1 Introduction

Following R. Brauer, the group algebra of a finite group $G$ over a field of characteristic $p$ (or a complete discrete valuation ring of residue characteristic $p$) splits into blocks. This leads to a distribution of the irreducible (ordinary and Brauer) characters of $G$ into blocks. For a block $B$, $k(B)$ denotes the number of irreducible ordinary characters of $G$ associated with $B$, and $l(B)$ denotes the number of irreducible Brauer characters of $G$ associated with $B$. Many of the central open problems in representation theory are concerned with these numbers. For example, Alperin’s Weight Conjecture [1] relates $l(B)$ to the number of $B$-weights. The number $k(B)$ appears in Brauer’s block-$k(B)$-Conjecture [2] which predicts $k(B) \leq |D|$ where $D$ is a defect group of $B$.

It is therefore an interesting task to determine the block invariants $k(B)$ and $l(B)$ with respect to a fixed defect group. Here it is often useful to study the heights of the irreducible characters. For an irreducible character $\chi$ of a block $B$ with defect group $D$ the height of $\chi$ is the largest integer $h(\chi) \geq 0$ such that $p^{h(\chi)}|G : D|_p$ divides $\chi(1)$. The number of characters of height $i$ is denoted by $k_i(B)$.

2 Block invariants

In my PhD thesis 2010, I determined the block invariants of 2-blocks with metacyclic defect groups [16]. It turned out that these numbers only depend on the fusion system of the block (this was independently obtained by Craven-Glesser [1]). The following result relies on preliminary work of Puig-Usami [12].

**Theorem 1.** Let $B$ be a 2-block of a finite group $G$ with a metacyclic defect group $D$. Then one of the following holds:

(i) $B$ is nilpotent. Then $k_1(B)$ is the number of ordinary characters of $D$ of degree $2^i$. In particular $k(B)$ is the number of conjugacy classes of $D$ and $k_0(B) = |D : D'|$. Moreover, $l(B) = 1$.

(ii) $D$ has maximal class. Then Theorem 3 below applies.

(iii) $D$ is a direct product of two isomorphic cyclic groups. Then $k(B) = k_0(B) = \frac{|D|+8}{3}$ and $l(B) = 3$.

It follows easily that the major counting conjecture are satisfied in this case.

Later in collaboration with Charles Eaton and Burkhard Külshammer, I obtained the block invariants of 2-blocks with minimal nonabelian defect groups [17] [5]. Here minimal nonabelian means that all proper subgroups are abelian, but the whole group is not. Rédei gave a classification of the minimal nonabelian $p$-groups [13]. We use the notation $[x, y] := xyx^{-1}y^{-1}$ and $[x, x, y] := [x, [x, y]]$.

**Theorem 2.** Let $B$ be a 2-block of a finite group $G$ with a minimal nonabelian defect group $D$. Then one of the following holds:

(i) $B$ is nilpotent. Then $k(B) = \frac{5}{3}|D|$, $k_0(B) = \frac{1}{2}|D|$, $k_1(B) = \frac{1}{8}|D|$ and $l(B) = 1$.

(ii) $|D| = 8$. Then Theorem 3 applies.

(iii) $D \cong \langle x, y \mid x^{2r} = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle$ for some $r \geq 2$. Then $k(B) = 5 \cdot 2^{r-1}$, $k_0(B) = 2^{r+1}$, $k_1(B) = 2^{r-1}$ and $l(B) = 2$.

(iv) $D \cong \langle x, y \mid x^{2r} = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle$ for some $r \geq 2$. Then $B$ is Morita equivalent to the group algebra of $D \times E$ where $E$ is a subgroup of $\text{Aut}(D)$ of order 3. In particular, $k(B) = \frac{5 \cdot 2^{2r-2} + 16}{3}$, $k_0(B) = \frac{2^{r+8}}{3}$, $k_1(B) = \frac{2^{r-2} + 8}{3}$ and $l(B) = 3$. 

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The last possibility in this theorem gives an example of Donovan’s Conjecture.

In recent papers [19, 13, 14], I was also able to handle 2-blocks with defect group $M \times C_{2^m}$ or $M \ast C_{2^m}$. Here $M$ is a 2-group of maximal class, $C_{2^m}$ is a cyclic group of order $2^m$ and $M \ast C_{2^m}$ denotes the central product. Moreover, $D_{2^n}$ (resp. $Q_{2^n}$, $SD_{2^n}$) is the dihedral (resp. quaternion, semidihedral) group of order $2^n$.

The following result generalizes work by Brauer [3] and Olsson [10].

**Theorem 3.** Let $B$ be a non-nilpotent 2-block of a finite group $G$ with defect group $D$, and let $m \geq 0$.

(i) If $D \cong D_{2^n} \times C_{2^m}$ for some $n \geq 3$, then $k(B) = 2^m(2^{n-2} + 3)$, $k_0(B) = 2^{m+2}$ and $k_1(B) = 2^m(2^{n-2} - 1)$.

According to two different fusion systems, $l(B)$ is 2 or 3.

(ii) If $D \cong Q_8 \times C_{2^m}$ or $D \cong Q_8 \ast C_{2^{m+1}}$, then $k(B) = 2^{m+7}$, $k_0(B) = 2^{m+2}$ and $k_1(B) = 2^{m} \cdot 3$ and $l(B) = 3$.

(iii) If $D \cong Q_{2^n} \times C_{2^m}$ or $D \cong Q_{2^n} \ast C_{2^{m+1}}$ for some $n \geq 4$, then $k_0(B) = 2^{m+2}$ and $k_1(B) = 2^m(2^{n-2} - 1)$.

According to two different fusion systems, one of the following holds

(a) $k(B) = 2^m(2^{n-2} + 4)$, $k_{n-2}(B) = 2^m$ and $l(B) = 2$.

(b) $k(B) = 2^m(2^{n-2} + 5)$, $k_{n-2}(B) = 2^{m+1}$ and $l(B) = 3$.

(iv) If $D \cong SD_{2^n} \times C_{2^m}$ for some $n \geq 4$, then $k_0(B) = 2^{m+2}$ and $k_1(B) = 2^m(2^{n-2} - 1)$. According to three different fusion systems, one of the following holds

(a) $k(B) = 2^m(2^{n-2} + 3)$ and $l(B) = 2$.

(b) $k(B) = 2^m(2^{n-2} + 4)$, $k_{n-2}(B) = 2^m$ and $l(B) = 2$.

(c) $k(B) = 2^m(2^{n-2} + 4)$, $k_{n-2}(B) = 2^m$ and $l(B) = 3$.

Notice that $Q_{2^n} \ast C_{2^m} \cong D_{2^n} \ast C_{2^m} \cong SD_{2^n} \ast C_{2^m}$ for $m \geq 2$. It should be pointed out that also the invariants for the defect group $D_4 \times C_{2^m}$ and $D_4 \ast C_{2^m}$ are known by work of Puig-Usami [12] and Kessar-Koshitan-Linckelmann [7].

These theorems together with one half of Brauer’s Height Zero Conjecture (which was proved recently by Kessar-Malle [8]) imply that the invariants of 2-blocks with defect at most 4 are known in almost all cases. Here, only for a block with elementary abelian defect group of order 16 and inertial index 15 it is not clear to my knowledge if Alperin’s Weight Conjecture holds (see [9]).

### 3 Conjectures

In the last two years I also made progress on some of the open conjectures in representation theory.

**Theorem 4.** Brauer’s $k(B)$-Conjecture holds for defect groups which contain a central, cyclic subgroup of index at most 9.

**Theorem 5.** Let $B$ be a block with a defect group which is a central extension of a group $Q$ of order 16 by a cyclic group. If $Q$ is not elementary abelian or if 9 does not divide the inertial index of $B$, then Brauer’s $k(B)$-conjecture holds for $B$.

As a corollary one gets Brauer’s $k(B)$-Conjecture for the 3-blocks of defect at most 3 and most 2-blocks of defect at most 5 (see [18]).

Another related conjecture was proposed by Olsson [11]: For a block $B$ with defect group $D$ it holds that $k_0(B) \leq |D : D'|$ where $D'$ is the commutator subgroup of $D$. In a joint work with László Héthelyi and Burkhard Külshammer, I verified Olsson’s Conjecture under certain hypotheses [6].

**Theorem 6.** Let $p > 3$. Then Olsson’s Conjecture holds for all $p$-blocks with defect groups of $p$-rank 2 and for all $p$-blocks with minimal non-abelian defect groups.
More detailed information is available if one involves the notion of subsections. A subsection for the block \( B \) is a pair \((u, b_u)\) where \( u \) is \( p \)-element of \( G \) and \( b_u \) is a Brauer correspondent of \( B \) in \( C_G(u) \). If \( b_u \) and \( B \) have the same defect, the subsection is called major.

**Theorem 7.** Let \( B \) be a \( p \)-block of a finite group \( G \) where \( p \) is an odd prime, and let \((u, b_u)\) be a \( B \)-subsection such that \( l(b_u) = 1 \) and \( b_u \) has defect \( d \). Moreover, let \( F \) be the fusion system of \( B \) and \( |\text{Aut}_F(\langle u \rangle)| = p^s r \), where \( p \nmid r \) and \( s \geq 0 \). Then we have

\[
k_0(B) \leq \frac{|\langle u \rangle| + p^s(r^2 - 1)}{|\langle u \rangle| \cdot r} p^d.
\]

If (in addition) \((u, b_u)\) is major, we can replace \( k_0(B) \) by \( \sum_{i=0}^{\infty} p^{2i} k_i(B) \) in (1).

**References**


[15] B. Sambale, *Blocks with defect group \( Q_{2^n} \times C_{2^m} \) and \( SD_{2^n} \times C_{2^m} \)*, submitted.


