Disclaimer: I won’t spoil how to solve Rubik’s cube!
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![Rotation diagram]

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![Cube before and after rotation]

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- The centers are fixed now (top $\rightarrow$ white, front $\rightarrow$ orange, ...).
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How “big” is the cube?
How many states can we reach by applying an arbitrary number of moves?
Facelets

- Idea: Enumerate the $6 \cdot 8 = 48$ edge and corner facelets:
Facelets

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Every move becomes a permutation in $S_{48}$, e.g. a clockwise $90^\circ$ turn of the front face:

The cube group

- Similarly, we define $b$ (back), $l$ (left), $r$ (right), $u$ (up), $d$ (down).
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Is $G$ transitive on the 48 facelets?

No: The $8 \cdot 3 = 24$ corner facelets and the $12 \cdot 2 = 24$ edge facelets form orbits $\Omega_C$ and $\Omega_E$. 

$\Omega_C$: 

$\Omega_E$: 

Hence,

\[ G \leq \text{Sym}(\Omega_C) \times \text{Sym}(\Omega_E) \cong S_{24}^2 \]

and \(|G| \leq (24!)^2 \approx 10^{48}\).
Action on $\Omega_C$

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- No: the three facelets of a corner **cubie** form a block $\Delta$ in $\Omega_C$. 
Hence,

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Is the action of \( G \) on \( \Omega_C \) primitive?

No: the three facelets of a corner cubie form a block \( \Delta \) in \( \Omega_C \).

We can permute the three facelets of \( \Delta \) only cyclically:
From the lecture:

**Satz 6.26.** Sei $G$ eine imprimitive Permutationsgruppe auf $\Omega$ mit Block $\Delta$. Sei $H := \{g \in G : g \Delta = \Delta\}$ und sei $\varphi : H \rightarrow \text{Sym}(\Delta)$ die Operation auf $\Delta$. Sei $\Gamma := \{g\Delta : g \in G\}$ und sei $\psi : G \rightarrow \text{Sym}(\Gamma)$ die Operation auf $\Gamma$. Dann ist $G$ zu einer Untergruppe von $\varphi(H) \wr \psi(G)$ isomorph.
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- This gives a homomorphism $G \to C_3 \wr S_8 \leq S_{24}$.
- Similarly, the two facelets of an edge cubie form a block of $\Omega_E$.
- Therefore,

$$G \leq C_3 \wr S_8 \times C_2 \wr S_{12}$$

and $|G| \leq 3^8 8! \cdot 2^{12} 12! \approx 5 \cdot 10^{20}$. 

---

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Action on corner cubies

Now we investigate the action of $G$ on the set $C$ of the eight corner cubies.

Let $\phi_C: G \to \text{Sym}(C)$ the corresponding homomorphism.

Consider the move sequence $x := u \circ r \circ f \in G$.

With suitable labeling:

$\phi_C(x) = (1, 2)(3, 4, 5, 6, 7)$

and

$\phi_C(x^5) = (1, 2)$.

By Exercise 31, $S_8$ is generated by adjacent transpositions.

Hence, $\phi_C(G) = S_8$. 
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\[ \begin{array}{c}
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![Rubik's Cube Diagram]
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![Diagram of Rubik's cube transformations]
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- In particular, $\varphi_E(G_C) \subseteq A_{12}$. 
Action on edge cubies

- Consider the commutator $y := [f, r] = f r f^{-1} r^{-1} \in G$. 

\[ \phi_{C}(y) = (1, 2, 3, 4)(4, 3, 5, 6)(1, 4, 3, 2)(4, 6, 5, 3) = (1, 4)(3, 5). \]

It follows that $y^2 \in G_C$.

Similarly, $\phi_{E}(y) = (1, 2, 3, 4)(4, 5, 6, 7)(1, 4, 3, 2)(4, 7, 6, 5) = (1, 5, 4)$. 

Therefore, $(1, 4, 5) = \phi_{E}(y^2) \in \phi_{E}(G_C) \subseteq A_{12}$. 

By Exercise 31, $A_{12} = \langle (1, 2, 3), \ldots, (10, 11, 12) \rangle \subseteq \phi_{E}(G_C) \subseteq A_{12}$. 

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Edge flips (computed)

- It remains to investigate $N := \ker(\varphi_C) \cap \ker(\varphi_E) \leq G$. 

  This is the set of states where each cubie is in the right spot, but might be flipped (edge) or twisted (corner).

  We have $N = N_3 \oplus N_2 \leq C_8 \times C_12$ (in fact: $F(G) = N$).

  A generator of $G$ is a product of two disjoint $4$-cycles on $\Omega_E$ and therefore an even permutation.

  For this reason it is impossible to flip only one edge and leave everything else fixed.

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- Hence, $|N_2| \leq 2^{11}$. 
Edge flips (realized)

On the other hand, we can flip just two (adjacent) edges:

\[ r^2 f^2 r^{-1} f r f r^2 b^{-1} u^{-1} f^{-1} u f b = \] (13 moves)
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Corner orientations (computed)

- Let $g \in G$ be a generator corresponding to

$$ (t, \pi) \in \langle \zeta \rangle^C \rtimes \text{Sym}(C) \cong C_3 \wr S_8. $$
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  \[
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  \]

- Since \( g \) has order 4, we obtain 
  
  \[
  1 = (t, \pi)^4 = (t \cdot \pi t, \pi^2) \ast (t, \pi) \ast (t, \pi) = (t \cdot \pi t \cdot \pi^2 t, \pi^3) \ast (t, \pi) \\
  = (t \cdot \pi t \cdot \pi^2 t \cdot \pi^3 t, \pi^4).
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- In particular,

\[1 = \prod_{c \in C} (t \pi t \pi^2 t \pi^3 t)(c) = \prod_{c \in C} t(c) \prod_{c \in C} t(\pi^{-1} c) \prod_{c \in C} t(\pi^{-2} c) \prod_{c \in C} t(\pi^{-3} c) = \left( \prod_{c \in C} t(c) \right)^4 = \prod_{c \in C} t(c).\]
Corner orientations (computed)

- If \((t, \pi), (t', \pi') \in \langle \zeta \rangle^C \rtimes \text{Sym}(C)\) such that \(\prod t(c) = \prod t'(c) = 1\), then also
  \[
  \prod_{c \in C} (t \cdot \pi t')(c) = \prod_{c \in C} t(c) \prod_{c \in C} t'(\pi^{-1} c) = 1.
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- In particular, \(|N_3| \leq 3^7\).
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- In particular, \(|N_3| \leq 3^7\).
- This can also be visualized as follows.
Corner orientations (visualized)

Fix an orientation of the corner facelets:
Corner orientations (visualized)

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Every move causes one of the following effects:
Corner orientations (visualized)

- No twists are introduced (move $f$).

\[
\begin{array}{ccc}
+ & - & \\
U & - & + \\
- & + & + \\
- & 0 & 0 & - & + & 0 & 0 \\
L & F & R & B \\
+ & - & 0 & 0 & + & - & 0 & 0 \\
+ & - & - \\
D & + \\
- & + \\
\end{array}
\]

The sum of all twists is always 0 modulo 3.
Corner orientations (visualized)

- No twists are introduced (move $f$).
- Two positive twists and two negative-twists are introduced (move $r$).

The sum of all twists is always $0$ modulo $3$. 
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- Three positive twists or three negative twists are introduced (move $b$).

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Benjamin Sambale (LUH)
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\[ + - 0 + 0 - + 0 + \]

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$\Rightarrow$ The sum of all twists is always 0 modulo 3.
Corner orientations (realized)

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\[ u^2bu^2b^{-1}lu^2f^{-1}u^2fl^2b^{-1}lb = \] (13 moves)
Corner orientations (realized)

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u^2 b u^2 b^{-1} u^2 f^{-1} u^2 f l^2 b^{-1} l b = (13 \text{ moves})\]

- This shows \( |N_3| = 3^7 \).
The order of $G$

We have proved:

Axiom
The order of $G$

We have proved:

**Theorem**

An element $(t, \pi, t', \pi') \in (C_3 \wr S_8) \times (C_2 \wr S_{12})$ belongs to $G$ if and only if

\[
\text{sgn}(\pi) = \text{sgn}(\pi'), \quad \prod_{c \in C} t(c) = \prod_{e \in E} t'(e) = 1.
\]

Hence, the index of $G$ in $(C_3 \wr S_8) \times (C_2 \wr S_{12})$ is 12 and

\[
|G| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 = 43.252.003.274.489.856.000.
\]
The order of $G$

We have proved:

**Theorem**

An element $(t, \pi, t', \pi') \in (C_3 \wr S_8) \times (C_2 \wr S_{12})$ belongs to $G$ if and only if

$$\text{sgn}(\pi) = \text{sgn}(\pi'), \quad \prod_{c \in \mathcal{C}} t(c) = \prod_{e \in \mathcal{E}} t'(e) = 1.$$ 

Hence, the index of $G$ in $(C_3 \wr S_8) \times (C_2 \wr S_{12})$ is 12 and

$$|G| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 = 43,252,003,274,489,856,000.$$ 

Interpretation: After taking apart and reassembling the cubies randomly, the cube is “solvable“ in only 1 out of 12 cases.
Consequences

\( G \cong (C_3^7 \times C_2^{11}) \rtimes (A_8 \times A_{12}) \rtimes C_2. \) Composition factors: 
\( C_2 \) (12 times), \( C_3 \) (7 times), \( A_8, A_{12}. \)
Consequences

- $G \cong (C_3^7 \times C_2^{11}) \rtimes (A_8 \times A_{12})$ × $C_2$. Composition factors: $C_2$ (12 times), $C_3$ (7 times), $A_8$, $A_{12}$.

- $Z(G) = \Phi(G) = \langle s \rangle \cong C_2$ where $s$ is the superflip: (all edges are flipped)
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- $Z(G) = \Phi(G) = \langle s \rangle \cong C_2$ where $s$ is the superflip: (all edges are flipped)

- $|G : G'| = 2$. 

$Z(G)$ is the center of the group $G$, and $\Phi(G)$ is the Frattini subgroup. The superflip is a permutation of the edges that flips all of them simultaneously.
Consequences

- $G \cong (C_3^7 \times C_2^{11}) \rtimes (A_8 \times A_{12}) \rtimes C_2$. Composition factors: $C_2$ (12 times), $C_3$ (7 times), $A_8$, $A_{12}$.
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- \( G \cong (C_3^7 \times C_2^{11}) \rtimes (A_8 \times A_{12}) \rtimes C_2 \). Composition factors: \( C_2 \) (12 times), \( C_3 \) (7 times), \( A_8 \), \( A_{12} \).

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- A chief series: \( 1 \trianglelefteq Z(G) \trianglelefteq N_2 \trianglelefteq N \trianglelefteq G_C \trianglelefteq G' \trianglelefteq G \).
Burnside’s Lemma

- Some states of the cube are symmetric to each other:
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- Applying Burnside’s Lemma with the symmetry group $S_4 \times C_2$ of the cube (in $\mathbb{R}^3$) yields:

\[901,083,918,813,616\]
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*Up to symmetries the cube has 901.083.404.981.813.616 states.*

Using the “symmetry” $g \leftrightarrow g^{-1}$, we get down to 450.541.810.590.509.978.
An upper bound

Optimal solutions

How many moves are required to solve any given cube state?

Theorem

Some states require at least 18 moves.

Proof.

Let $s_n$ be the number of states that can be reached with exactly $n$ moves. Obviously, $s_0 = 1$ and $s_1 = 3 \cdot 6 = 18$. On the second move, it makes no sense to turn the same face again. This leaves 15 moves.
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Proof (continued).

- If the first two moves turn opposite faces, their order does not matter. Hence, \( s_2 = 15s_1 - 9 \cdot 3 = 3^5 \).
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• Now suppose that \( n - 1 \) moves have been carried out.

• If the next two moves turn opposite faces, both should differ from face \( n - 1 \). So we reach at most \( 18s_{n-1} \) new states in this case.

\[ s_{n+1} \leq 12s_n + 18s_{n-1}. \]

Solving the recurrence yields

\[ \sum_{n=0}^{17} s_n < |G|. \]
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- Otherwise, we reach at most \( 12s_n \) new states. Altogether,

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God’s number is 20


Every state of the cube can be solved with at most 20 moves and the superflip cannot be solved with less than 20 moves.
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Proof.

God’s number is 20

**Theorem (Rokicki–Kociemba–Davidson–Dethridge, 2010)**

*Every state of the cube can be solved with at most 20 moves and the superflip cannot be solved with less than 20 moves.*

**Proof.**


- Most states require 18 moves and the average is slightly below 18.
God’s number is 20


Every state of the cube can be solved with at most 20 moves and the superflip cannot be solved with less than 20 moves.

Proof.


- Most states require 18 moves and the average is slightly below 18.
- If only quarter turn moves are allowed, God’s number increases to 26.
Finding an optimal solution is **NP-complete** (2018).
Algorithms

- Finding an optimal solution is NP-complete (2018).
- Korf’s algorithm finds an optimal solution, but can take hours for a single state.

Kociemba’s algorithm finds “short” solutions (less than 20 moves on average) within seconds. Implementation: [http://kociemba.org/cube.htm](http://kociemba.org/cube.htm)

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Human achievements

- There are frequent international speedcubing competitions. Some official world records:
  - Fastest solve: 5.53 s on average!
  - Fewest moves: 21 on average!
  - Blindfold: 59 cubes solved in 59:46 minutes including memorization time!
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Variations

Is the following cube any different?

Yes, but not so much harder to solve (→ Christmas exercise!).

Benjamin Sambale (LUH)
Variations

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Variations

The invention of $n \times n \times n$-cubes:

<table>
<thead>
<tr>
<th>$n$</th>
<th>Inventor</th>
<th>Product name</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Larry D. Nichols</td>
<td>Pocket Cube</td>
<td>1970</td>
</tr>
<tr>
<td>3</td>
<td>Ernő Rubik</td>
<td>Rubik’s Cube</td>
<td>1974</td>
</tr>
<tr>
<td>4</td>
<td>Péter Sebestény</td>
<td>Rubik’s Revenge</td>
<td>1981</td>
</tr>
<tr>
<td>5</td>
<td>Udo Krell</td>
<td>Professor’s Cube</td>
<td>1981</td>
</tr>
<tr>
<td>6</td>
<td>Panagiotis Verdes</td>
<td>V-Cube 6</td>
<td>2004</td>
</tr>
</tbody>
</table>
Variations

For $n \geq 7$ there is a fundamental design problem:

$$\sqrt{2}n > n + 2\sqrt{2}$$
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- For $n \geq 7$ there is a fundamental design problem:

$$\sqrt{2} \{ n \} \quad \sqrt{2} \{ \sqrt{2} \} \quad 2\sqrt{2} > n + 2\sqrt{2}$$

- The red square falls off!
For $n \geq 7$ there is a fundamental design problem:

$$\sqrt{2}n > n + 2\sqrt{2}$$

The red square falls off!

Remedy:

V-Cube 9:  
ShengShou 9:
Endless other variations

Square-1  Skewb Master  Ghost cube  Pyraminx
Skewb Diamond  Skewb Ultimate  Gigaminx  Hypercube

With the computer...

Let's use the open-source computer algebra system GAP. Rubik's group can be copied from http://www.gap-system.org/Doc/Examples/rubik.html

GAP-Code

\begin{verbatim}
f:=(6,25,43,16,2,14,32,5,13,31,4,12,30,3,11,29,1,10,28)\;
f:=6\cdot25\cdot43\cdot16
\end{verbatim}

\begin{verbatim}
G:=Group(f,b,l,r,u,d);
\end{verbatim}

\begin{verbatim}
Order(G);
\end{verbatim}

\begin{verbatim}
G=Group(u*l,f*r*b); #returns true
\end{verbatim}

Interpretation: Every state can be solved using only the two sequences \(ul\) and \(frb\) (never turning the down face)!
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orb:=Orbits(G);
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```gap
corners:=Blocks(G,orb[1]);
edges:=Blocks(G,orb[2]);
phiC:=ActionHomomorphism(G,corners,OnSets);
phiE:=ActionHomomorphism(G,edges,OnSets);
StructureDescription(Image(phiC)); #returns "S8"
StructureDescription(Image(phiE,Kernel(phiC))); #"A12"
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FG:=FreeGroup("f","b","l","r","u","d");
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# Satz 8.7
PreImagesRepresentative(hom,s); # solution of the superflip
Length(last); # number of quarter turn moves
PreImagesRepresentative(hom,Random(G));
StringTime(time); # how long did it take?
BrowseRubikCube(); # interactive mode
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FG:=FreeGroup("f","b","l","r","u","d");
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Merry Christmas!