Rubik's Group

My final lecture

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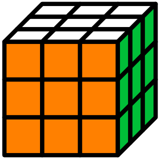

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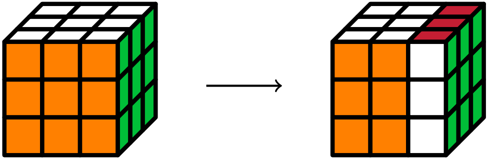

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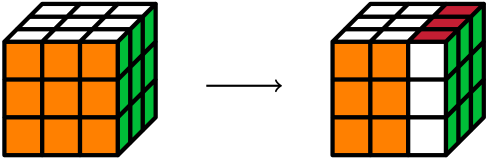

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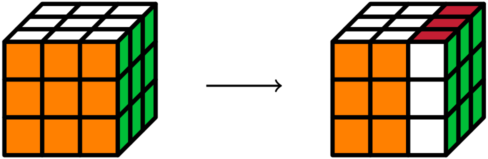

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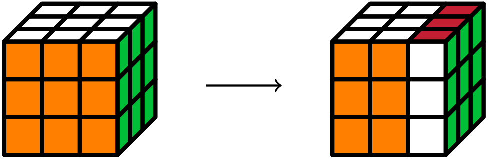

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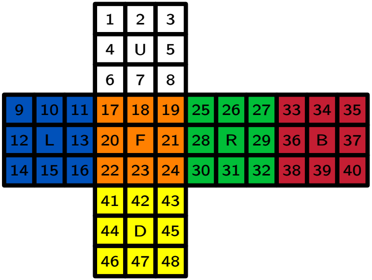
How "big" is the cube?

How many states can we reach by applying an arbitrary number of moves?

Facelets

- Idea: Enumerate the $6 \cdot 8 = 48$ edge and corner facelets:



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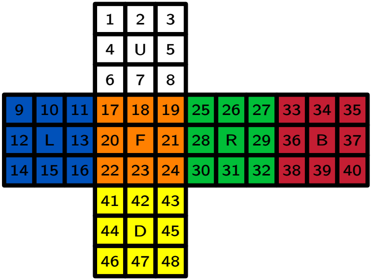

- Every move becomes a permutation in S_{48}, e. g.
  a clockwise 90° turn of the front face:

$$f := (6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)(17, 19, 24, 22)(18, 21, 23, 30).$$

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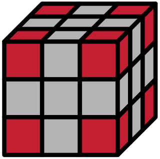
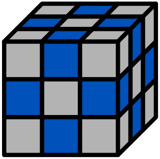
- Is G transitive on the 48 facelets?
- No: The $8 \cdot 3 = 24$ corner facelets and the $12 \cdot 2 = 24$ edge facelets form orbits Ω_C and Ω_E.

Ω_C: 

Ω_E: 

Action on Ω_C

- Hence,
$$G \leq \mathrm{Sym}(\Omega_C) \times \mathrm{Sym}(\Omega_E) \cong S_{24}^2$$
and $|G| \leq (24!)^2 \approx 10^{48}$.

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and $|G| \leq (24!)^2 \approx 10^{48}$.
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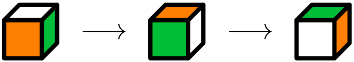
- No: the three facelets of a corner cubie form a block Δ in Ω_C.
- We can permute the three facelets of Δ only cyclically:



Action on Ω_E

From the lecture:

Satz 6.26. *Sei G eine imprimitive Permutationsgruppe auf Ω mit Block Δ. Sei $H := \{g \in G : {}^g\Delta = \Delta\}$ und sei $\varphi : H \to \mathrm{Sym}(\Delta)$ die Operation auf Δ. Sei $\Gamma := \{{}^g\Delta : g \in G\}$ und sei $\psi : G \to \mathrm{Sym}(\Gamma)$ die Operation auf Γ. Dann ist G zu einer Untergruppe von $\varphi(H) \wr \psi(G)$ isomorph.*

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- This gives a homomorphism $G \to C_3 \wr S_8 \leq S_{24}$.
- Similarly, the two facelets of an edge cubie form a block of Ω_E.
- Therefore,

$$G \leq C_3 \wr S_8 \times C_2 \wr S_{12}$$

and $|G| \leq 3^8 8! \cdot 2^{12} 12! \approx 5 \cdot 10^{20}$.

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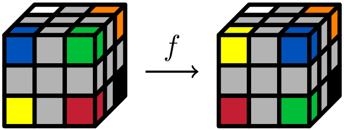

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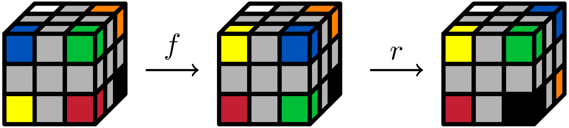

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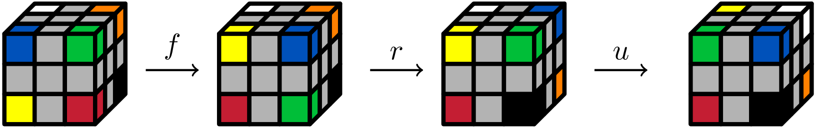

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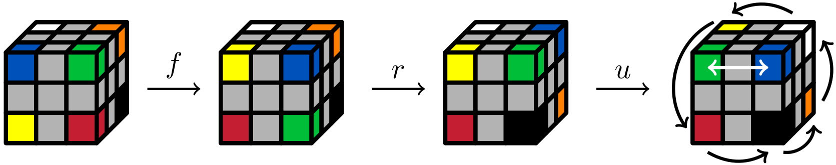

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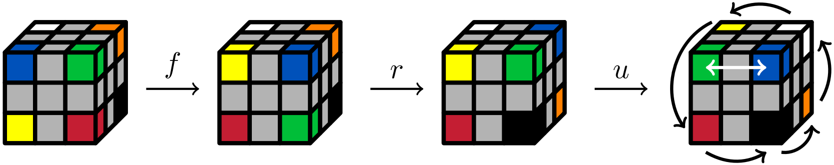

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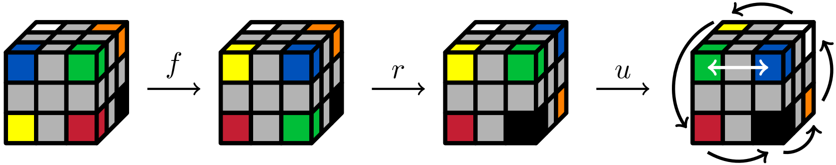

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- Since $S_8 = \langle (1,2), \ldots, (7,8) \rangle$ (exercise), $\varphi_C(G) = S_8$.

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- Therefore, $A_{12} = \langle (1,2,3), \ldots, (10,11,12) \rangle \subseteq \varphi_E(G_C) \subseteq A_{12}$.

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- For this reason it is impossible to flip only one edge and leave everything else fixed.
- Hence, $|N_2| \leq 2^{11}$.

Edge flips (realized)

- On the other hand, we can flip just two (adjacent) edges:

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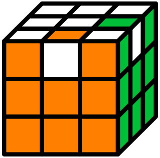

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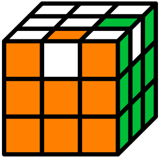
$$r^2 f^2 r^{-1} f r f r^2 b^{-1} u^{-1} f^{-1} u f b \quad =$$  (13 moves)

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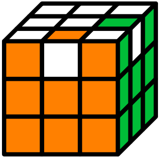

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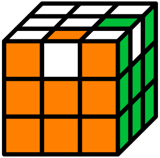

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- The group $N_2 \rtimes S_{12} \leq G$ is the reflection group with Dynkin diagram D_{12}.

Corner orientations (computed)

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- Since g has order 4, we obtain

$$1 = (t, \pi)^4 = (t \cdot {}^{\pi}t, \pi^2) * (t, \pi) * (t, \pi) = (t \cdot {}^{\pi}t \cdot {}^{\pi^2}t, \pi^3) * (t, \pi)$$
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- In particular,

$$\begin{aligned} 1 &= \prod_{c \in \mathcal{C}} (t\, {}^{\pi}t\, {}^{\pi^2}t\, {}^{\pi^3}t)(c) = \prod_{c \in \mathcal{C}} t(c) \prod_{c \in \mathcal{C}} t({}^{\pi^{-1}}c) \prod_{c \in \mathcal{C}} t({}^{\pi^{-2}}c) \prod_{c \in \mathcal{C}} t({}^{\pi^{-3}}c) \\ &= \Big(\prod_{c \in \mathcal{C}} t(c) \Big)^4 = \prod_{c \in \mathcal{C}} t(c). \end{aligned}$$

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- If $(t,\pi),(t',\pi')\in\langle\zeta\rangle^{\mathcal{C}}\rtimes\mathrm{Sym}(\mathcal{C})$ such that $\prod t(c)=\prod t'(c)=1$, then also
$$\prod_{c\in\mathcal{C}}(t\cdot{}^{\pi}t')(c)=\prod_{c\in\mathcal{C}}t(c)\prod_{c\in\mathcal{C}}t'({}^{\pi^{-1}}c)=1.$$

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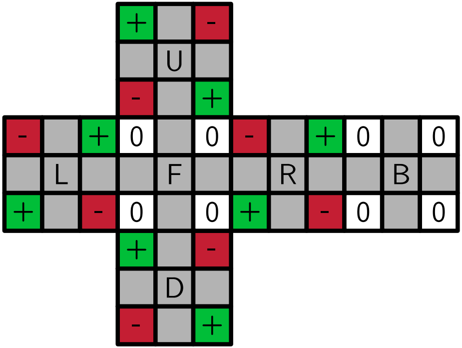
- This can also be visualized as follows.

Corner orientations (visualized)

Fix an orientation of the corner facelets:



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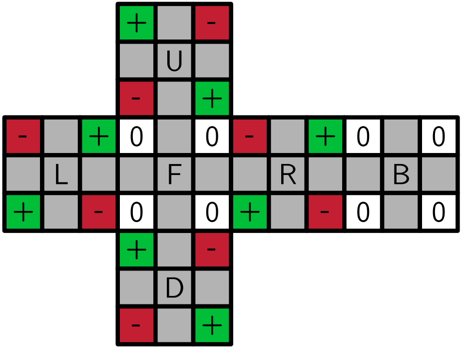

Every move causes one of the following effects:

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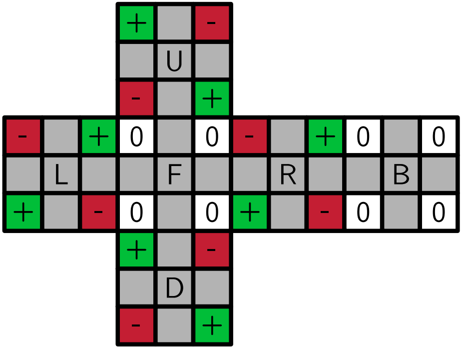

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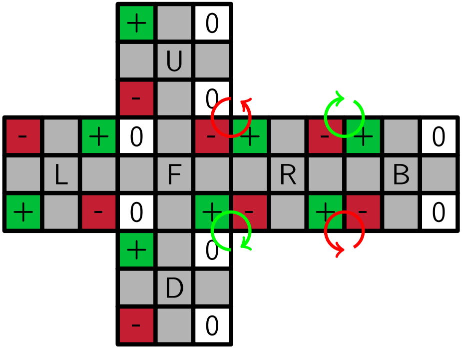

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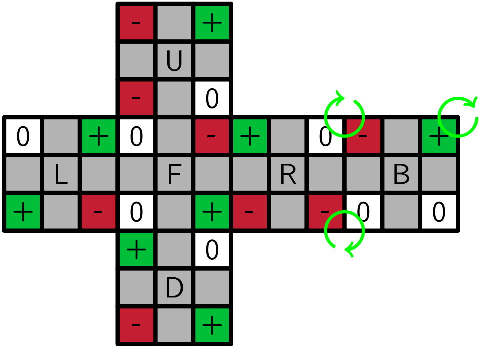

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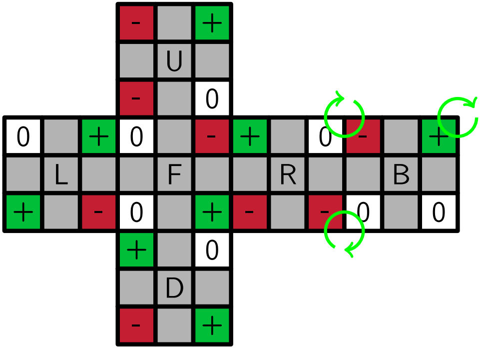

$\Longrightarrow$ The sum of all twists is always 0 modulo 3.

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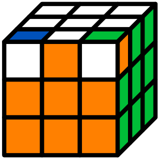

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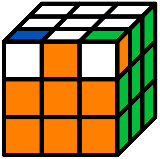

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- This shows $|N_3| = 3^7$.

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An element $(t, \pi, t', \pi') \in (C_3 \wr S_8) \times (C_2 \wr S_{12})$ *belongs to* G *if and only if*

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Hence, the index of G *in* $(C_3 \wr S_8) \times (C_2 \wr S_{12})$ *is* 12 *and*

$$\boxed{|G| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 = 43.252.003.274.489.856.000.}$$

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Interpretation: After taking apart and reassembling the cubies randomly, the cube is “solvable“ in only 1 out of 12 cases.

Consequences

- $G \cong (C_3^7 \times C_2^{11}) \rtimes (A_8 \times A_{12}) \rtimes C_2$. Composition factors: C_2 (12 times), C_3 (7 times), A_8, A_{12}.

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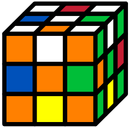

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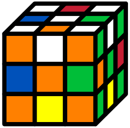

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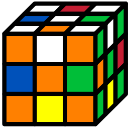

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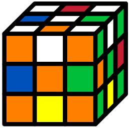

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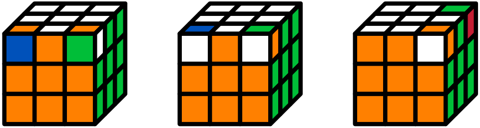
Burnside's Lemma

- Some states of the cube are symmetric to each other:



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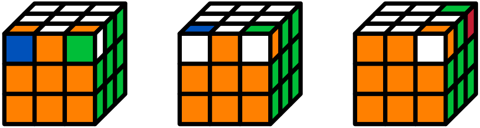

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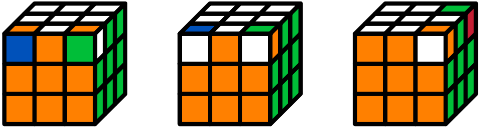

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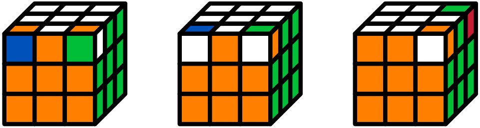

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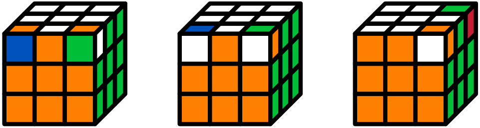

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Using the "symmetry" $g \leftrightarrow g^{-1}$, we get down to $450.541.810.590.509.978$.

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- On the second move, it makes no sense to turn the same face again. This leaves 15 moves.

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Proof (continued).

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- Solving the recurrence yields $\sum_{n=0}^{17} s_n < |G|$. □

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- Most states require 18 moves and the average is slightly below 18.
- If only quarter turn moves are allowed, God's number increases to 26.

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- There is a zero-knowledge AI algorithm in the spirit of AlphaZero which finds solution with 30 quarter turn moves on average.

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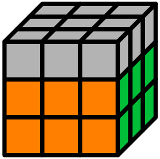

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- Tip: Write down s and s^{-1} on paper! If you mess up $[s, u]$, you have to start from the very beginning.

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Visit: `https://www.worldcubeassociation.org`,
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Variations

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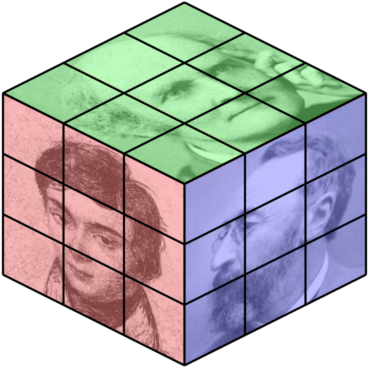

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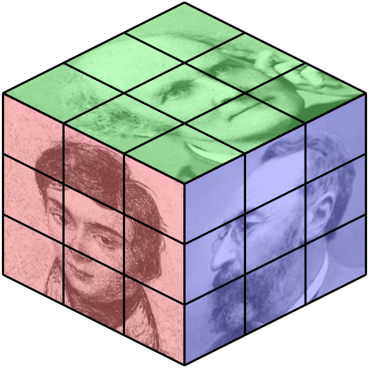

Answer: Yes.

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- On the other hand, $(uf)^{105} = (1_G, \zeta, \zeta, 1, \ldots, 1) \in \hat{G}$. Hence,

$$|\hat{G}| = 2^{11}|G| = 88.580.102.706.155.225.088.000.$$

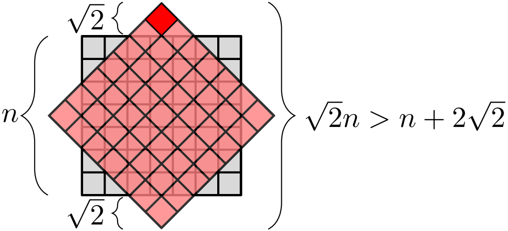
Variations

The invention of $n \times n \times n$-cubes:

n	Inventor	Product name	Year
2	Larry D. Nichols	Pocket Cube	1970
3	Ernő Rubik	Rubik's Cube	1974
4	Péter Sebestény	Rubik's Revence	1981
5	Udo Krell	Professor's Cube	1981
6	Panagiotis Verdes	V-Cube 6	2004

Variations

- For $n \geq 7$ there is a fundamental design problem:



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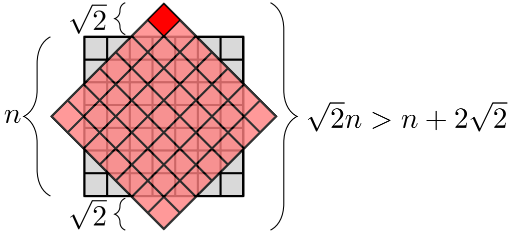

- The red square falls off!

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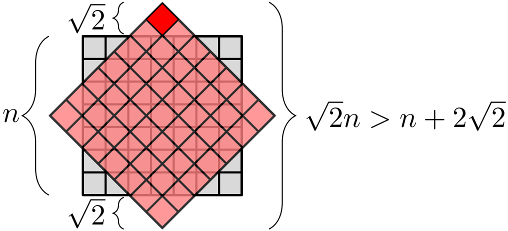
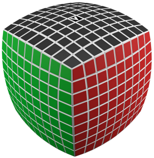
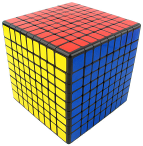
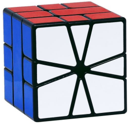


- The red square falls off!
- Remedy:

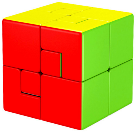
V-Cube 9: 

ShengShou 9: 

Endless other variations

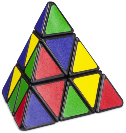


Square-1

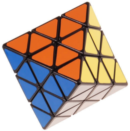


Puppet cube

Ghost cube



Pyraminx



Skewb Diamond

Skewb Ultimate

Gigaminx

Hypercube

Visit: `www.thecubicle.com`, `ruwix.com`, `mastercubestore.de`

With the computer...

Lets use the open-source computer algebra system GAP.
Rubik's group can be copied from
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Interpretation: Every state can be solved using only the two sequences ul and frb (never turning the down face)!

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orb:=Orbits(G);
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```
orb:=Orbits(G);
corners:=Blocks(G,orb[1]);
```

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phiC:=ActionHomomorphism(G,corners,OnSets);
```

With the computer. . .

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orb:=Orbits(G);
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StructureDescription(Image(phiC)); #returns "S8"
```

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StructureDescription(Image(phiE,Kernel(phiC))); #"A12"
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ZG:=Center(G);
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StructureDescription(Image(phiC)); #returns "S8"
StructureDescription(Image(phiE,Kernel(phiC))); #"A12"
ZG:=Center(G);
s:=ZG.1; #first generator = superflip
```

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```
FG:=FreeGroup("f","b","l","r","u","d");
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FG:=FreeGroup("f","b","l","r","u","d");
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```

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StringTime(time); #how long did it take?
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StringTime(time); #how long did it take?
BrowseRubikCube(); #interactive mode
```

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Happy semester break!