Character theory of symmetric groups

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1 Ordinary characters

A *partition* of $n \in \mathbb{N}_0$ is a sequence $\lambda = (\lambda_i)_{i \in \mathbb{N}}$ of non-negative integers such that $\lambda_1 \ge \lambda_2 \ge \ldots$ and $|\lambda| := \sum_{i \in \mathbb{N}} \lambda_i = n$. The non-zero λ_i are called *parts* of λ, while the $\lambda_i = 0$ are usually omitted. The number of parts is called the *length* of λ. Every partition λ can be visualized with a *Young diagram* with λ_i boxes in the i-th row. By "transposing" the Young diagram (i. e. reflecting on the diagonal) we obtain the Young diagram of the *conjugate* partition $\lambda' = (\lambda'_i)$ with $\lambda'_i := |\{j : \lambda_j \ge i\}|$ for $i \in \mathbb{N}$. Obviously, $\lambda'' = \lambda$. We call λ *symmetric* if $\lambda' = \lambda$. A *Young tableau* (of λ) is a Young diagram (of λ) where every box contains exactly one of the numbers $1, \ldots, n$ and the numbers in each row are increasingly ordered.

Example 1. Let $\lambda = (4, 2, 2, 1) = (4, 2^2, 1)$ be a partition of $n = 9$. Then the Young diagram of λ, a Young tableau and the conjugate Young diagram are given by:

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 3 & 6 & 7 \\ \hline 1 & 8 \\ \cline{1-2} 5 & 9 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \end{array}$$

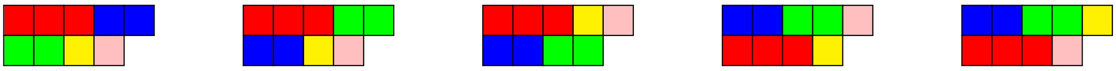
Every conjugacy class of the symmetric group S_n consists of the elements with a common cycle type. Therefore, the conjugacy classes of S_n can be identified with the partition of n. The Young tableaux of a partition λ are in one-to-one correspondence with the (ordered) partitions $Y = (Y_1, Y_2, \ldots)$ of the set $\{1, \ldots, n\}$ such that $|Y_i| = \lambda_i$ for $i \in \mathbb{N}$. Hence, S_n acts transitively on the set of Young tableaux of λ via ${}^gY = ({}^gY_i)$ for $g \in S_n$. The stabilizer of Y is the *Young subgroup* $S_Y := \prod \mathrm{Sym}(Y_i) \le S_n$ and the permutation character is $\psi_\lambda := (1_{S_Y})^{S_n}$. The characters ψ_λ and $\mathrm{sgn}\,\psi_{\lambda'}$ (where sgn is the *sign* character) have exactly one irreducible constituent χ_λ. Then $\chi_{\lambda'} = \mathrm{sgn}\,\chi_\lambda$ and

$$\mathrm{Irr}(S_n) = \{\chi_\lambda : \lambda \text{ partition of } n\}.$$

Example 2. We have $\psi_{(n)} = 1_{S_n} = \chi_{(n)}$ and $\chi_{(1^n)} = \chi_{(n)'} = \mathrm{sgn}$. The Young tableaux of $(n-1, 1)$ can be identified with the numbers $1, \ldots, n$. Hence, $\psi_{(n-1,1)}$ is the natural (2-transitive) permutation character of S_n and $\chi_{(n-1,1)} = \psi_{(n-1,1)} - 1_{S_n}$ for $n \ge 2$.

Let λ and μ be partitions of n. If $g \in S_n$ has type μ, then $\psi_\lambda(g)$ is the number of ways to distribute the parts of μ onto the parts of λ.

Example 3. For $\lambda = (5,4)$ and $\mu = (3, 2^2, 1^2)$, we obtain $\psi_\lambda(g) = 5$ as follows:



Starting with $\psi_{(n)} = \chi_{(n)} = 1_{S_n}$, one can compute $\mathrm{Irr}(S_n)$ recursively via

$$\chi_\lambda = \psi_\lambda - \sum_{\mu > \lambda} [\psi_\lambda, \chi_\mu]\chi_\mu = \psi_\lambda - 1_{S_n} - \sum_{(n) > \mu > \lambda} [\psi_\lambda, \chi_\mu]\chi_\mu$$

where $>$ denotes the lexicographical order. In fact, χ_μ can only occur in ψ_λ if $\mu \trianglerighteq \lambda$, i. e.

$$\sum_{i=1}^{s} \mu_i \geq \sum_{i=1}^{s} \lambda_i \qquad (s = 1, 2, \ldots)$$

(*dominance order*).

The *hook* $h_{ij}(\lambda) = h_{ij}$ of a box (i,j) of the Young diagram Y of a partition λ is the union of the boxes (i,j), $(i,j+1)$, $\ldots$ and the boxes $(i+1,j)$, $(i+2,j)$, $\ldots$. Then $|h_{ij}| = \lambda_i + \lambda'_j - i - j + 1$ is the *hook length* and the *hook length formula* holds

$$\chi_\lambda(1) = \frac{n!}{\prod\limits_{(i,j) \text{ box of } Y} h_{ij}}.$$

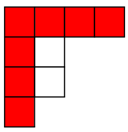
Let t_k be the number of k-cycles of some $g \in S_n$. *Frobenius' character formula* states that $\chi_\lambda(g)$ is the coefficient of $X_1^{h_{11}} X_2^{h_{21}} \ldots$ in the polynomial

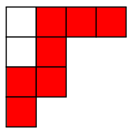
$$\prod_{i<j} (X_i - X_j) \prod_{k \geq 1} (X_1^k + X_2^k + \ldots)^{t_k}.$$

Let $l_{ij} := \lambda'_j - i$ (resp. $a_{ij} := \lambda_i - j$) be the *leg length* (resp. *arm length*). Removing h_{ij} from Y yields a Young diagram of a partition $\lambda \setminus h_{ij}$ of $n - |h_{ij}|$. Equivalently, one can remove the corresponding *rim hook*.

Example 4. A Young diagram filled with hook lengths, the hook h_{11} and its rim hook:

7	5	2	1
4	2		
3	1		
1			





Next, let $g \in S_n$ of type μ and let $h \in S_{n-\mu_k}$ be of type $(\mu_1, \ldots, \mu_{k-1}, \mu_{k+1}, \ldots)$. Let Y be the Young tableau of λ. Then the *Murnaghan-Nakayama formula* states that

$$\chi_\lambda(g) = \sum_{\substack{(i,j) \text{ box of } Y \\ h_{ij} = \mu_k}} (-1)^{l_{ij}} \chi_{\lambda \setminus h_{ij}}(h).$$

The special case $\mu_k = 1$ is called *branching rule*

$$(\chi_\lambda)_{S_{n-1}} = \sum_{\substack{(i,j) \text{ box of } Y \\ h_{ij} = 1}} \chi_{\lambda \setminus h_{ij}}.$$

2 Specht modules

Let $T_1, \ldots, T_k$ be the Young tableaux of a given partition λ of n. Note that $k = \frac{n!}{\prod \lambda_i!}$. The $\mathbb{Q}$-vector space M with basis $T_1, \ldots, T_k$ is the $\mathbb{Q}S_n$-permutation module with character ψ_λ as defined above. Let Y_i' be the set partition of $\{1, \ldots, n\}$ corresponding to the conjugate tableau T_i' of λ'. The *Specht module* S^λ associated with λ is the submodule of M generated by the elements

$$t_i := \sum_{\pi \in S_{Y_i'}} \operatorname{sgn}(\pi)^\pi T_i \qquad (i = 1, \ldots, k)$$

(it is easy to see that ${}^\pi T_i \neq {}^\sigma T_i$ for $\pi \neq \sigma$). It turns out that S^λ is simple with character χ_λ. In particular, all irreducible characters of S_n can be realized over $\mathbb{Z}$. Therefore, the Frobenius-Schur indicators are always 1. A basis of S^λ is given by those t_i such that T_i is *standard*, i. e. also the columns of T_i are increasingly ordered. Thus, the hook formula also counts the number of standard Young tableaux of λ.

3 Blocks

Let p be a prime. A *p-hook* is a hook of length p. Starting from a partition λ we can successively remove all p-hooks from the corresponding Young diagram to obtain the *p-core* which is a partition of $n - wp$ where w is the *weight* of λ (this does not depend on the way the hooks are removed). Characters $\chi_\lambda, \chi_\mu \in \operatorname{Irr}(S_n)$ lie in the same *p-block* if and only if they have the same p-core (*Nakayama's conjecture*). In this way, the p-blocks of S_n can be labeled by p-cores. The *weight* of a block B is the weight of any λ with $\chi_\lambda \in \operatorname{Irr}(B)$. Note that conjugate characters (and blocks) have conjugate cores. The principal block containing $1_{S_n} = \chi_{(n)}$ corresponds to the core (r) where $r \in \{0, \ldots, p-1\}$ such that $n \equiv r \pmod p$. The blocks of weight 0 contain only one irreducible character χ_λ where λ is a core. By the hook formula, $|S_n|_p = \chi_\lambda(1)_p$. Hence, these are the blocks of p-defect 0. Note that the 2-cores are the *staircase* partitions $(k, k-1, \ldots, 1)$. In particular, S_n has at most one 2-block of weight w and in that case $n - 2w = \binom{k+1}{2}$ is a triangular number.

In general, the fusion system of a p-block B of weight w is the fusion system of S_{pw} with respect to its Sylow p-subgroup P of order $p^{w + \lfloor w/p \rfloor + \cdots}$ (Legendre's formula). In particular, P is a defect group of B. If $w = \sum a_i p^{i-1}$ is the p-adic expansion (i. e. $0 \leq a_i \ldots < p$), then $P \cong \prod P_i^{a_i}$ where $P_i := C_p \wr \ldots \wr C_p$ (i copies). Moreover, B is splendid derived equivalent to the principal block of S_{wp} and

$$k(B) := |\operatorname{Irr}(B)| = \sum_{\substack{(w_1, \ldots, w_p) \in \mathbb{N}_0^p \\ \sum w_i = w}} \pi(w_1) \ldots \pi(w_p)$$

where $\pi(m)$ is the number of partitions of $m \in \mathbb{N}_0$. Obviously, P is abelian if and only if $w < p$ and in this case *Broué's conjecture* holds.

The *(p)-abacus* $A_\lambda \subseteq \{0, \ldots, p-1\} \times \mathbb{N}_0$ of a partition λ is defined by $(r, s) \in A_\lambda \Leftrightarrow \exists i : r + sp = h_{i1}$. The elements of A_λ can be visualized as *beads* on a matrix with infinitely many columns. The rows of this matrix are called *runners*. Removing a box from the Young diagram of λ is the same as moving a bead of A_λ up to the previous runner (modulo p). Removing a p-hook slides a bead to the left by one (in particular this spot must be vacant beforehand). Hence, the abacus of a core has no "holes" and its first runner is empty.

Let B be a block of weight w with core μ. Let a_i be the number of beads on runner i of A_μ. Suppose that $a_{i+1} - a_i \geq w$ for some $i \in \{0, \ldots, p-2\}$. Then, interchanging runner i and $i+1$ yields a core of a block $\hat{B}$ of $S_{n-a_{i+1}+a_i}$ which is Morita equivalent to B (*Scopes' reduction*). Thus, in order to determine the Morita equivalence class of B we may assume that $a_{i+1}-a_i<w$ for $i=0,\ldots,p-2$. Since $a_0=0$, it follows that $a_i\le i(w-1)$ for all i. The number of blocks with these restrictions is $\frac{1}{p}\binom{wp}{p-1}$. If $\mu\ne\mu'$, then B is also Morita equivalent to the block B' of S_n with core μ' (note that $\operatorname{Irr}(B')=\operatorname{sgn}\operatorname{Irr}(B)$). Therefore the number of Morita equivalence classes of p-blocks of symmetric groups of weight w is at most

$$\frac{1}{2p}\binom{wp}{p-1}+\frac{1}{2}\binom{\lfloor wp/2\rfloor}{\lfloor p/2\rfloor}.$$

If $a_i=i(w-1)$ for $i=0,\ldots,p-1$, then B is called *RoCK block* and

$$n=\frac{p}{24}\Big((w-1)^2p(p^2-1)+2(w-1)p^2+22w+2\Big).$$

In the case $w<p$ the RoCK is Morita equivalent to its Brauer correspondent in $\mathrm{N}_{S_n}(D)$ where D is a defect group of B. Moreover, B is Morita equivalent to the principal block of $S_p\wr S_w$.

Example 5. The Morita equivalence classes of 3-blocks of S_n of weight (defect) 2 are represented by the principal blocks of S_6, S_7 and a non-principal block of S_{11}. The cores and abaci are given as follows:

empty abacus/core $\quad \begin{array}{c|c}0&\cdot\\1&\bullet\\2&\cdot\end{array}\ \yng(1) \qquad \begin{array}{c|c}0&\cdot\\1&\cdot\\2&\bullet\end{array}\ \yng(2) \qquad \begin{array}{c|cc}0&\cdot&\cdot\\1&\bullet&\cdot\\2&\bullet&\bullet\end{array}\ \yng(3,1,1)$

4 Decomposition numbers

In general, the number of irreducible Brauer characters of a finite group equals the number of conjugacy classes of p-regular elements. For S_n this is the number of partitions with no non-zero part divisible by p. A partition is called *p-regular* if it has no p parts of the same non-zero length (for $p=2$ this means that all parts are distinct). By *Glaisher's Theorem*, also the number of these partitions is the number of irreducible Brauer characters (for $p=2$ this is *Euler's Theorem*: the number of partitions with distinct parts is the number of partitions with odd parts). Starting from an arbitrary partition λ we construct a p-regular partition λ^0 by successively removing p-hooks with arm length 0. For a p-block B with weight w and core μ the number of irreducible Brauer characters in B equals the number of p-regular partitions with core μ. We write $\operatorname{IBr}(B):=\{\varphi_\lambda:\chi_\lambda\in\operatorname{Irr}(B),\ \lambda^0=\lambda\}$ and

$$l(B):=|\operatorname{IBr}(B)|=\sum_{\substack{(w_1,\ldots,w_{p-1})\in\mathbb{N}_0^{p-1}\\ \sum w_i=w}}\pi(w_1)\ldots\pi(w_{p-1}).$$

Unlike in the ordinary case there is no formula for the degrees of Brauer characters. In fact, for $p=2$ and $n\ge 20$ (say) these degrees are unknown. We denote the decomposition numbers of B by $d_{\lambda\tau}:=d_{\chi_\lambda\varphi_\tau}$. If the irreducible characters of B are ordered in such a way that the p-regular partitions in decreasing lexicographical order come first, then the decomposition matrix $(d_{\lambda\tau})$ has unitriangular shape.

For partitions λ and μ of n let

$$t_{\lambda\mu}:=-\sum_{\lambda\setminus h_{ij}(\lambda)=\mu\setminus h_{kl}(\mu)}(-1)^{l_{ij}(\lambda)+l_{kl}(\mu)}\nu_p(|h_{ij}(\lambda)|)$$

where ν_p is the p-adic valuation. Then the *Jantzen-Schaper formula* states that

$$d_{\lambda\tau} \leq \sum_{\mu > \lambda} t_{\lambda\mu} d_{\mu\tau}$$

for $\lambda \neq \tau$. Moreover, $d_{\lambda\tau} = 0$ if and only if the right hand side is 0. For blocks of weight at most 3 it is known that $d_{\lambda\tau} \leq 1$ and therefore $(d_{\lambda\tau})$ can be computed recursively.

5 Cartan invariants

We have seen above that $k(B)$ and $l(B)$ only depend on the weight w of a block B of S_n. We therefore write $k(w) := k(B)$ and $l(w) := l(B)$. The elementary divisors of the Cartan matrix $C(B)$ of B will also depend solely on w (but $C(B)$ itself depends on more than that). We make use of the generating function $P(x) := \sum_{k \geq 0} \pi(k) x^k$. A formula of Euler states that

$$P(x) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}.$$

Moreover, if $\pi_0(n)$ is the number of p-regular partitions of n, then

$$\sum_{n \geq 0} \pi_0(n) x^n = P(x) P(x^p)^{-1}.$$

The results above can be rephrased as

$$\sum_{w \geq 0} k(w) x^w = P(x)^p, \tag{5.1}$$

$$\sum_{w \geq 0} l(w) x^w = P(x)^{p-1}. \tag{5.2}$$

Let $m(w)$ be the multiplicity of 1 as an elementary divisor of $C(B)$. Then

$$\sum_{w \geq 0} m(w) x^w = P(x)^{p-2} P(x^p).$$

In particular, $m(w) > 0$ if $p > 2$ and

$$m(w) = \begin{cases} \pi(w/2) & \text{if } w \equiv 0 \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

if $p = 2$. For a partition $\lambda = (\lambda_1, \ldots)$ let

$$e(\lambda) = \sum_{k \geq 1} \frac{p^{\nu_p(\lambda_k)+1} - 1}{p - 1}.$$

Let $\pi_0^e(n)$ be the number of p-regular partitions λ of n such that $e(\lambda) = e$. A theorem of Olsson says that the multiplicity of p^e as an elementary divisor of $C(B)$ is

$$\sum_{s=0}^{w} m(w - s) \pi_0^e(s).$$

It is also possible to express the multiplicities of lower defect groups of B.

Example 6. The principal 2-block B of S_{10} has weight $w = 5$. We only need the 2-regular partitions of 1, 3, 5:

λ	(1)	$(2,1)$	(3)	$(3,2)$	$(4,1)$	(5)
$e(\lambda)$	1	4	1	4	8	1

Hence, 2^e can only occur as elementary divisor if $e \in \{1, 4, 8\}$. The multiplicity of $2^8 = |D|$ is always 1. The multiplicities of 2 and 16 are

$$\begin{aligned} m(4)\pi_0^1(1) + m(2)\pi_0^1(3) + m(0)\pi_0^1(5) &= 2 + 1 + 1 = 4, \\ \pi_0^4(3) + \pi_0^4(5) &= 1 + 1 = 2 \end{aligned}$$

respectively. In general, the multiplicity of 2 is $\pi(0) + \ldots + \pi(k)$ if $w = 2k + 1$ and 0 otherwise.

6 Heights

Let $n = \sum a_i p^i$ is the p-adic expansion where p is a prime. For any expansion $n = \sum b_i p^i$ with $b_0, b_1, \ldots \geq 0$ let

$$\delta(b_0, b_1, \ldots) := \frac{\sum_i b_i - a_i}{p - 1} \geq 0.$$

Let $E_d(n)$ be the set of those sequences $(b_0, \ldots)$ such that $\delta(b_0, \ldots) = d$.

Next let $c(n)$ be the number of p-core partitions of n (= number of blocks of defect 0 of S_n). Set $C(x) := \sum_{n \geq 0} c(n) x^n$. Generalizing (5.1) we define

$$P(x)^s = \sum_{t=0}^{\infty} k(s, t) x^t,$$
$$C(x)^s = \sum_{t=0}^{\infty} c(s, t) x^t.$$

Note that if $t < p$, then $c(t) = \pi(t)$ and $c(s, t) = k(s, t)$ for all $s \geq 0$. Let $m_d(n)$ be the number of $\chi \in \mathrm{Irr}(S_n)$ such that $\chi(1)_p = p^d$. Olsson has shown that

$$m_d(n) = \sum_{(b_0,\ldots) \in E_d(n)} c(1, b_0) c(p, b_1) c(p^2, b_2) \ldots.$$

For $d = 0$ we have $E_0(n) = \{(a_0, \ldots)\}$ and this yields *MacDonald's Theorem*

$$m_0(n) = k(1, a_0) k(p, a_1) \ldots.$$

If additionally $p = 2$, then $a_i \leq 1$ and $m_0(n) = 2^{a_0 + \cdots}$. In particular, if $n = 2^k$, then $m_0(n) = n$ and the corresponding characters $\chi_\lambda \in \mathrm{Irr}(S_n)$ (of odd degree) are labeled by the *hook partitions* $\lambda = (s, 1^{n-s})$ for $s = 1, \ldots, n$.

Now let B be a p-block of S_n with weight w and defect d. The *height* $h(\chi) \geq 0$ of $\chi \in \mathrm{Irr}(B)$ is defined by $\chi(1)_p p^{d-h(\chi)} = |S_n|_p$. Let $k_h(w)$ be the number of $\chi \in \mathrm{Irr}(B)$ of height h (depends only on w). Then

$$k_h(w) = \sum_{(b_0,\ldots) \in E_h(w)} c(p, b_0) c(p^2, b_1) \ldots.$$

Since for $n = pw$ there is only one block of maximal defect in S_n, we recover $k_0(w) = m_0(pw)$. The maximal possible height of some $\chi \in \operatorname{Irr}(B)$ is

$$h = \frac{w - \sum a_i}{p-1}$$

where $w = \sum a_i p^i$ is the p-adic expansion. Then $k_h(w) = c(p,w)$ since $E_h(w) = \{(w,0,\ldots)\}$. For $p = 2$ it can happen that $k_h(w) = 0$ (e. g. $k_3(5) = c(2,5) = 0$). In general, Olsson's Conjecture $k_0(w) \le |D : D'|$ holds where D is a defect group of B.

Example 7. For $p = 2$ and $n = 7 = 1 + 2 + 4$ we have $(a_0, a_1, a_2) = (1,1,1)$ and

$$E_1(7) = \{(3,0,1),(1,3)\}, \qquad E_2(7) = \{(3,2)\}, \qquad E_3(7) = \{(5,1)\}, \qquad E_4(7) = \{(7)\}.$$

Moreover,

$$C(x) = 1 + x + x^3 + x^6 + x^{10} + \ldots, \quad C(x)^2 = 1 + 2x + x^2 + 2x^3 + \ldots, \quad C(x)^4 = 1 + 4x + \ldots$$

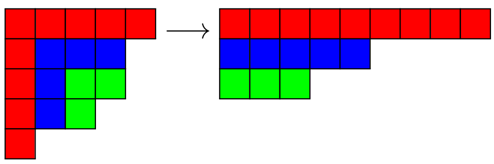
Consequently,

$$\begin{aligned}
m_0(7) &= 2^{1+1+1} = 8,\\
m_1(7) &= c(1,3)c(4,1) + c(1,1)c(2,3) = 4 + 2 = 6,\\
m_2(7) &= c(1,3)c(2,2) = 1,\\
m_3(7) &= c(1,5)c(2,1) = 0,\\
m_4(7) &= c(1,7) = 0
\end{aligned}$$

7 Alternating groups

A conjugacy class C of A_n lies in a conjugacy class of S_n and therefore belongs to a partition λ of n. More precisely, C is not a conjugacy class of S_n if and only if λ has distinct odd parts. In this case $C \,\dot\cup\, C^{(12)}$ is a conjugacy class of S_n. By *Sylvester's Theorem* there is a bijection Γ between the symmetric partitions and the partitions with distinct odd parts:

$$(\lambda_1, \lambda_2, \ldots) \xrightarrow{\Gamma} (2\lambda_1 - 1, 2\lambda_2 - 3, \ldots)$$



If $\lambda \neq \lambda'$, then $(\chi_\lambda)_{A_n} \in \operatorname{Irr}(A_n)$. Now suppose that $\lambda = \lambda'$ and $\mu := \Gamma(\lambda)$. Then by Clifford theory, $(\chi_\lambda)_{A_n} = \xi_\lambda + \xi_\lambda^{(12)}$ for some $\xi_\lambda \in \operatorname{Irr}(A_n)$ with $\xi_\lambda^{S_n} = \chi_\lambda$. We fix $g \in A_n$ of type μ. Then for $h \in A_n$ and $\epsilon := (-1)^{\frac{n-l(\mu)}{2}}$ we have

$$\xi_\lambda(h) = \begin{cases} \frac{1}{2}\chi_\lambda(h) & \text{if } h \text{ is not of type } \mu,\\ \frac{1}{2}\big(\epsilon + \sqrt{\epsilon\mu_1\ldots\mu_{l(\mu)}}\big) & \text{if } h \text{ is conjugate to } g \text{ in } A_n,\\ \frac{1}{2}\big(\epsilon - \sqrt{\epsilon\mu_1\ldots\mu_{l(\mu)}}\big) & \text{if } h \text{ is conjugate to } g^{(12)} \text{ in } A_n. \end{cases}$$

This allows to compute the character table of A_n from $\mathrm{Irr}(S_n)$.

Similarly, if B is a p-block of S_n with core $\mu \neq \mu'$, then B is isomorphic to a block B' of A_n via restriction. In this case, $p > 2$ and B and B' have the same fusion system. Now suppose that B has core $\mu = \mu'$, weight w and defect group D. Then B covers a block B' of A_n with defect group $D \cap A_n$ and fusion system A_{wp}. If $p = 2$, every core has the form $\mu = (a, a-1, \dots, 1) = \mu'$. If in addition w is odd, then every $\chi \in \mathrm{Irr}(B)$ restricts to $\mathrm{Irr}(B')$. Hence, in this case, $k(B) = 2k(B')$ and the decomposition matrix of B consists of two copies of the decomposition matrix of B'.

For an odd prime p let $p^* = (-1)^{\frac{p-1}{2}}p$. Robinson and Thompson have shown that if $n \geq 25$, then

$$\mathbb{Q}(A_n) = \mathbb{Q}(\sqrt{p^*} : 3 \leq p \leq n \text{ prime }, p \neq n-2).$$

8 Wreath products

Generalizing the abacus we call any strictly decreasing sequence $a = (a_i) \in \mathbb{N}_0^l$ a *β-set* of *length* $l(a) = l$. We often identify β-sets with finite subsets of $\mathbb{N}_0$. For $s \in \mathbb{N}_0$ also

$$a^{+s} := (a_1 + s, \dots, a_l + s, s-1, s-2, \dots, 0)$$

is a β-set (of length $l+s$). Any β-set a determines a partition $\lambda := P(a) := (a_1-(l-1), a_2-(l-2), \dots, a_l)$ (note that a is the set of first column hook lengths of λ). Since $P(a) = P(a^{+s})$, we may assume that $l(a) \equiv 0 \pmod p$ in the following. We define $a_i^{(p)} := \{b \in \mathbb{N}_0 : bp + i \in a\}$ for $i = 0, \dots, p-1$ (that is, we look at each runner of the abacus individually). Then the sequence of partitions $\lambda^{(p)} := (P(a_0^{(p)}), \dots, P(a_{p-1}^{(p)}))$ is called the *p-quotient* of λ. The number $\sum |P(a_i^{(p)})|$ equals the weight of λ. Conversely, λ is uniquely determined by its p-core and p-quotient. If μ is the p-core of λ, the *p-sign* of λ is defined by $\delta_p(\lambda) = (-1)^{\sum l_i}$ where the l_i are the leg lengths of the p-hooks removed from λ to obtain μ.

Example 8. For $\lambda = (5, 4, 1^2)$ and $p = 2$ we obtain

$$a = (8, 6, 2, 1), \qquad (a_i^{(p)}) = (\{4, 3, 1\}, \{0\}), \qquad \lambda^{(p)} = ((2, 2, 1), ()).$$

Hence, λ has weight 5 and the p-core is (1).

Let B be a p-block of S_n with weight w. Let $\mathrm{Irr}(C_p) = \{\varphi_1, \dots, \varphi_p\}$ and let $\tau = (\tau_1, \dots, \tau_p)$ a tuple of partitions such that $\sum |\tau_i| = w$. The linear characters $\varphi^{\otimes|\tau_i|} := \varphi_i \otimes \ldots \otimes \varphi_i \in \mathrm{Irr}(C_p^{|\tau_i|})$ extend to $C_p \wr S_{|\tau_i|}$ and we can define $\varphi_{\tau_i} := \varphi^{\otimes|\tau_i|}\chi_{\tau_i} \in \mathrm{Irr}(C_p \wr S_{|\tau_i|})$ where $\chi_{\tau_i} \in \mathrm{Irr}(S_{|\tau_i|})$. Finally let $\varphi_\tau := \big(\bigotimes_{i=1}^p \varphi_{\tau_i}\big)^{C_p \wr S_w} \in \mathrm{Irr}(C_p \wr S_w)$. Then $\mathrm{Irr}(B) \to \mathrm{Irr}(C_p \wr S_w)$, $\chi_\lambda \mapsto \varphi_{\lambda^{(p)}}$ is a height preserving bijection.

Now we label the conjugacy classes of $C_p \wr S_w$ where we consider C_p as $\mathbb{Z}/p\mathbb{Z}$. For $(x_1 \dots x_w, \sigma) \in C_p \wr S_w$ we define a tuple of partitions $\tau = (\tau_0, \dots, \tau_{p-1})$ as follows: For every cycle $(a_1, \dots, a_s)$ in σ let $s \in \tau_{x_{a_1} + \ldots + x_{a_s}}$. Then $\sum |\tau_i| = w$. Let $g_1, \dots, g_l \in C_p \wr S_w$ be representatives for the classes of $C_p \wr S_w$ corresponding to the partition tuples τ with $\tau_0 = ()$ (note that these elements are non-trivial). Osima has shown that there exists $S \in \mathrm{GL}(l(B), \mathbb{C})$ such that

$$(d_{\chi_\lambda, i}) = (\delta_p(\lambda)\varphi_{\lambda^{(p)}}(g_i))S$$

where $(d_{\chi_\lambda,i})_{\lambda,i}$ is the decomposition matrix of B. It follows that the so-called *contributions* of B can be computed inside the smaller group $C_p \wr S_w$. More precisely,

$$[\chi_\lambda, \chi_\mu]^0 = \frac{1}{n!} \sum_{g \in S_n^0} \chi_\lambda(g)\chi_\mu(g^{-1}) = \delta_p(\lambda)\delta_p(\mu) \sum_{i=1}^{l} \frac{1}{|\mathrm{C}_{C_p \wr S_w}(g_i)|} \varphi_{\lambda^{(p)}}(g_i)\varphi_{\mu^{(p)}}(g_i^{-1})$$

for every $\chi_\lambda, \chi_\mu \in \mathrm{Irr}(B)$.

9 Double covers and spin blocks