p$:
    - $k(B) - l(B)$ known by Gao-Zeng
    - Holloway, Koshitani and Kunugi determined $k(B)$, $k_i(B)$, $l(B)$ under additional assumptions on $G$
    - Partial results in case $|D| = p^3$ by Hendren
    - $k(B)$, $k_i(B)$, $l(B)$ determined for $|D| = 3^3$ by S.

\textbf{Conclusion}: Many things are open in case $p > 2$. 
Let \( p > 2 \) and

\[
D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, \ yxy^{-1} = x^{1+p^{m-1}} \rangle \cong C_{p^m} \rtimes C_{p^n}
\]

where \( m \geq 2 \) and \( n \geq 1 \).

- These are precisely the metacyclic defect groups \( D \) such that \( |D'| = p \).
- These are precisely the metacyclic, minimal non-abelian groups, i.e. all proper subgroups of \( D \) are abelian.
- The family includes the groups \( D \cong C_{p^m} \rtimes C_p \) mentioned above.
New results

Theorem (S., 2014)

*The Alperin-McKay Conjecture holds for all blocks with metacyclic, minimal non-abelian defect groups.*
Sketch of proof

**Idea:** Compute $k_0(B)$ in terms of the fusion system $\mathcal{F}$ only.

- By a result of Stancu, $\mathcal{F}$ is controlled, i.e. any conjugation on subpairs is induced from Aut($D$).
- Moreover, $\text{Out}_{\mathcal{F}}(D)$ is cyclic of order dividing $p - 1$ by a result of Sasaki.
- In particular, $\mathcal{F}$ only depends on the inertial index $e(B) := |\text{Out}_{\mathcal{F}}(D)|$.
- Let
  
  $$\text{foc}(B) := \langle f(a)a^{-1} : a \in Q \leq D, f \in \text{Aut}_{\mathcal{F}}(Q) \rangle$$

  be the *focal subgroup* of $B$.
- It follows that $\text{foc}(B)$ lies in the cyclic normal subgroup $\langle x \rangle$. In particular $p^n \mid |D : \text{foc}(B)|$. 
Sketch of proof

- By Broué-Puig and Robinson, $D/\mathfrak{soc}(B)$ acts freely on $\text{Irr}_0(B)$ via the $\ast$-construction.
- In particular $p^n \mid k_0(B)$.
- On the other hand, we have upper bounds for $k_0(B)$ and $\sum p^{2i}k_i(B)$ from Héthelyi-Külshammer-S. (using properties of decomposition numbers)
- Finally, a formula by Brauer gives a lower bound for $k(B)$.
- The claim follows by a combination of these estimates. □
The proof does not rely on the classification.

As a corollary, one gets $k_1(B) = k(B) - k_0(B)$.

This confirms less-known conjectures by Eaton, Eaton-Moretó, Robinson and Malle-Navarro, for $B$. 
Isaacs and Navarro proposed a refinement of the Alperin-McKay-Conjecture:

**Conjecture (Isaacs-Navarro, 2002)**

Let $b_D$ be the Brauer correspondent of $B$ in $N_G(D)$. Then for every $p$-automorphism $\gamma \in \text{Gal}(\mathbb{Q}|_G|\mathbb{Q}|_G|_{p'})$ we have

$$|\{\chi \in \text{Irr}_0(B) : \gamma \chi = \chi\}| = |\{\chi \in \text{Irr}_0(b_D) : \gamma \chi = \chi\}|.$$
Remarks

Proposition (S.)

The Isaacs-Navarro Conjecture holds for all blocks with defect group $C_p^2 \rtimes C_p$.

In fact every $p$-automorphism $\gamma \in \text{Gal}(\mathbb{Q}|_G|\mathbb{Q}|_G|_p)$ acts trivially on $\text{Irr}_0(B)$. 
The case $p = 3$

Theorem (S., 2014)

Let $B$ be a non-nilpotent 3-block with metacyclic, minimal non-abelian defect groups. Then

\[
\begin{align*}
  k_0(B) &= \frac{3^{m-2} + 1}{2} 3^{n+1}, \\
  k_1(B) &= 3^{m+n-3}, \\
  k(B) &= \frac{11 \cdot 3^{m-2} + 9}{2} 3^{n-1}, \\
  l(B) &= 2
\end{align*}
\]

where $m$ and $n$ are the parameters in the presentation of $D$. 
Since $B$ is non-nilpotent and $e(B) \mid p - 1$, we have $e(B) = 2$.

The theory of lower defect groups implies $l(B) \in \{2, 3\}$.

Use induction on $n$. In case $n = 1$ and $l(B) = 3$, decomposition numbers are in exceptional configuration → contradiction.

Let $n \geq 2$. Then induction gives $k(B) - l(B)$.

By a result of Robinson, $Z(D)\text{foc}(B)/\text{foc}(B)$ acts freely on $\text{Irr}(B)$ by the $\ast$-construction.

In particular $3^{n-1} \mid k(B)$, and the result follows. □
The proof is still classification-free.
The induction argument works for any prime $p > 2$.
Therefore, it suffices to handle the defect groups $D \cong C_{p^m} \rtimes C_p$ for $m \geq 2$. 
Remarks

- Since $\mathcal{F}$ is controlled and $\text{Out}_F(D)$ is cyclic, Alperin’s Weight Conjecture asserts $l(B) = e(B)$.
- The Ordinary Weight Conjecture is equivalent Dade’s Projective Conjecture and predicts $k_i(B)$ in terms of $\mathcal{F}$.

Corollary

*Alperin’s Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every 3-block with metacyclic, minimal non-abelian defect groups.*
The case $p = 5$

**Theorem (S., 2014)**

Let $B$ be a 5-block of a finite group with non-abelian defect group $C_{25} \rtimes C_{5^n}$ where $n \geq 1$. Then

\[
\begin{align*}
  k_0(B) &= \left( \frac{4}{e(B)} + e(B) \right) 5^n, \\
  k_1(B) &= \frac{4}{e(B)} 5^{n-1}, \\
  k(B) &= \left( \frac{24}{e(B)} + 5e(B) \right) 5^{n-1}, \\
  l(B) &= e(B).
\end{align*}
\]

Again Alperin’s Weight Conjecture and the Ordinary Weight Conjecture are satisfied in this special case.
Proposition

Let \( p \in \{7, 11, 13, 17, 23, 29\} \) and let \( B \) be a \( p \)-block with defect group \( C_{p^2} \rtimes C_{p^n} \) where \( n \geq 1 \). If \( e(B) = 2 \), then

\[
\begin{align*}
  k_0(B) &= \frac{p + 3}{2} p^n, \\
  k_1(B) &= \frac{p - 1}{2} p^{n-1}, \\
  k(B) &= \frac{p^2 + 4p - 1}{2} p^{n-1}, \\
  l(B) &= 2.
\end{align*}
\]
Let $B$ be a block with defect group $D$ and fusion system $\mathcal{F}$.

Then the hyperfocal subgroup of $B$ is defined by

$$\mathfrak{h}(B) := \langle f(a)a^{-1} : a \in Q \leq D, f \in O^p(\text{Aut}_\mathcal{F}(Q)) \rangle$$

By a result of Puig the source algebra $iBi$ of $B$ can be expressed as a crossed product:

$$iBi = \bigoplus_{x \in D/\mathfrak{h}(B)} \mathcal{H}x$$

where $\mathcal{H}$ is the hyperfocal subalgebra of $iBi$.

$\mathcal{H}$ is unique up to $(iBi^D)^\times$-conjugation as $D$-stable unitary subalgebra of $iBi$. 

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**Final remarks**

- Let $B$ be a block with defect group $D$ and fusion system $\mathcal{F}$.
- Then the hyperfocal subgroup of $B$ is defined by

$$\mathfrak{h}(B) := \langle f(a)a^{-1} : a \in Q \leq D, f \in O^p(\text{Aut}_\mathcal{F}(Q)) \rangle$$

By a result of Puig the source algebra $iBi$ of $B$ can be expressed as a crossed product:

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$\mathcal{H}$ is unique up to $(iBi^D)^\times$-conjugation as $D$-stable unitary subalgebra of $iBi$. 

Final remarks

- Moreover, $\mathcal{H} \cap Di = \eta \eta p(B)i$.
- If $D$ is non-abelian, metacyclic for an odd prime $p$, then $\eta \eta p(B) \subseteq \text{foc}(B)$ are cyclic.
- Assume that $F = \mathcal{O}/\text{Rad}(\mathcal{O})$ is an algebraically closed field of characteristic $p$.
- It follows from Watanabe that $\mathcal{H}$, considered as an algebra over $F$, has finite representation type.