Kondo's Fusion Theorem

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The aim of these notes is to present a strong version of Alperin's fusion theorem due to Kondo [11].

Let G be a finite group. We call two Sylow p-subgroups S and T of G *equivalent* if there exist Sylow p-subgroups $S = S_0, \ldots, S_n = T$ such that $S_i \cap S_{i-1} \ne 1$ for $i = 1, \ldots, n$. This defines an equivalence relation $\sim$ on $\mathrm{Syl}_p(G)$. If G has more than one $\sim$-class, then G is called *p-isolated.*

A proper subgroup $H < G$ is called *strongly p-embedded* if p divides $|H|$, but $H \cap H^g$ is a p'-group for every $g \in G \setminus H$.

Lemma 1.

(a) If G has a strongly p-embedded subgroup H, then G is p-isolated and the Sylow p-subgroups of H form a union of $\sim$-classes of G.

(b) If G is p-isolated, then the stabilizer of a $\sim$-class is strongly p-embedded in G.

Proof. Let $H < G$ be strongly p-embedded and $P \in \mathrm{Syl}_p(H)$. Let $S \in \mathrm{Syl}_p(G)$ such that $P \le S$. If $P < S$, then there exists $g \in \mathrm{N}_S(P) \setminus H$ such that $P = P \cap P^g \in H \cap H^g$. This contradiction shows that $P = S$. Let $T \in \mathrm{Syl}_p(G)$ such that $S \cap T \ne 1$. Let $g \in G$ with $S^g = T$. Then $S \cap T \le H \cap H^g$ and it follows that $g \in H$. Thus, $T = S^g \in \mathrm{Syl}_p(H)$. Hence, $\mathrm{Syl}_p(H)$ is a union of $\sim$-classes. Since $\mathrm{N}_G(S) \le H < G$, there must be at least one $\sim$-class outside H. In particular, G is p-isolated.

Suppose conversely that G is p-isolated and let H be the stabilizer of the $\sim$-class of $S \in \mathrm{Syl}_p(G)$. Then $1 \ne S \le H < G$. Let $g \in G \setminus H$ and let P be a Sylow p-subgroup of $H \cap H^g$. Let $h, h' \in H$ be such that $P \le S^h \cap S^{h'g}$. Since $g \notin H$, S^h and $S^{h'g}$ are not equivalent. In particular, $P \le S^h \cap S^{h'g} = 1$. This shows that H is strongly p-embedded. □

Lemma 2. *Let $H < G$ be strongly p-embedded.*

(a) Let $K \le H$ such that p divides $|K|$. Then $\mathrm{N}_G(K) \le H$.

(b) Let $N \trianglelefteq G$ such that p divides $|N|$. Then $G = H\mathrm{O}^p(N)$.

Proof.

(a) For $g \in \mathrm{N}_G(K)$, p divides the order of $K = K \cap K^g \le H \cap H^g$. Hence, $g \in H$.

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(b) Let $S \in \mathrm{Syl}_p(H)$. By Lemma 1, $S \in \mathrm{Syl}_p(G)$ and $1 \ne N \cap S \in \mathrm{Syl}_p(N)$. Since $N = (N \cap S)\mathrm{O}^p(N)$, $\mathrm{O}^p(N)$ acts transitively on $\mathrm{Syl}_p(N)$. The Frattini argument yields $G = \mathrm{N}_G(N \cap S)\mathrm{O}^p(N)$. By (a), $\mathrm{N}_G(N \cap S) \le H$. □

The following lemma is well-known.

Lemma 3. *Let A be a non-cyclic abelian group acting coprimely on a group G. Then*

$$G = \langle \mathrm{C}_G(x) : x \in A \setminus \{1\}\rangle.$$

Proof. We may assume that A is a p-group. Let q be a prime divisor of $|G|$. Since the number of Sylow q-subgroups of G divides the p'-number $|G|$, there exists an A-invariant Sylow q-subgroup Q of G. We may assume that $G = Q$. Suppose that G has an A-invariant normal subgroup N such that $1 < N < G$. By induction on $|G|$, we may assume that $N = \langle \mathrm{C}_N(x) : x \in A \setminus \{1\}\rangle$ and $G/N = \langle \mathrm{C}_{G/N}(x) : x \in A \setminus \{1\}\rangle$. Since A acts coprimely, $\mathrm{C}_{G/N}(x) = \mathrm{C}_G(x)N/N$. Hence,

$$G = \langle \mathrm{C}_G(x)N : x \in A \setminus \{1\}\rangle = \langle \mathrm{C}_G(x) : x \in A \setminus \{1\}\rangle.$$

Therefore, we may assume that G is elementary abelian and A acts irreducibly. By Schur's Lemma, the endomorphism ring E of the simple $\mathbb{F}_qA$-module G is a finite division algebra, so E is a field. In particular, the multiplicative group of E is cyclic. Hence, A cannot act faithfully on G. Thus, there exists $x \in A \setminus \{1\}$ such that $G = \mathrm{C}_G(x)$. □

The next result is not needed in the sequel.

Proposition 4. *Let G be p-solvable for some prime divisor p of $|G|$. Then G is p-isolated if and only if $\mathrm{O}_p(G) = 1$ and the Sylow p-subgroups are cyclic or quaternion groups.*

Proof. Suppose first that $\mathrm{O}_p(G) = 1$ and the Sylow p-subgroups of G are cyclic or quaternion groups. These p-groups have only one subgroup of order p. Hence, $S \sim T$ if and only if $S \cap T \ne 1$. Since $\mathrm{O}_p(G)$ is the intersection of all Sylow p-subgroups, there must exist $S, T \in \mathrm{Syl}_p(G)$ such that $S \cap T = 1$. In particular, G is p-isolated (note that we do not need the p-solvability of G).

Now assume conversely that G is p-isolated. Then obviously $\mathrm{O}_p(G) = 1$. Let N be a minimal normal subgroup of G. Since G is p-solvable, N is a p'-group. By induction on $|G|$, we may that G/N is not p-isolated. Let H be the stabilizer of a $\sim$-class of G. Then there exist $S \in \mathrm{Syl}_p(H)$ and $T \in \mathrm{Syl}_p(G)$ such that $H \cap T = 1$, but $(SN \cap TN)/N = SN/N \cap TN/N \ne 1$. Let $S_0 \le S$ and $T_0 \le T$ with $SN \cap TN = S_0N = T_0N$. If $N \le H$, then we obtain the contradiction

$$H \cap T \ge S_0N \cap T = T_0N \cap T \ge T_0 > 1.$$

Hence, $N \nleq H$. Let q be a prime divisor of $|N : N \cap H|$. Since the number of Sylow q-subgroups of N divides the p'-number $|N|$, there exists a S-invariant Sylow q-subgroup $Q \ne 1$ of N. Then S normalizes $Q_0 := Q \cap H$ and $Q_1 := \mathrm{N}_Q(Q_0)/Q_0 \ne 1$. By Lemma 2, $\mathrm{C}_Q(x) = \mathrm{C}_G(x) \cap Q \le H \cap Q = Q_0$ for every $x \in S \setminus \{1\}$. Since the action of S on Q_1 is coprime, $\mathrm{C}_{Q_1}(x) = 1$. By Lemma 3, every abelian subgroup of S is cyclic. This implies the claim as is well-known. □

If G is a non-abelian simple group with a cyclic Sylow p-subgroup P, then $\mathrm{N}_G(P)$ is strongly p-embedded by a theorem of Blau [5]. In this situation the $\sim$-classes are singletons, i. e. P is a *trivial intersection set*. Note that by Brauer–Suzuki there are no simple groups with a cyclic or quaternion Sylow 2-subgroup. Bender [4] has classified all 2-isolated groups. In general, the p-isolated groups are determined in principle via the classification of finite simple groups (see [16, Theorem 6.4]).

Lemma 5. *Let G be p-isolated with normal subgroups $N, M \trianglelefteq G$ such that p divides $|N|$ and $|M|$. Then p divides $|N \cap M|$.*

Proof. Let $H < G$ be strongly p-embedded and $S \in \mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$. Then $S \cap N \neq 1 \neq S \cap M$ by hypothesis. By way of contradiction, suppose that $N \cap M$ is a p'-group. Then $[S \cap N, S \cap M] \leq S \cap N \cap M = 1$ and S contains a non-cyclic abelian subgroup. By Lemma 3 and Lemma 2 we conclude that

$$N \cap M = \langle C_{N \cap M}(x) : x \in S \setminus \{1\} \rangle \leq \langle \mathrm{N}_G(\langle x \rangle) : x \in S \setminus \{1\} \rangle \leq H.$$

Since N normalizes $(S \cap M)(N \cap M) \leq H$, we obtain $N \leq H$ again by Lemma 2. But now $G = \mathrm{N}_G(N) \leq H$, a contradiction. □

Lemma 5 implies that every p-isolated group G has a unique minimal normal subgroup $M(G) \trianglelefteq G$ such that p divides $|M(G)|$.

Two Sylow p-subgroups S and T of G have a *tame intersection* if $\mathrm{N}_S(S \cap T)$ and $\mathrm{N}_T(S \cap T)$ are Sylow p-subgroups of $\mathrm{N}_G(S \cap T)$.

Lemma 6. *Let $P, Q \in \mathrm{Syl}_p(G)$ be distinct such that $P \cap Q \neq 1$. Then there exist Sylow p-subgroups $P = P_0, P_1, \ldots, P_n = Q$ of G with the following properties:*

(a) P_{i-1} and P_i have a tame intersection $H_i := P_{i-1} \cap P_i$ for $i = 1, \ldots, n$,

(b) $\mathrm{N}_G(H_i)/H_i$ is p-isolated for $i = 1, \ldots, n$. Define $X(H_i)/H_i := M(\mathrm{N}_G(H_i)/H_i)$,

(c) there exists $x_i \in \mathrm{O}^p(X(H_i))$ such that $P_i^{x_i} = P_{i-1}$ for $i = 1, \ldots, n$,

(d) $P \cap Q = H_1 \cap \ldots \cap H_n$.

Proof. We argue by induction on $|P : P \cap Q|$. Suppose first that $|P : P \cap Q| = p$. Since $P \cap Q$ is normal in $P = P_0$ and in $Q = P_1$, the intersection $H := H_1 = P_0 \cap P_1$ is tame. Moreover, $\mathrm{N}_G(H)/H$ has two distinct Sylow subgroups P/H and Q/H of order p. Hence, $\mathrm{N}_G(H)/H$ is p-isolated and

$$K/H := (\mathrm{N}_G(H) \cap \mathrm{N}_G(Q))/H$$

is strongly p-embedded in $\mathrm{N}_G(H)/H$ by Lemma 1. An application of Lemma 2 with $N := X(H)/H$ yields

$$\mathrm{N}_G(H)/H = K/H \cdot \mathrm{O}^p(X(H)/H) = K\mathrm{O}^p(X(H))/H.$$

Since $P, Q \leq \mathrm{N}_G(H)$, there exists $x \in \mathrm{N}_G(H)$ such that $P^x = Q$. We may write $x = yx_1$ with $y \in K \leq \mathrm{N}_G(Q)$ and $x_1 \in \mathrm{O}^p(X(H))$. It follows that $P = Q^x = Q^{x_1}$. Now all four conditions are fulfilled.

For the induction step let $H := P \cap Q$. Choose $R, S \in \mathrm{Syl}_p(G)$ such that $\mathrm{N}_P(H) \leq \mathrm{N}_R(H) \in \mathrm{Syl}_p(\mathrm{N}_G(H))$ and $\mathrm{N}_Q(H) \leq \mathrm{N}_S(H) \in \mathrm{Syl}_p(\mathrm{N}_G(H))$. Then $H < \mathrm{N}_P(H) \leq P \cap R$ and $H < \mathrm{N}_Q(H) \leq S \cap Q$. If $H < R \cap S$, then we apply induction to the pairs (P, R), (R, S) and (S, Q) to obtain a series of Sylow subgroups satisfying the four conditions. Hence, we may assume that $H = R \cap S$ in the following. In particular, R and S have a tame intersection.

Suppose next that $\mathrm{N}_R(H)/H \sim \mathrm{N}_S(H)/H$ in $\mathrm{N}_G(H)/H$. Then there exist $R = R_0, R_1, \ldots, R_m = S \in \mathrm{Syl}_p(G)$ such that $H < R_{i-1} \cap R_i$ for $i = 1, \ldots, m$. Again we apply induction to the pairs (R_{i-1}, R_i) to obtain the desired sequence of Sylow p-subgroups. Therefore, we may assume that $\mathrm{N}_R(H)/H \not\sim \mathrm{N}_S(H)/H$. In particular, $\mathrm{N}_G(H)/H$ is p-isolated. Let K/H be the stabilizer of the $\sim$-class containing $\mathrm{N}_S(H)/H$. By Lemma 1, K/H is strongly p-embedded in $\mathrm{N}_G(H)/H$. As above we obtain $\mathrm{N}_G(H) =$

$K\mathrm{O}^p(X(H))$ via Lemma 2. By Sylow's theorem there exists $x \in \mathrm{N}_G(H)$ such that $\mathrm{N}_S(H)^x = \mathrm{N}_R(H)$. We write $x = yx_1$ with $y \in K$ and $x_1 \in \mathrm{O}^p(X(H))$. Since $\mathrm{N}_R(H)/H \not\sim \mathrm{N}_S(H)/H$, also $\mathrm{N}_R(H)/H \not\sim \mathrm{N}_{S^y}(H)/H$. In particular,

$$H \le S^x \cap S^y \le \mathrm{N}_R(H) \cap \mathrm{N}_{S^y}(H) = H$$

and S^x and S^y have a tame intersection. On the other hand, $H < \mathrm{N}_R(H) \le R \cap S^x$. Since $\mathrm{N}_S(H)/H \sim \mathrm{N}_{S^y}(H)/H$, there exist $S^y = S_0, S_1, \ldots, S_n = S \in \mathrm{Syl}_p(G)$ such that $\mathrm{N}_{S_i}(H) \in \mathrm{Syl}_p(\mathrm{N}_G(H))$ and $H < S_{i-1} \cap S_i$ for $i = 1, \ldots, n$. Finally, we apply induction to the pairs (P, R), (R, S^x), $(S_0, S_1), \ldots, (S_{n-1}, S_n)$, (S, Q). The gap between S^x and $S_0 = S^y$ is bridged with the element $x_1 \in \mathrm{O}^p(X(H))$ constructed above. □

Theorem 7 (Kondo). *Let P be a Sylow p-subgroup of G. Let $A, B \subseteq P$ and $g \in G$ such that $A^g = B \not\subseteq \{1\}$. Then there exist $H_1, \ldots, H_n \le P$, $x_1, \ldots, x_n \in G$ and $y \in \mathrm{N}_G(P)$ with the following properties for $i = 1, \ldots, n$:*

(a) $\mathrm{N}_P(H_i) \in \mathrm{Syl}_p(\mathrm{N}_G(H_i))$,

(b) $\mathrm{N}_G(H_i)/H_i$ *is p-isolated,*

(c) $x_i \in \mathrm{O}^p(X(H_i))$,

(d) $A^{x_1 \ldots x_{i-1}} \subseteq H_i$,

(e) $g = x_1 \ldots x_n y$.

Proof. For $H \le P$ we abbreviate $K_H := \mathrm{O}^p(X(H))$ whenever $\mathrm{N}_G(H)/H$ is p-isolated. By the uniqueness of $X(H)$ (Lemma 5), it is easy to see that $K_H^x = K_{H^x}$ for $x \in G$. By hypothesis, $A \subseteq P \cap P^{g^{-1}} \ne 1$. By Lemma 6, there exist $P = P_0, \ldots, P_n = P^{g^{-1}} \in \mathrm{Syl}_p(G)$ such that

- P_{i-1} and P_i have a tame intersection $L_i := P_{i-1} \cap P_i$ for $i = 1, \ldots, n$,
- $\mathrm{N}_G(L_i)/L_i$ is p-isolated for $i = 1, \ldots, n$.
- there exists $y_i \in K_{L_i}$ such that $P_i^{y_i} = P_{i-1}$ for $i = 1, \ldots, n$,
- $P \cap P^{g^{-1}} = L_1 \cap \ldots \cap L_n$.

Define

$$x_i := y_i^{y_{i-1} \ldots y_1}, \qquad H_i := L_i^{y_i \ldots y_1} \qquad (i = 1, \ldots, n).$$

Then $x_1 \ldots x_i = y_i \ldots y_1$ and $\mathrm{N}_G(H_i)/H_i$ is p-isolated for $i = 1, \ldots, n$. Moreover, $H_i \le P_i^{y_i \ldots y_1} = P_0 = P$ and

$$\mathrm{N}_P(H_i) = \mathrm{N}_{P_i}(L_i)^{y_i \ldots y_1} \in \mathrm{Syl}_p(\mathrm{N}_G(L_i)^{y_i \ldots y_1}) = \mathrm{Syl}_p(\mathrm{N}_G(H_i)),$$

since L_i is a tame intersection. Next we note that $A^{x_1 \ldots x_{i-1}} \subseteq L_i^{y_{i-1} \ldots y_1} = L_i^{y_i \ldots y_1} = H_i$ and

$$x_i = y_i^{y_{i-1} \ldots y_1} \in K_{L_i}^{y_{i-1} \ldots y_1} = K_{H_i}$$

for $i = 1, \ldots, n$. Now $P = P_0 = P_n^{y_n \ldots y_1} = (P^{g^{-1}})^{x_1 \ldots x_n}$ implies $g = x_1 \ldots x_n y$ for some $y \in \mathrm{N}_G(P)$. This completes the proof. □

Lemma 8. *Let $H \le P \in \mathrm{Syl}_p(G)$ such that $\mathrm{N}_P(H) \in \mathrm{Syl}_p(\mathrm{N}_G(H))$. Let*

$$N/\mathrm{C}_G(H) := \mathrm{O}_p(\mathrm{N}_G(H)/\mathrm{C}_G(H)).$$

Then the following assertions are equivalent:

(a) $H \in \mathrm{Syl}_p(N)$.

(b) $\mathrm{C}_P(H) \le H$ *and* $\mathrm{O}_{p'p}(\mathrm{N}_G(H)) = H\mathrm{C}_G(H)$.

Proof. Note that $\mathrm{N}_P(H) \in \mathrm{Syl}_p(\mathrm{N}_G(H))$ implies $\mathrm{C}_P(H) \in \mathrm{Syl}_p(\mathrm{C}_G(H))$. Suppose first that $H \in \mathrm{Syl}_p(N)$. By the Schur–Zassenhaus Theorem, $N = H\mathrm{C}_G(H) = H \times Q$ where $Q = \mathrm{O}_{p'}(\mathrm{C}_G(H)) \le \mathrm{O}_{p'}(\mathrm{N}_G(H))$. Hence, $\mathrm{C}_P(H) = \mathrm{Z}(H) \le H$. Since $\mathrm{O}_{p'}(\mathrm{N}_G(H))$ acts trivially on H, we also have $\mathrm{O}_{p'}(\mathrm{N}_G(H)) = Q$. Now let $M := \mathrm{O}_{p'p}(\mathrm{N}_G(H))$. Then $HQ/Q \le M/Q$ and

$$N/\mathrm{C}_G(H) = HQ/\mathrm{C}_G(H) \le M/\mathrm{C}_G(H) \le \mathrm{O}_p(\mathrm{N}_G(H)/\mathrm{C}_G(H)) = N/\mathrm{C}_G(H).$$

This shows that $M = N = H\mathrm{C}_G(H)$.

Suppose conversely that (b) holds. Then again $\mathrm{C}_G(H)H = H \times Q$ with $Q = \mathrm{O}_{p'}(\mathrm{N}_G(H))$. Since

$$|N/Q| = |N/\mathrm{C}_G(H)||\mathrm{C}_G(H)/Q| = |\mathrm{O}_p(\mathrm{N}_G(H)/\mathrm{C}_G(H))||\mathrm{Z}(H)|$$

is a p-power, we obtain $N/Q \le \mathrm{O}_p(\mathrm{N}_G(H)/Q)$, i. e. $N \le \mathrm{O}_{p'p}(\mathrm{N}_G(H)) = HQ$. Hence, $H \in \mathrm{Syl}_p(N)$. □

Theorem 9 (Kondo)**.** *Let P be a Sylow p-subgroup of G. Let $A, B \subseteq P$ and $g \in G$ such that $A^g = B \nsubseteq \{1\}$. Then there exist $H_1, \ldots, H_n \le P$, $x_1, \ldots, x_n \in G$, $c \in \mathrm{C}_G(A)$ and $y \in \mathrm{N}_G(P)$ with the following properties for $i = 1, \ldots, n$:*

(a) $\mathrm{N}_P(H_i) \in \mathrm{Syl}_p(\mathrm{N}_G(H_i))$,

(b) $\mathrm{C}_P(H_i) \le H_i$,

(c) $\mathrm{O}_{p'p}(\mathrm{N}_G(H_i)) = H_i\mathrm{C}_G(H_i)$,

(d) $\mathrm{N}_G(H_i)/H_i$ *is p-isolated,*

(e) $x_i \in \mathrm{O}^p(X(H_i))$,

(f) $A^{x_1 \ldots x_{i-1}} \subseteq H_i$,

(g) $g = cx_1 \ldots x_n y$.

Proof. We choose $H_1, \ldots, H_n \le P$, $x_1, \ldots, x_n \in G$ and $y \in \mathrm{N}_G(P)$ as in Theorem 7. Suppose that (b) or (c) does not hold for some $H := H_i$. Then by Lemma 8, $|N/H|$ is divisible by p where $N/\mathrm{C}_G(H) := \mathrm{O}_p(\mathrm{N}_G(H)/\mathrm{C}_G(H))$. The definition of $X(H)$ yields $X(H) \le N$. Since $N/\mathrm{C}_G(H)$ is a p-group, $\mathrm{O}^p(N) \le \mathrm{C}_G(H)$. Since

$$X(H)/\mathrm{O}^p(N) \cap X(H) \cong X(H)\mathrm{O}^p(N)/\mathrm{O}^p(N) \le N/\mathrm{O}^p(N),$$

we conclude that

$$x_i \in \mathrm{O}^p(X(H)) \le \mathrm{O}^p(N) \le \mathrm{C}_G(H).$$

Therefore, $c := x_i^{(x_1 \ldots x_{i-1})^{-1}} \in \mathrm{C}_G(H_i^{(x_1 \ldots x_{i-1})^{-1}}) \le \mathrm{C}_G(A)$ and $g := cx_1 \ldots x_{i-1}x_{i+1} \ldots x_n y$. We repeat this process until every H_i fulfills the stated conditions. □

Since $O^p(X(H_i))$ is generated by p'-elements, we may require that $x_1, \ldots, x_n$ are p'-elements. Alternatively, we may assume that $x_i \in X(H_i)$ are p-elements since by definition, $X(H_i)$ is generated by p-elements.

Theorem 9 generalizes Alperin's original fusion theorem [1] as well as Goldschmidt's extension [9]. A similar result was obtained by Puig [14] (see also [15, Chapter 5]). A readable account of the fusion theorem for fusion systems can be found in [7, Theorem 4.51]. Some more specific fusion theorems were given in [10, (9.1)] and [17, Theorem 3.3]. Alperin and Gorenstein [3] developed a fusion theorem using an abstract conjugacy functor. A graph-theoretical proof of their result was provided by Stellmacher [18]. In the latter paper and in [13] it was shown which subgroups need to appear in every fusion theorem. Dolan [8] has proved that the number of elements $x_1, \ldots, x_n$ used in the fusion theorem can be bounded in terms of the nilpotency class of a Sylow p-subgroup. Using his techniques, Collins [6] gave another proof of the fusion theorem. Finally, Alperin [2] derived a fusion theorem where the orders $|C_P(H_i)|$ are unimodal. A corresponding version for fusion systems was obtained by Lynd [12, Proposition 3.1].

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