Further evidence for conjectures in block theory

Benjamin Sambale, FSU Jena

DMV-Tagung Saarbrücken, September 18, 2012
Let $G$ be a finite group and $p$ be a prime.

Let $B$ be a $p$-block of $G$ with defect $d$.

We denote the number of irreducible characters of $B$ by $k(B)$, and the number of irreducible Brauer characters by $l(B)$.

**Theorem (Olsson, 1981)**

(i) If $l(B) \leq 2$, then $k(B) \leq p^d$.
(ii) If $p = 2$ and $l(B) \leq 3$, then $k(B) \leq 2^d$.

*In particular Brauer’s $k(B)$-Conjecture holds in these cases.*
Remarks

- Usually the knowledge of $l(B)$ implies the exact value of $k(B)$.
- Hence, Olsson’s result is more of theoretical nature.
- In order to improve Olsson’s theorem, the idea is to replace $l(B)$ by something “local”.
- Let $D$ be a defect group of $B$, and let $u \in Z(D)$.
- Then there is a Brauer correspondent $b_u$ of $B$ in $C_G(u)$.
- The pair $(u, b_u)$ is called major subsection (for $B$).
A generalization

**Theorem (S., 2012)**

Let $p = 2$, and let $(u, b_u)$ a major $B$-subsection such that $l(b_u) \leq 3$. Then

$$k(B) \leq k_0(B) + \frac{2}{3} \sum_{i=1}^{\infty} 2^i k_i(B) \leq 2^d.$$ 

In particular Brauer’s $k(B)$-Conjecture holds for $B$.

Here $k_i(B)$ denotes the number of irreducible characters of height $i \geq 0$ of $B$. 
The proof relies on calculations with so-called "contributions" which were introduced by Brauer.

In contrast to Olsson’s proof, the contributions are not always integers in this general setting.

Applying Galois theory fixes this issue.

However, for odd primes $p$ things are more difficult, since the cyclotomic fields behave differently.
Olsson also proved the $k(B)$-Conjecture under the hypothesis $k(B) - l(B) \leq 3$ for $p = 2$.

This can also be carried over to $k(B) - l(b_u) \leq 3$.

However, in many cases we have $k(B) - l(B) \leq k(B) - l(b_u)$; so the general result adds little.

Let us go over to arbitrary subsections, i.e. $u \in D$ does not necessarily belong to the center of $D$. 
Theorem (Robinson, 1992)

Let \((u, b_u)\) be a \(B\)-subsection such that \(b_u\) has defect \(q\). Then

\[ k_0(B) \leq p^q \sqrt{l(b_u)}. \]

- This is useful for proving Olsson’s Conjecture \(k_0(B) \leq |D : D'|\).
- For \(p = 2\) this can be slightly improved to:
Theorem (S., 2012)

Let \((u, b_u)\) be a subsection of a 2-block \(B\) such that \(b_u\) has defect \(q\). Set

\[
\alpha := \begin{cases} \left\lfloor \sqrt{l(b_u)} \right\rfloor & \text{if } \left\lfloor \sqrt{l(b_u)} \right\rfloor \text{ is odd,} \\ \frac{l(b_u)}{\left\lfloor \sqrt{l(b_u)} \right\rfloor + 1} & \text{otherwise.} \end{cases}
\]

Then \(k_0(B) \leq 2^q \alpha\). In particular \(k_0(B) \leq 2^q\) if \(l(b_u) \leq 3\).
Last year I developed a similar method for bounding $k(B)$ and $k_0(B)$ using the Cartan matrix:

**Theorem (S., 2011)**

Let $(u, b_u)$ be a $B$-subsection such that $b_u$ has Cartan matrix $C_u = (c_{ij})$ up to basic sets. Then

$$k_0(B) \leq \sum_{i=1}^{l(b_u)} c_{ii} - \sum_{i=1}^{l(b_u)-1} c_{i,i+1}.$$ 

If $(u, b_u)$ is major, we can replace $k_0(B)$ by $k(B)$ in this formula.
Examples

- So in case $p = 2$ and $l(b_u) \leq 3$ we do not need the Cartan matrix anymore.
- This implies Brauer’s $k(B)$-Conjecture in many more cases which will be shown later.

**Theorem (S., 2011)**

Let $B$ be a 2-block with defect group $M \times C$ or $M \ast C$ where $C$ is cyclic and $M$ is a nonabelian group of maximal class. Then $l(B) \leq 3$. 
Recently, I extended this result to the following similar defect groups:

\[ \langle v, x, a \mid v^{2n} = x^2 = a^{2m} = 1, \; xv = av = v^{-1}, \; ax = vx \rangle \]
\[ \cong D_{2n+1} \rtimes C_{2m} \quad (n, m \geq 2), \]
\[ \langle v, x, a \mid v^{2n} = 1, \; a^{2m} = x^2 = v^{2n-1}, \; xv = av = v^{-1}, \; ax = vx \rangle \]
\[ \cong D_{2n+1} \cdot C_{2m} \cong Q_{2n+1} . C_{2m} \quad (n, m \geq 2 \text{ and } m \neq n), \]
\[ \langle v, x, a \mid v^{2n} = a^{2m} = 1, \; x^2 = v^{2n-1}, \; xv = av = v^{-1}, \; ax = vx \rangle \]
\[ \cong Q_{2n+1} \rtimes C_{2m} \quad (n, m \geq 2). \]

In fact all block invariants \((k(B), k_i(B)\text{ and } l(B))\) could be determined precisely. This gives new evidence for Alperin’s Weight Conjecture.
These groups are examples of bicyclic 2-groups, i.e. they can be written in the form $D = \langle x \rangle \langle y \rangle$ for some $x, y \in D$.

I also classified all saturated fusion systems on bicyclic 2-groups using results of Janko.

It turns out that they are not exotic, i.e. they occur in finite groups.

Instead of the whole classification, I state some corollaries.

**Theorem (S., 2012)**

Let $P$ be a bicyclic, nonmetacyclic 2-group. Then $P$ admits a non-trivial fusion system if and only if $P'$ is cyclic. The number of these groups (and fusion systems) grows with $\log^2 |P|$.
Corollary

Let $G$ be a finite group with bicyclic Sylow 2-subgroup $P$. If $P'$ is noncyclic, then $P$ has a normal complement in $G$.

Theorem (Yang for odd primes, 2011)

Olsson’s Conjecture holds for all blocks with bicyclic defect groups.

Here it is important to observe that bicyclic $p$-groups for odd primes $p$ are always metacyclic (Huppert).
Theorem (S., 2012)

Let $D$ be a cyclic central extension of one of the following groups

1. a metacyclic group,
2. a minimal nonabelian group,
3. a group of order at most 16,
4. $M \rtimes C$ where $M$ has maximal class and $C$ is cyclic,
5. $D_{2n} \rtimes C_{2^m}$, $Q_{2n} \rtimes C_{2^m}$ and $D_{2n}.C_{2^m}$ as above,
6. $\prod_{i=1}^{n} C_{2^{m_i}}$ where $|\{m_i : i = 1, \ldots, n\}| \geq n - 1$,
7. SmallGroup(32, $i$) for $i \in \{11, 22, 28, 29, 33, 34\}$,
8. a group which admits only the trivial fusion system.

Then Brauer’s $k(B)$-Conjecture holds for every 2-block with defect group $D$. 
Let $B$ be a 2-block of a finite group $G$ with defect group $D$ as above.

Let $u \in Z(D)$ such that $D/\langle u \rangle$ has one of the stated isomorphism types.

Let $(u, b_u)$ be the corresponding (major) subsection.

Then $b_u$ dominates a block $\overline{b_u}$ of $C_G(u)/\langle u \rangle$ with defect group $D/\langle u \rangle$.

Moreover, the Cartan matrices of $b_u$ and $\overline{b_u}$ differ only by the factor $|\langle u \rangle|$. In particular $l(b_u) = l(\overline{b_u})$. 
In most cases Brauer’s $k(B)$-Conjecture follows from $l(b_u) = l(\overline{b_u}) \leq 3$.

In case $D/\langle u \rangle \cong C_2^4$ we can apply the “inverse Cartan method” introduced by Brauer.

In the remaining cases we can compute (by computer) a list of possible Cartan matrices for $\overline{b_u}$ and the Cartan method implies the result.

For defect groups of order at most 32 the $k(B)$-Conjecture follows at once.

Using the result above, I verified the $k(B)$-Conjecture for 244 of the 267 defect groups of order 64.
Corollary

Let $B$ be a 2-block with defect group $D$ of order at most 64. If $D$ is generated by two elements, then Brauer’s $k(B)$-Conjecture holds for $B$.

Corollary

Let $D$ be a 2-group containing a cyclic subgroup of index at most 4. Then Brauer’s $k(B)$-Conjecture holds for every block with defect group $D$.

For every $n \geq 6$ there are exactly 33 groups of order $2^n$ satisfying the hypothesis of the last corollary (Ninomiya).
A similar theorem for arbitrary subsections yields the following result.

**Theorem (S., 2012)**

*Olsson’s Conjecture holds for all 2-blocks of defect at most 5.*
The invariants of 2-blocks with metacyclic defect groups are known by work of several authors.

In particular most of the conjectures are fulfilled.

For odd primes the situation is more complicated.

We already saw that Olsson’s Conjecture holds for metacyclic defect groups (even bicyclic).

**Theorem (Gao for odd primes, 2011)**

Brauer’s $k(B)$-Conjecture is satisfied for all blocks with metacyclic defect groups.
**Brauer’s Height Zero Conjecture** asserts that \( k(B) = k_0(B) \) if and only if \( B \) has abelian defect groups.

**Theorem (S., 2012)**

*Brauer’s Height Zero Conjecture is satisfied for all blocks with metacyclic defect groups.*
Let $B$ be a $p$-block with metacyclic defect group $D$.

We may assume that $p$ is odd and $D$ is nonabelian (Kessar-Malle).

Then the fusion of subsections is controlled by the inertial group of $B$ (Stancu).

The theory of lower defect groups implies $l(B) \geq e(B) | p - 1$ where $e(B)$ is the inertial index of $B$.

If $\mathcal{R}$ is a set of representatives of the $G$-conjugacy classes of subsections, then we have

$$k(B) = \sum_{(u,b_\mathcal{R}) \in \mathcal{R}} l(b_\mathcal{R}).$$
Sketch of the proof (2)

- This gives a lower bound for $k(B)$.
- We can always find a subsection $(u, b_u)$ such that $|C_D(u)| = |D : D'|$ and $C_D(u)/\langle u \rangle$ is cyclic.
- Since $b_u$ has defect group $C_D(u)/\langle u \rangle$, the Cartan method implies an upper bound for $k_0(B)$.
- Now $k_0(B) < k(B)$ follows.