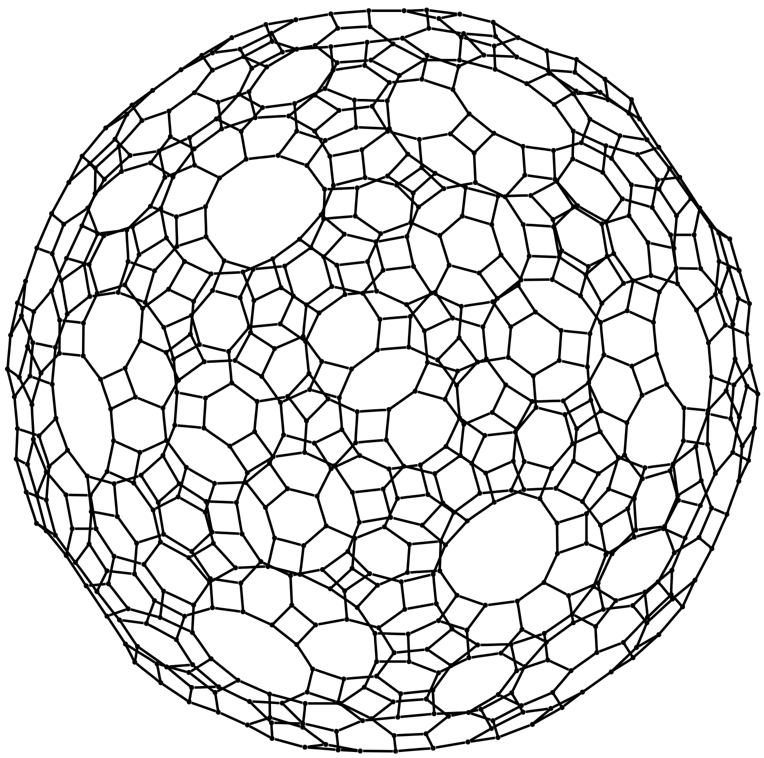
Finite Weyl groupoids and crystallographic arrangements



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von

Michael Cuntz
aus Berlin

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Contents

CHAPTER 1

Introduction

Reflections appear in many areas of mathematics. For instance, certain groups generated by involutions may be investigated by representing them as reflection groups. In particular, the Weyl groups belong to this class. They appear naturally inside semisimple algebraic groups and are fundamental for their classification. In their reflection representation, the Weyl groups are in fact subgroups of $\mathrm{GL}(\mathbb{Z}^r)$ for some r. This integrality is a very strong and important restriction; reflection groups with this property are also called *crystallographic*.

Closely related to the algebraic group is another important structure, the Lie algebra. Lie algebras arise in nature as vector spaces of linear transformations, for example differential operators. It turns out that finite dimensional semisimple complex Lie algebras decompose into a direct sum labeled by *roots* and a Cartan subalgebra. These roots are (up to signs) the normal vectors defining the reflection hyperplanes of a Weyl group. Again, we have an integrality property for the roots: Let $\mathcal{A}$ be the real hyperplane arrangement given by the orthogonal complements of the roots. Then this is a simplicial arrangement and for each chamber K, the roots labeling the walls of K form a simple system Δ, and in particular all other roots are integer linear combinations of the roots in Δ.

So apparently the combinatorics of root systems and Weyl groups play an important role in mathematics and moreover, a certain integrality is an essential feature of these structures. In the last decades, the concept of a Lie algebra has been generalized in many directions. For example, deformations of Lie algebras called quantum groups have proved to be useful in physics. More generally the theory of Hopf algebras seems to be a further natural direction. Recent results on pointed Hopf algebras have led to yet another symmetry structure, the *Weyl groupoid*. Again one has vectors called “roots”, but this time the object acting on the roots is in general a groupoid and not a group anymore. A remarkable fact which is one of my results [**14**] (Chapt. 7) is that even in this much more general setting, the above integrality still plays a crucial role.

Although the theory of Weyl groupoids has been more and more separated from its origin due to the newly-found connection to simplicial arrangements (see below), let us first review the motivation that has led to these insights: the theory of pointed Hopf algebras and the classification of Nichols algebras.

A Hopf algebra A over an algebraically closed field $\Bbbk$ of characteristic zero is called *pointed* if all its simple left or right comodules have dimension one. If A is pointed then the coradical A_0 of A is a group algebra $\Bbbk\Gamma$. Further, there is a natural filtration $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ of A given by the coradical because A_0 is conilpotent. This filtration defines a graded Hopf algebra $\operatorname{gr} A = \bigoplus_{i>0} A_i/A_{i-1}$. A theorem by Radford [**43**] states that there exists a graded braided Hopf algebra R in the category of Yetter-Drinfeld modules over $\Bbbk\Gamma$ such that $\operatorname{gr} A$ is a biproduct, $\operatorname{gr} A \cong R\#\Bbbk\Gamma$.
The subalgebra generated by the $x \in R$ with $\Delta(x) = 1\otimes x + x\otimes 1$ is a so-called *Nichols algebra* (see below for the definition). One natural problem is now to determine for a given Nichols algebra R all pointed Hopf algebras A such that $\operatorname{gr} A \cong R\#\Bbbk\Gamma$ where Γ is the group of group-like elements of A. The *Lifting method* of Andruskiewitsch and Schneider [**5**] gives an answer to this question. Thus to obtain a classification of pointed Hopf algebras it is natural to classify Nichols algebras first.

Let H be a Hopf algebra, V a Yetter-Drinfeld module over H, and $T(V)$ the tensor algebra on V. Let $\mathcal{S}$ be the set of all homogeneous ideals $J \trianglelefteq T(V)$ generated by elements of degree greater or equal 2 which are coideals and Yetter-Drinfeld modules over H. If $I(V)$ is the maximal element in $\mathcal{S}$, then $\mathcal{B}(V) := T(V)/I(V)$ is a Nichols algebra, and in fact we can use this as a definition because any Nichols algebra is of this form. Thus to understand Nichols algebras it will be important to find invariants associated to a pair (H, V) as above.
In recent papers, Andruskiewitsch, Heckenberger and Schneider [**2**] and Heckenberger and Schneider [**32**] gave a construction which associates to a Yetter-Drinfeld module V over H a so-called *Weyl groupoid*. This Weyl groupoid is the symmetry structure of the Nichols algebra $\mathcal{B}(V)$ in the same way as the Weyl group is the symmetry structure of the universal enveloping algebra of a semisimple Lie algebra. A Weyl groupoid is a groupoid generated by reflections which are not viewed as endomorphisms of a vector space but as isomorphisms between two (possibly different) objects labeled by the same vector space. A good way to define this groupoid is by the *Cartan scheme* (see [**17**] or Chapt. 2) which is a collection of generalized Cartan matrices together with a labeled regular graph ρ connecting these matrices. The edges of the graph are the generating morphisms.
For the classification of Nichols algebras of diagonal type it is crucial to be able to decide if a given Cartan scheme admits a finite root system. Therefore the Weyl groupoids of Cartan schemes which admit a finite root system are particularly interesting (see Chapt. 2 for details). Weyl groupoids and their root systems were introduced by Heckenberger and Yamane in [**34**], where a generalization of Matsumoto's Theorem is proved which specializes to the classical theorem for the case of crystallographic Coxeter groups. Hence Weyl groupoids inherit many nice properties.

So the Weyl groupoid historically appeared as an invariant needed to classify Nichols algebras. However, the situation has changed during the last year: Recent observations have considerably increased the importance of the Weyl groupoid.

If the *real roots* of a Weyl groupoid form a finite root system (as defined in Chapt. 2), then we will say that the Weyl groupoid is finite. As for root systems from Coxeter groups, a finite Weyl groupoid $\mathcal{W}$ defines a hyperplane arrangement: Fix an object a and its set of positive roots R_+^a. The arrangement associated to $\mathcal{W}$ and a is the set of orthogonal complements of the elements of R_+^a in $\mathbb{R}^r$. It turns out that these arrangements are simplicial, i.e. the complement of the union of these hyperplanes decomposes into open simplicial cones. This was first observed in [**19**] (Chapt. 5) in the case of rank three and then proved for finite Weyl groupoids of arbitrary rank in [**33**]. Moreover, it turned out that in terms of simplicial arrangements the axioms of a finite Weyl groupoid reduce to one single integrality property (I) (see [**14**] or Chapt. 7 for details). We call simplicial arrangements satisfying this axiom (I) *crystallographic arrangements*.

Among other things, this explains why the class of arrangements obtained from Weyl groupoids is so large. In fact this class is so large that for example in rank three, 53 of the 67 known sporadic simplicial arrangements (see [**23**]) over $\mathbb{Q}$ are crystallographic. Since the classification of simplicial arrangements in the real projective plane is still an open problem, it could be very valuable to have a complete classification of a large subclass. But also in higher rank generalizations of crystallographic arrangements could have a great impact. Like reflection arrangements, all crystallographic arrangements are free [**10**]. They could provide examples or counterexamples in geometry or topology.

The present “Habilitationsschrift” is a collection of papers concerned with finite Weyl groupoids and crystallographic arrangements. The series begins with a definition of Weyl groupoids in terms of categories and ends with a complete classification of the “universal finite” case, i.e. the classification of connected simply connected Cartan schemes for which the real roots are a finite root system in the sense of Chapt. 2. There are three side trips: A connection to cluster algebras (Chapt. 4), observations on simplicial arrangements in the real projective plane (Chapt. 6), and the so-called crystallographic arrangements (Chapt. 7). The papers treating the classification (Chapt. 2,3,4,5,8) were obtained in joint work with István Heckenberger.

Here is an overview of the contents of the papers:

Weyl groupoids with at most three objects:
We introduce the notions of Cartan schemes and root systems and classify those which admit a finite root system and which have up to three objects. Although the classification up to three objects is not used in the remaining papers on the classification, this paper introduces most of the structures needed later on as well as some general propositions about the decomposition of Cartan schemes.

Weyl groupoids of rank two and continued fractions:
A theory of coverings of Cartan schemes is developed (in arbitrary rank). In the

remaining papers on the classification we concentrate on the "universal" or "simply connected" case. In terms of coverings, all other finite Weyl groupoids are obtained as quotients. They correspond to the subgroups of the *automorphism group* of the smallest quotient. Further, we establish a connection between "universal finite Weyl groupoids" of rank two and certain continued fractions: In the Weyl groupoid, a chain of morphisms $\ldots s_i s_j s_i^a$ has finite order if the corresponding continued fraction is not convergent. This implies the existence of a Cartan entry 1 and eventually yields an algorithm to enumerate all "finite Weyl groupoids of rank two" or more precisely, the connected Cartan schemes of rank two for which the real roots form a finite root system.

Reflection groupoids of rank two and Cluster algebras of type A:
We refine the above algorithm for universal finite Weyl groupoids of rank two to an explicit classification: There is a natural bijection to the set of triangulations of convex polygons by non-intersecting diagonals. Further we introduce a generalization that yields the cluster algebras of type A.

Finite Weyl groupoids of rank three:
This is perhaps the deepest result of the series. Using several theorems and a computer proof we obtain a complete classification of "finite Weyl groupoids" of rank three: The algorithm is based on the theorem that any positive root is either simple or a sum of two positive roots. It terminates thanks to a theorem giving a bound for the entries of the Cartan matrices. Finally, one obtains an efficient implementation by using a theorem about a certain "weak convexity". One sees that the classification result is highly non-trivial by looking at the output which appears to be very heterogeneous: There are 55 such groupoids (up to equivalences) and most of them are "sporadic".

Minimal fields of definition for simplicial arrangements in the real projective plane:
This is a sideline to the classification series. A first step towards a classification of simplicial arrangements coming from "Cartan schemes" with Cartan matrices with entries which are not necessarily integers is the determination of the "smallest" extensions of $\mathbb{Q}$ which are needed to construct arrangements having the "same" incidence structures as the known simplicial arrangements of rank three.

Crystallographic arrangements: Weyl groupoids and simplicial arrangements:
In this paper we explain the relation between the fact that an arrangement is simplicial and the fact that the base changes from a chamber to an adjacent chamber is a reflection, as well as the implications of a certain integrality axiom (Chapt. 7, Def. 2.3 (I)). We call simplicial arrangements satisfying this axiom *crystallographic* and construct a "finite universal Weyl groupoid" associated to such an arrangement. Together with [**33**], this paper yields a one-to-one correspondence between the "finite

Weyl groupoids" classified in the present series of papers and the large class of crystallographic arrangements.

Finite Weyl groupoids:
We achieve the ultimate goal for the theory of "finite Weyl groupoids": a complete classification. As for the classification of rank three, this paper is divided into a theoretical part and a computational part. The theory (analysis of Dynkin diagrams) is needed to handle the infinite series. A computer proof yields the 74 remaining "sporadic universal finite Weyl groupoids", it is based on the classification of rank three. In addition to the enumeration of all root systems we also list many invariants as for instance the automorphism groups of all finite Weyl groupoids.

Acknowledgments

I am very grateful to István Heckenberger both for introducing me to the theory of Weyl groupoids and for his large part in developing the subject which I find so fascinating. Special thanks are also due to Gunter Malle for his steady support and for his confidence in me.

CHAPTER 2

Weyl groupoids with at most three objects

M. CUNTZ AND I. HECKENBERGER
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We adapt the generalization of root systems of the second author and H. Yamane to the terminology of category theory. We introduce Cartan schemes, associated root systems and Weyl groupoids. After some preliminary general results, we completely classify all finite Weyl groupoids with at most three objects. The classification yields that there exist infinitely many "standard", but only 9 "exceptional" cases.

1. Introduction

In the last decades, root systems and their generalizations have continuously led to many remarkable new results. In most cases the motivation was to understand the structure of some generalization of Lie algebras, for example Kac-Moody algebras or Lie superalgebras.

Following the plan of Andruskiewitsch and Schneider for a classification of pointed Hopf algebras [**3**],[**4**], a new type of root systems emerged [**25**]. These are fundamental invariants of Nichols algebras of diagonal type, and are crucial for the full classification of finite-dimensional Nichols algebras of diagonal type [**30**]. In [**34**] an axiomatic definition of a generalization of root systems was introduced, based on the main properties of the root systems of Nichols algebras of diagonal type. The class of these root systems includes properly the reduced root systems in the sense of Bourbaki [**11**] and the root systems of Kac-Moody algebras [**37**], but contains many exceptional cases. The reduced root systems of simple Lie superalgebras also fit naturally into the new framework [**34**]. The results in [**2**] indicate that a large class of Nichols algebras of non-diagonal type, such as those of finite group type, presumably admits a root system satisfying the axioms given in [**34**].

The main aspect of the novel root systems is, that one starts with a family of Cartan matrices instead of a single Cartan matrix. Consequently, the symmetry object is not a group but a groupoid, the so called *Weyl groupoid*.

There have also been many efforts to find "root systems" for complex reflection groups [**13**], the main achievement being the cyclotomic Hecke algebras. In a connected groupoid, if one fixes an object a, then the morphisms from a to a form a group $\mathrm{Hom}(a)$ which is isomorphic for all choices of a. It is easy to see that any finite group appears as $\mathrm{Hom}(a)$ for some finite Weyl groupoid. So in particular, it will be interesting to investigate the root systems for which $\mathrm{Hom}(a)$ is a complex reflection group.

Matsumoto's theorem holds for Coxeter groupoids [**34**] and hence there will be many nice properties of Coxeter groups which may be translated to this new situation. However, there are some important differences, for example the exchange condition only holds in a weak version.

In this article, we focus on various different aspects of Weyl groupoids. To stress the naturality of the construction, we introduce new concepts for the definitions using the language of category theory. A Weyl groupoid is based on a set of Cartan matrices $\mathcal{C}$ called a *Cartan scheme*. For such a scheme $\mathcal{C}$, we define root systems of type $\mathcal{C}$ and their Weyl groupoid. We stay in the general setting and deduce many useful results about root systems, Cartan schemes and Weyl groupoids, extending the analysis in [**34**]. We discuss standard Cartan schemes and their Weyl groupoids: Regardless of the number of objects, these are defined with the help of a single Cartan matrix, and are closely related to the crystallographic Weyl groups. Then decompositions of Cartan schemes, root systems, and their Weyl groupoids are investigated and characterized. In Prop. 4.6 we prove that a finite root system for a given Cartan scheme is reducible if and only if the family of Cartan matrices is simultaneously decomposable, or equivalently, if one of the Cartan matrices of the family is decomposable.

Then we turn our attention to the case of finite root systems. The main theorems merge to the result:

THEOREM 1.1. *Let $\mathcal{W}$ be the Weyl groupoid of a finite irreducible connected root system with at most* 3 *objects. Then one of the following holds:*

(1) *$\mathcal{W}$ is standard, i.e. all Cartan matrices are equal.*
(2) *$\mathcal{W}$ is one of* 9 *exceptional Weyl groupoids.*

The main tool in our proofs is Thm. 2.6, the proof of which is given in [**34**]. It states that $\mathcal{W}$ is generated by reflections and Coxeter relations. For details on which Cartan matrices actually yield standard Weyl groupoids and a description of the exceptional cases, see the theorems 5.4, 6.1, 6.3 and 6.5. As a consequence of our classification, we conclude in Rems. 5.5 and 6.2 that there exist root systems associated to some non-standard Cartan schemes which cannot be obtained as a root system of a finite-dimensional Nichols algebra of diagonal type.

There are many open questions left. It is conceivable that there are only finitely many non-standard irreducible connected Weyl groupoids for a fixed number of objects. Notice that there are infinitely many standard irreducible connected Weyl groupoids with two objects, but that all irreducible connected Weyl groupoids with three objects have rank less or equal four.

We use the symbols $\mathbb{N}$ and $\mathbb{N}_0$ for the set of positive and nonnegative integers, respectively.

We want to thank G. Malle for providing us with Ex. 3.2, and H.-J. Schneider for many interesting discussions on the subject and for his help in looking for a good terminology.

2. Cartan schemes, root systems, and their Weyl groupoids

First the generalization of root systems given in [**34**, Def. 2] is reformulated in terms of category theory.

Let I be a non-empty finite set and $\{\alpha_i \,|\, i \in I\}$ the standard basis of $\mathbb{Z}^I$. Recall from [**37**, §1.1] that a generalized Cartan matrix $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \le 0$ for all $i, j, k \in I$ with $j \ne k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

DEFINITION 2.1. Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

We say that $\mathcal{C}$ is *connected*, if the group $\langle \rho_i \,|\, i \in I\rangle \subset \mathrm{Aut}(A)$ acts transitively on A, that is, if for all $a, b \in A$ with $a \ne b$ there exist $n \in \mathbb{N}$, $a_1, a_2, \dots, a_n \in A$, and $i_1, i_2, \dots, i_{n-1} \in I$ such that

$$a_1 = a, \quad a_n = b, \quad a_{j+1} = \rho_{i_j}(a_j) \quad \text{for all } j = 1, \dots, n-1.$$

Two Cartan schemes $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I', A', (\rho'_i)_{i\in I'}, (C'^a)_{a\in A'})$ are termed *equivalent*, if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1(\rho_i(a)) = \rho'_{\varphi_0(i)}(\varphi_1(a)), \qquad c^{\varphi_1(a)}_{\varphi_0(i)\varphi_0(j)} = c^a_{ij}$$

for all $i, j \in I$ and $a \in A$.

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma_i^a \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma_i^a(\alpha_j) = \alpha_j - c_{ij}^a \alpha_i \qquad \text{for all } j \in I. \tag{2.1}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are generated by the maps $\sigma_i^a \in \mathrm{Hom}(a, \rho_i(a))$ with $i \in I$, $a \in A$. Formally, for $a, b \in A$ the set $\mathrm{Hom}(a, b)$ consists of the triples (b, f, a), where

$$f = \sigma_{i_n}^{\rho_{i_{n-1}}\cdots\rho_{i_1}(a)} \cdots \sigma_{i_2}^{\rho_{i_1}(a)} \sigma_{i_1}^a$$

and $b = \rho_{i_n} \cdots \rho_{i_2}\rho_{i_1}(a)$ for some $n \in \mathbb{N}_0$ and $i_1, \ldots, i_n \in I$. The composition is induced by the group structure of $\mathrm{Aut}(\mathbb{Z}^I)$:

$$(a_3, f_2, a_2) \circ (a_2, f_1, a_1) = (a_3, f_2 f_1, a_1)$$

for all $(a_3, f_2, a_2), (a_2, f_1, a_1) \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$. By abuse of notation we will write $f \in \mathrm{Hom}(a, b)$ instead of $(b, f, a) \in \mathrm{Hom}(a, b)$.

The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$.

The Weyl groupoid $\mathcal{W}(\mathcal{C})$ of a Cartan scheme $\mathcal{C}$ is a groupoid. Indeed, (M1) implies that $\sigma_i^a \in \mathrm{Aut}(\mathbb{Z}^I)$ is a reflection for all $i \in I$ and $a \in A$, and hence the inverse of $\sigma_i^a \in \mathrm{Hom}(a, \rho_i(a))$ is $\sigma_i^{\rho_i(a)} \in \mathrm{Hom}(\rho_i(a), a)$ by (C1) and (C2). Therefore each morphism of $\mathcal{W}(\mathcal{C})$ is an isomorphism.

If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes, then $\mathcal{W}(\mathcal{C})$ and $\mathcal{W}(\mathcal{C}')$ are isomorphic groupoids.

Recall that a groupoid G is called *connected*, if for each $a, b \in \mathrm{Ob}(G)$ the class $\mathrm{Hom}(a, b)$ is non-empty. Hence $\mathcal{W}(\mathcal{C})$ is a connected groupoid if and only if $\mathcal{C}$ is a connected Cartan scheme.

DEFINITION 2.2. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subset \mathbb{Z}^I$, and define $m_{i,j}^a = |R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R_+^a \cup -R_+^a$, where $R_+^a = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma_i^a(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m_{i,j}^a$ is finite, then $(\rho_i\rho_j)^{m_{i,j}^a}(a) = a$.

If $\mathcal{R}$ is a root system of type $\mathcal{C}$, then we say that $\mathcal{W}(\mathcal{R}) = \mathcal{W}(\mathcal{C})$ is the *Weyl groupoid of* $\mathcal{R}$. Further, $\mathcal{R}$ is called *connected*, if $\mathcal{C}$ is a connected Cartan scheme. If $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ is a root system of type $\mathcal{C}$ and $\mathcal{R}' = \mathcal{R}'(\mathcal{C}', ({R'}^a_{a\in A'}))$ is a root system of type $\mathcal{C}'$, then we say that $\mathcal{R}$ and $\mathcal{R}'$ are *equivalent*, if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes given by maps $\varphi_0 : I \to I'$, $\varphi_1 : A \to A'$ as in Def. 2.1, and if the map $\varphi_0^* : \mathbb{Z}^I \to \mathbb{Z}^{I'}$ given by $\varphi_0^*(\alpha_i) = \alpha_{\varphi_0(i)}$ satisfies $\varphi_0^*(R^a) = {R'}^{\varphi_1(a)}$ for all $a \in A$.

REMARK 2.3. (1) Reduced root systems with a fixed basis, see [**11**, Ch. VI,§1.5], are examples of root systems of type $\mathcal{C}$ in the following way. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme, such that $A = \{a\}$ has only one element, and C^a is a Cartan matrix of finite type. Then $\rho_i = \mathrm{id}$ for all $i \in I$. Let R^a be the reduced root system associated to C^a, where the basis $\{\alpha_i \,|\, i \in I\}$ of $\mathbb{Z}^I$ is identified with a basis of R^a. Then $\mathcal{R} = \mathcal{R}(\mathcal{C}, R^a)$ is a root system of type $\mathcal{C}$.

(2) In root systems 0 is never a root. The same holds for the sets R^a in root systems of type $\mathcal{C}$. Indeed, if $0 \in R^a$, then $0 \in R^a \cap \mathbb{Z}\alpha_i$ for all $i \in I$, and since I is non-empty, this is a contradiction to (R2).

(3) Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. For $i, j \in I$ with $i \neq j$, (C1) and (R4) imply that the relations

$$(\rho_j \rho_i)^{m^a_{i,j}}(a) = a \tag{2.2}$$

hold for all $a \in A$. Further,

$$(\sigma_i \sigma_j)^{m^a_{i,j}} 1_a = (\sigma_j \sigma_i)^{m^a_{i,j}} 1_a = 1_a \tag{2.3}$$

for all $a \in A$ and $i, j \in I$ with $i \neq j$, see Thm. 2.6 below. Here 1_a is the identity of the object a, and we use the convention that upper indices referring to objects are neglected if they are uniquely determined by the context.

REMARK 2.4. In [**34**, Def. 2] it is assumed that a root system $\mathcal{R}$ of type $\mathcal{C}$ is *connected*. We omit this axiom to have a definition which is compatible with passing to restrictions, see Def. 4.1. Note that a restriction of $\mathcal{R}$ is generally not connected, even if $\mathcal{R}$ is connected.

The following lemma states that in root systems of type $\mathcal{C}$ some axioms are redundant.

LEMMA 2.5. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a quadruple, where A is a non-empty set, $\rho_i : A \to A$ is a map for all $i \in I$, and $C^a = (c^a_{ij})_{i,j\in I} \in \mathbb{Z}^{I\times I}$ for all $a \in A$, such that $c^a_{ii} = 2$ for all $i \in I$, $a \in A$, and that (C1) holds. For all $a \in A$ let $R^a \subset \mathbb{Z}^I$ satisfying (R1)–(R4). Then $\mathcal{C}$ is a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ is a root system of type $\mathcal{C}$.*

PROOF. We have to prove that (M1), (C2), and (M2) hold for C^a for all $a \in A$. Let $a \in A$ and $j, k \in I$ with $j \neq k$. Then $\alpha_k \in R^a$ by (R2), hence $\sigma_j^a(\alpha_k) \in R_+^{\rho_j(a)}$ by (R1) and (R3). Therefore $c_{jk}^a \leq 0$ by Eq. (2.1). This proves (M1) for C^a.

Let now $a \in A$ and $i, j \in I$. Then

$$\begin{aligned}\sigma_i^{\rho_i(a)}\sigma_i^a(\alpha_j) =&\sigma_i^{\rho_i(a)}(\alpha_j - c_{ij}^a\alpha_i)\\ =&\alpha_j + (c_{ij}^a - c_{ij}^{\rho_i(a)})\alpha_i \in R_+^{\rho_i^2(a)} = R_+^a\end{aligned}$$

by Eq. (2.1), (R1)–(R3), and (C1), and hence $c_{ij}^a \geq c_{ij}^{\rho_i(a)}$. Replacing a by $\rho_i(a)$, we obtain in the same way that $c_{ij}^a \leq c_{ij}^{\rho_i(a)}$. Hence (C2) holds. Assume now that $c_{ij}^a = 0$. Then $i \neq j$ by (M1). Further, Eq. (2.1), (C2), and relation $c_{ij}^a = 0$ imply that

$$\sigma_i^a\sigma_j^{\rho_j(a)}(\alpha_i) =\sigma_i^a(\alpha_i - c_{ji}^{\rho_j(a)}\alpha_j) = -\alpha_i - c_{ji}^a\alpha_j,$$

and $\sigma_i^a\sigma_j^{\rho_j(a)}(\alpha_i) \in -R_+^{\rho_i(a)}$ by (R1)–(R3). Since $c_{ji}^a \leq 0$ by (M1), this gives that $c_{ji}^a = 0$, hence (M2) is proven. □

Recall the convention in Rem. 2.3(3). The Weyl groupoid of a root system of type $\mathcal{C}$ is a generalization of the notion of a Weyl group, as the following theorem shows.

THEOREM 2.6. [**34**, Thm. 1] *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Let $\mathcal{W}$ be the abstract groupoid with $\mathrm{Ob}(\mathcal{W}) = A$ such that $\mathrm{Hom}(\mathcal{W})$ is generated by abstract morphisms $s_i^a \in \mathrm{Hom}(a, \rho_i(a))$, where $i \in I$ and $a \in A$, satisfying the relations*

$$s_i s_i 1_a = 1_a, \quad (s_j s_k)^{m_{j,k}^a} 1_a = 1_a, \qquad a \in A,\ i, j, k \in I,\ j \neq k.$$

Here 1_a is the identity of the object a, and $(s_j s_k)^\infty 1_a$ is understood to be 1_a. The functor $\mathcal{W} \to \mathcal{W}(\mathcal{R})$, which is the identity on the objects, and on the set of morphisms is given by $s_i^a \mapsto \sigma_i^a$ for all $i \in I$, $a \in A$, is an isomorphism of groupoids.

One says that $\mathcal{W}(\mathcal{R})$ is a *Coxeter groupoid*. Thus it makes sense to speak about the *length*

$$\ell(\omega) = \min\{m \in \mathbb{N}_0 \mid \omega = \sigma_{i_1} \cdots \sigma_{i_m} 1_a, i_1, \dots, i_m \in I\} \tag{2.4}$$

of a morphism $\omega \in \mathrm{Hom}(a, b) \subset \mathrm{Hom}(\mathcal{W}(\mathcal{R}))$. The most essential difference between Coxeter groupoids and Coxeter groups is the presence of several objects in the former.

DEFINITION 2.7. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. Let Γ be a nondirected graph, such that each edge is labelled by an element of I, and any two edges between two fixed vertices are labelled differently. Assume that there is a bijection φ from A to the set of vertices of Γ, and two vertices $\varphi(a)$, $\varphi(b)$, where $a, b \in A$, are connected by an edge labelled by $i \in I$ if and only if $a \neq b$ and $\rho_i(a) = b$.

The graph Γ is called the *object change diagram* of $\mathcal{C}$. If $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ is a root system of type $\mathcal{C}$, then we also say that Γ is the object change diagram of $\mathcal{R}$.

The object change diagram of a reduced root system is a single vertex without any edges. Other examples will appear in later sections. Note that the object change diagram of a Cartan scheme $\mathcal{C}$ is connected as a graph if and only if the Cartan scheme $\mathcal{C}$ is connected, or equivalently, if the Weyl groupoid $\mathcal{W}(\mathcal{C})$ is a connected groupoid.

DEFINITION 2.8. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. For all $a \in A$ let $(R^a)^{\mathrm{re}} = \{\omega(\alpha_i) \,|\, \omega \in \mathrm{Hom}(b, a), b \in A, i \in I\}$, and call $(R^a)^{\mathrm{re}}$ the *set of real roots* of a.

Note that by (R2) and (R3) we have $(R^a)^{\mathrm{re}} \subset R^a$ for all $a \in A$. The sets of real roots are interesting for various reasons, one of them is the following.

PROPOSITION 2.9. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme, and let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ be a root system of type $\mathcal{C}$. Then $\mathcal{R}^{\mathrm{re}} = \mathcal{R}^{\mathrm{re}}(\mathcal{C}, ((R^a)^{\mathrm{re}})_{a\in A})$ is a root system of type $\mathcal{C}$, and $\mathcal{W}(\mathcal{R}^{\mathrm{re}}) = \mathcal{W}(\mathcal{R})$.*

PROOF. Since $\mathcal{C}$ is a Cartan scheme, it suffices to show that the sets $(R^a)^{\mathrm{re}}$ satisfy axioms (R1)–(R4) for all $a \in A$. Let $a \in A$. Since $\omega\sigma_i^{\rho_i(b)}(\alpha_i) = -\omega(\alpha_i)$ for all $i \in I$, $b \in A$, and $\omega \in \mathrm{Hom}(b, a)$, we obtain that $(R^a)^{\mathrm{re}} = -(R^a)^{\mathrm{re}}$. Let $(R^a)^{\mathrm{re}}_+ = (R^a)^{\mathrm{re}} \cap \mathbb{N}_0^I$ and $(R^a)^{\mathrm{re}}_- = (R^a)^{\mathrm{re}} \cap -\mathbb{N}_0^I$. Then $(R^a)^{\mathrm{re}}_- = -(-(R^a)^{\mathrm{re}} \cap \mathbb{N}_0^I) = -(R^a)^{\mathrm{re}}_+$, and hence (R1) implies that $(R^a)^{\mathrm{re}} = (R^a)^{\mathrm{re}}_+ \cup -(R^a)^{\mathrm{re}}_+$. Since $\alpha_i \in (R^a)^{\mathrm{re}}$, $(R^a)^{\mathrm{re}} \subset R^a$, and $(R^a)^{\mathrm{re}} = -(R^a)^{\mathrm{re}}$ by (R1), Axiom (R2) implies that $(R^a)^{\mathrm{re}} \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$. Axiom (R3) holds for $\mathcal{R}^{\mathrm{re}}$ by definition. Finally, if $m^a_{i,j}$ is finite for an $a \in A$ and elements $i, j \in I$ with $i \neq j$, then

$$R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) = (R^a)^{\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)$$

by [**34**, Lemma 4]. Thus (R4) holds for $\mathcal{R}^{\mathrm{re}}$, since it holds for $\mathcal{R}$. The equation $\mathcal{W}(\mathcal{R}^{\mathrm{re}}) = \mathcal{W}(\mathcal{R})$ follows since $\mathcal{W}(\mathcal{R}^{\mathrm{re}}) = \mathcal{W}(\mathcal{C}) = \mathcal{W}(\mathcal{R})$ by definition. □

Now we discuss the finiteness of root systems of type $\mathcal{C}$.

DEFINITION 2.10. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. We say that $\mathcal{R}$ is *finite* if R^a is finite for all $a \in A$.

The finiteness of $\mathcal{R}$ does not mean that $\mathcal{W}(\mathcal{R})$ is finite, since A may be infinite. But the following holds.

LEMMA 2.11. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Then the following are equivalent.*

(1) *$\mathcal{R}$ is finite.*
(2) *R^a is finite for at least one $a \in A$.*
(3) *$\mathcal{R}^{\mathrm{re}}$ is finite.*
(4) *$\mathcal{W}(\mathcal{R})$ is finite.*

PROOF. (1)⇒(2) is trivial.

(2)⇒(1). Assume that $a \in A$ such that R^a is finite. Since $\mathcal{R}$ is connected, for each $b \in A$ there exists $\omega \in \mathrm{Hom}(a, b)$. Then $R^b = \omega(R^a)$ by (R3), and hence R^b is finite for all $b \in A$.

(1)⇒(4). Let $a, b \in A$. Since R^b is finite, R^a contains the standard basis of $\mathbb{Z}^I$ by (R2), and since $\omega(R^a) \subset R^b$ for all $\omega \in \mathrm{Hom}(a, b)$, the set $\mathrm{Hom}(a, b)$ is finite. Assume now that A is infinite, and let $a \in A$. The finiteness of R^a implies that there exist $b, c \in A$ and $f \in \mathrm{Hom}(a, b)$, $g \in \mathrm{Hom}(a, c)$ with $f \neq g$ and

$$Y := \{\alpha \in R^a_+ \,|\, f(\alpha) \in R^b_+\} = \{\beta \in R^a_+ \,|\, g(\beta) \in R^c_+\}.$$

Then $fg^{-1} \in \mathrm{Hom}(c, b)$ and

$$\begin{aligned} fg^{-1}(g(Y)) =& f(Y) \subset R^b_+,\\ fg^{-1}(R^c_+ \setminus g(Y)) =& f(-(R^a_+ \setminus Y)) \subset R^b_+ \end{aligned}$$

by (R1) and (R3). Therefore $fg^{-1}(R^c_+) \subset R^b_+$, and hence $b = c$ and $fg^{-1} = 1_b$ by [**34**, Lemma 8(iii)]. This is a contradiction to $f \neq g$, and hence A is finite. This proves (1)⇒(4).

(4)⇒(1). We prove that if $\mathcal{R}$ is infinite, then $\mathcal{W}(\mathcal{R})$ is infinite. For this we show by induction on m that for all $m \in \mathbb{N}$ there exist $a, b \in A$ and $\omega \in \mathrm{Hom}(a, b)$ such that

$$|\{\alpha \in R^a_+ \,|\, \omega(\alpha) \in -R^b_+\}| = m. \tag{2.5}$$

The latter holds for $m = 1$, since $\omega = \sigma_i^a$ fulfills Eq. (2.5) for all $a \in A$, $i \in I$, and $b = \rho_i(a)$. Suppose now that $m \in \mathbb{N}$, $a, b \in A$, and $\omega \in \mathrm{Hom}(a, b)$ such that Eq. (2.5) holds. Since $|R^a| = \infty$, there exists $\alpha \in R^a_+$ with $\omega(\alpha) \in R^b_+$. Since ω is linear, there exists $i \in I$ such that $\omega(\alpha_i) \in R^b_+$. Then $\ell(\omega\sigma_i^{\rho_i(a)}) = m+1$ by [**34**, Cor. 3], and hence

$$\left|\{\alpha \in R^{\rho_i(a)}_+ \,|\, \omega\sigma_i^{\rho_i(a)}(\alpha) \in -R^b_+\}\right| = m + 1$$

by [**34**, Lemma 8(iii)]. Thus the induction step is proved.

Finally, the equivalence of (3) and (4) follows from the equivalence of (1) and (4) and the equation $\mathcal{W}(\mathcal{R}^{\mathrm{re}}) = \mathcal{W}(\mathcal{R})$, see Prop. 2.9. □

Now we prove that if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then all roots are real, that is, $\mathcal{R}$ is uniquely determined by $\mathcal{C}$.

PROPOSITION 2.12. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Let $a \in A$, $m \in \mathbb{N}_0$, and $i_1, \dots, i_m \in I$ such that $\omega = 1_a\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}$ and $\ell(\omega) = m$. Then the elements*

$$\beta_n = 1_a\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{n-1}}(\alpha_{i_n}) \in R^a_+,$$

where $n \in \{1, 2, \dots, m\}$ (and $\beta_1 = \alpha_{i_1}$), are pairwise different. In particular, if $\mathcal{R}$ is finite and $\omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{R}))$ is a longest element, see [**34**, Cor. 5]*, then*

$$\{\beta_n \mid 1 \le n \le \ell(\omega) = |R^a|/2\} = R^a_+.$$

PROOF. For all $n \in \mathbb{N}_0$ with $n \le m$ let $a_n = \rho_{i_n}\cdots\rho_{i_2}\rho_{i_1}(a)$. Since $\ell(\sigma^{a_1}_{i_1}\sigma^{a_2}_{i_2}\cdots\sigma^{a_n}_{i_n}) = n$ for all $n \in \{1, 2, \dots, m\}$, one has $\beta_n \in R^a_+$ for these n by [**34**, Cor. 3]. By the same argument one has for all $k, n \in \mathbb{N}$ with $k < n \le m$ the relation

$$\sigma^{a_{k+1}}_{i_{k+1}}\sigma^{a_{k+2}}_{i_{k+2}}\cdots\sigma^{a_{n-1}}_{i_{n-1}}(\alpha_{i_n}) \in R^{a_k}_+.$$

Thus $\sigma_{i_k}\sigma_{i_{k+1}}\cdots\sigma_{i_{n-1}}1_{a_{n-1}}(\alpha_{i_n}) \ne \alpha_{i_k}$, and hence $\beta_n \ne \beta_k$ for all $k, n \in \mathbb{N}$ with $k < n \le m$. □

For any groupoid G and any $a \in \mathrm{Ob}(G)$ let $\mathrm{Hom}(a) = \mathrm{Hom}(a, a) \subset \mathrm{Hom}(G)$. Then $\mathrm{Hom}(a)$ is a subgroup of G, which depends on a. However, the following is true.

PROPOSITION 2.13. *Let G be a connected groupoid and $a, b \in \mathrm{Ob}(G)$. Then $\mathrm{Hom}(a)$ and $\mathrm{Hom}(b)$ are isomorphic groups.*

PROOF. Choose $X_b \in \mathrm{Hom}(a, b)$. This exists since G is connected. Then the map

$$\phi_{a,b} : \mathrm{Hom}(a) \to \mathrm{Hom}(b), \quad g \mapsto X_b g X_b^{-1}$$

is a group homomorphism with inverse given by $\phi^{-1}_{a,b}(g) = X_b^{-1} g X_b$. □

The map $\phi_{a,b}$ in the previous proof is a piece of a more general structure. Namely, let G be a connected groupoid, $a \in \mathrm{Ob}(G)$, and for each $b \in \mathrm{Ob}(G)$ let $X_b \in \mathrm{Hom}(a, b)$ be a fixed morphism. Then the assignment $F_{a,X} : G \to \mathrm{Hom}(a)$,

$$\begin{aligned} F_{a,X}(b) &= a && \text{for all } b \in \mathrm{Ob}(G), \\ F_{a,X}(g) &= X_c^{-1} g X_b && \text{for all } g \in \mathrm{Hom}(b, c), \end{aligned} \tag{2.6}$$

defines a fully faithful functor. In fact, G is as a groupoid isomorphic to the transformation groupoid $H := \mathrm{Hom}(a) \times \mathrm{Ob}(G)$ given by $\mathrm{Ob}(H) = \mathrm{Ob}(G)$, $\mathrm{Hom}(H) = \mathrm{Ob}(G) \times \mathrm{Hom}(a) \times \mathrm{Ob}(G)$ with composition

$$(b, g, c)(b', g', c') = \begin{cases} \text{not defined} & \text{if } c \ne b', \\ (b, gg', c') & \text{if } c = b'. \end{cases}$$

The isomorphism $G \to H$ is given by $g \mapsto (c, F_{a,X}(g), b)$ for $g \in \mathrm{Hom}(b,c)$. In particular, G is uniquely determined by the cardinality of $\mathrm{Ob}(G)$ and by $\mathrm{Hom}(a)$ for any $a \in \mathrm{Ob}(G)$.

If a connected groupoid G is presented by generators and relations, then for any $a \in \mathrm{Ob}(G)$ the group $\mathrm{Hom}(a)$ also can be presented by generators and relations. To do so, let $F_{a,X} : G \to \mathrm{Hom}(a)$ be the functor defined above. The following proposition then follows from the discussion above.

PROPOSITION 2.14. *Let G be a connected groupoid and let $a \in \mathrm{Ob}(G)$. Suppose that J, K are index sets and $\mathrm{Hom}(G)$ is generated by $s_j \in \mathrm{Hom}(a_j, b_j)$, where $j \in J$, and relations $r_k = 1_{c_k} \in \mathrm{Hom}(c_k)$, where $k \in K$. Then $\mathrm{Hom}(a)$ is generated by $F_{a,X}(s_j)$, where $j \in J$, and relations $F_{a,X}(r_k) = 1$, where $k \in K$ and 1 is the neutral element of $\mathrm{Hom}(a)$.*

3. Standard Cartan schemes and their Weyl groupoids

In this section we study root systems of type $\mathcal{C}$, where the Cartan matrices are identical for all objects. The structure of more general root systems of type $\mathcal{C}$ seems to be much more complicated, as the classification results in the next sections show.

DEFINITION 3.1. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme such that $C^a = C^b$ for all $a, b \in A$. Then we say that $\mathcal{C}$ is *standard*, and that the Weyl groupoid $\mathcal{W}(\mathcal{C})$ is standard. If $\mathcal{R}$ is a root system of type $\mathcal{C}$, then we say that $\mathcal{R}$ is standard, if $\mathcal{C}$ is a standard Cartan scheme.

The terminology stems from [**1**, Sect. 3.3], where related more concrete groupoids are studied.

The standard Cartan schemes in the next example show that the class of root systems of type $\mathcal{C}$, where $\mathcal{C}$ is running over all Cartan schemes, is richer than the one of finite groups.

EXAMPLE 3.2. Let H be a finite group. Then there exists a connected Cartan scheme $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and a finite root system $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ of type $\mathcal{C}$, such that $\mathrm{Hom}(a) \cong H$ for all $a \in A$. Indeed, H can be considered as a subgroup of a symmetric group $\mathbb{S}_{n+1}$. Let A be the set of left cosets gH, where $g \in \mathbb{S}_{n+1}$, and let $I = \{1, 2, \ldots, n\}$. For all $gH \in A$ and $i \in I$ let $\rho_i(a) = (i, i+1)gH$, where $(i, i+1)$ is the transposition of i and $i+1$, and $C^{gH} = (c_{ij}^{gH})_{i,j\in I}$ with

$$c_{ij}^{gH} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ is a standard Cartan scheme. For all $a \in A$ let $R^a = R^a_+ \cup -R^a_+$, where

$$R^a_+ := \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \,|\, 1 \le i \le j \le n\}, \tag{3.1}$$

be the set of roots associated to $\mathbb{S}_{n+1}$. Then $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ is a root system of type $\mathcal{C}$, and $\mathrm{Hom}(eH) \subset \mathrm{Hom}(\mathcal{W}(\mathcal{R}))$ is isomorphic to H.

The structure of finite connected standard root systems of type $\mathcal{C}$ is very close to the structure of reduced root systems in the sense of [**11**, Ch. VI, §1.4].

THEOREM 3.3. *Let I be a non-empty finite set, $C = (c_{ij})_{i,j\in I}$ a generalized Cartan matrix, and $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ a connected standard Cartan scheme with $C^a = C$ for all $a \in A$, and let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ be a root system of type $\mathcal{C}$.*

(1) For all $a \in A$ the set $\cup_{b\in A} \mathrm{Hom}(a, b) \subset \mathrm{Aut}(\mathbb{Z}^\theta)$ is a group, and as such it is isomorphic to the Weyl group $W(C)$ associated to the generalized Cartan matrix C.

(2) $\mathcal{R}$ is finite if and only if C is of finite type.

(3) Assume that $\mathcal{R}$ is finite. Then for all $a \in A$, R^a is the set of roots corresponding to $W(C)$, and hence independent of the choice of $a \in A$.

PROOF. Since $\mathcal{C}$ is standard, the maps $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^\theta)$ do not depend on the object $a \in A$, and generate the Weyl group $W(C) \subset \mathrm{Aut}(\mathbb{Z}^\theta)$ associated to the generalized Cartan matrix C. Let $a \in A$. Since

$$\cup_{b\in A} \mathrm{Hom}(a, b) = \{(\rho_{i_1} \cdots \rho_{i_n}(a), \sigma_{i_1} \cdots \sigma_{i_n} 1_a, a) \,|\, n \in \mathbb{N}_0,\, i_1, \ldots, i_n \in I\},$$

Thm. 2.6 implies that (1) holds.

Assume that $\mathcal{R}$ is finite. Since $\mathcal{C}$ is standard, Prop. 2.12 tells that R^a_+ is the set of positive roots corresponding to $W(C)$. This implies (3).

Now we prove (2). If C is of finite type, then $W(C)$ is finite. Since $\mathcal{C}$ is connected, A is finite by Part (1). Thus $\mathcal{W}(\mathcal{C})$ is finite by Part (1), and hence $\mathcal{R}$ is finite by Lemma 2.11.

Conversely, assume that $\mathcal{R}$ is finite. Then Lemma 2.11 implies that $\mathcal{W}(\mathcal{R})$ is finite, and hence $W(C)$ is finite by Part (1). Thus C is of finite type. □

4. Decomposition of finite root systems

In this section we study the reducibility of root systems of type $\mathcal{C}$. An analogous notion exists for root systems, see [**11**, Ch. 6, §1.2], and it is crucial for classification results.

DEFINITION 4.1. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. Let $J \subset I$ be a non-empty subset, and identify $\{\alpha_i \mid i \in J\}$ with the standard basis of $\mathbb{Z}^J$. For all $a \in A$ let $C'^a = (c^a_{ij})_{i,j\in J}$. Then $\mathcal{C}' = \mathcal{C}'(J, A, (\rho_i)_{i\in J}, (C'^a)_{a\in A})$ is a Cartan scheme, called the *restriction of* $\mathcal{C}$ to J, and will be denoted by $\mathcal{C}|_J$.

Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ be a root system of type $\mathcal{C}$. Define $R'^a = R^a \cap \sum_{i\in J} \mathbb{Z}\alpha_i$. Then $\mathcal{R}' = \mathcal{R}'(\mathcal{C}|_J, (R'^a)_{a\in A})$ is a root system of type $\mathcal{C}|_J$, and will be denoted by $\mathcal{R}|_J$.

Restrictions are helpful to decide if a root system of type $\mathcal{C}$ is standard.

REMARK 4.2. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Assume that for each pair $(i, j) \in I \times I$ with $i \neq j$ there exists a subset $J \subset I$ such that $i, j \in J$ and $\mathcal{R}|_J$ is standard. Then $\mathcal{R}$ is standard. Indeed, $\mathcal{R}$ is standard if and only if $c^a_{ij} = c^b_{ij}$ for all $a, b \in A$ and $i, j \in I$. The latter holds by assumption on the pairs $(i, j) \in I \times I$, and since $c^a_{ii} = 2$ for all $i \in I$ and $a \in A$.

DEFINITION 4.3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. Assume that $I', I'' \subset I$ are non-empty disjoint subsets such that $I = I' \cup I''$ and $c^a_{ij} = 0$ for all $i \in I'$, $j \in I''$. Then we write $\mathcal{C} = \mathcal{C}|_{I'} \oplus \mathcal{C}|_{I''}$, and say that $\mathcal{C}$ is the *direct sum of* $\mathcal{C}|_{I'}$ *and* $\mathcal{C}|_{I''}$.

Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ be a root system of type $\mathcal{C}$. Assume that

$$R^a = \Big(R^a \cap \sum_{i\in I'} \mathbb{Z}\alpha_i\Big) \cup \Big(R^a \cap \sum_{j\in I''} \mathbb{Z}\alpha_j\Big) \qquad \text{for all } a \in A.$$

Then we write $\mathcal{R} = \mathcal{R}|_{I'} \oplus \mathcal{R}|_{I''}$, and $\mathcal{R}$ is called the *direct sum of* $\mathcal{R}|_{I'}$ *and* $\mathcal{R}|_{I''}$. We also say that $\mathcal{R}$ is *reducible*. If $\mathcal{R} \neq \mathcal{R}|_{I_1} \oplus \mathcal{R}|_{I_2}$ for all non-empty disjoint subsets $I_1, I_2 \subset I$, then $\mathcal{R}$ is termed *irreducible*.

From now on let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. We are going to give criteria for the reducibility of $\mathcal{R}$.

LEMMA 4.4. *Let* $a \in A$ *and* $i, j \in I$ *with* $i \neq j$. *The following are equivalent.*

(1) $c^a_{ij} = c^a_{ji} = 0$.
(2) $R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) = \{\alpha_i, \alpha_j\}$.

(3) $m^a_{i,j} = 2$.

PROOF. Since $\alpha_i, \alpha_j \in R^a$ by (R2), (2) is equivalent to (3). Further, from (R1)–(R3) and Eq. (2.1) we conclude that (2) implies (1). Assume now that (1) holds, and let $\alpha := r_i\alpha_i + r_j\alpha_j \in R^a_+$, where $r_i, r_j \in \mathbb{N}_0$. Then (R1) and relation $\sigma^a_i(\alpha) = -r_i\alpha_i + r_j\alpha_j \in R^{\rho_i(a)}$ imply that $r_i = 0$ or $r_j = 0$. Hence $\alpha \in \{\alpha_i, \alpha_j\}$ by (R2). This proves (1)⇒(2). □

LEMMA 4.5. *Suppose that $a \in A$ and $i, j \in I$, where $i \neq j$. If $c^a_{ij} = 0$ then $c^a_{jl} = c^{\rho_i(a)}_{jl}$ for all $l \in I$.*

PROOF. Let $l \in I$. If $l = j$, then $c^a_{jl} = c^{\rho_i(a)}_{jl} = 2$, and if $l = i$, then $c^a_{jl} = c^a_{lj} = 0 = c^{\rho_i(a)}_{lj} = c^{\rho_i(a)}_{jl}$ by (M2) and (C2). Assume now that $l \in I \setminus \{i, j\}$. Then $\sigma^{\rho_j(a)}_i \sigma^a_j(\alpha_l) = \sigma^{\rho_i(a)}_j \sigma^a_i(\alpha_l)$ by Thm. 2.6. Explicit calculation gives

$$\sigma^{\rho_j(a)}_i \sigma^a_j(\alpha_l) = \sigma^{\rho_j(a)}_i(\alpha_l - c^a_{jl}\alpha_j) = \alpha_l - c^{\rho_j(a)}_{il}\alpha_i - c^a_{jl}\alpha_j,$$

and similarly $\sigma^{\rho_i(a)}_j \sigma^a_i(\alpha_l) = \alpha_l - c^{\rho_i(a)}_{jl}\alpha_j - c^a_{il}\alpha_i$. Comparing the coefficients of α_j gives the claim of the lemma. □

PROPOSITION 4.6. *Let $I' \subset I$ be a subset of I, and let $I'' = I \setminus I'$. Assume that $I', I'' \neq \emptyset$. The following are equivalent.*

(1) *There exists $a \in A$ such that $c^a_{ij} = 0$ for all $i \in I'$ and $j \in I''$.*
(2) *For all $a \in A$ and $i \in I'$, $j \in I''$ one has $c^a_{ij} = c^a_{ji} = 0$.*
(3) *For all $a \in A$ let $\hat{R}^a = (R^a \cap \sum_{i\in I'} \mathbb{Z}\alpha_{i'}) \cup (R^a \cap \sum_{i\in I''} \mathbb{Z}\alpha_{i''})$. Then $\hat{\mathcal{R}} = \hat{\mathcal{R}}(\mathcal{C}, (\hat{R}^a)_{a\in A})$ is a root system of type $\mathcal{C}$.*

If $\mathcal{R}$ is finite then (1)–(3) are equivalent to the following.

(4) *$\hat{\mathcal{R}} = \mathcal{R}$, where $\hat{\mathcal{R}}$ is as in (3).*
(5) *$\mathcal{R} = \mathcal{R}|_{I'} \oplus \mathcal{R}|_{I''}$ with respect to the permutation $\phi = \mathrm{id}$ of A.*

PROOF. The implication (2)⇒(1) is trivial.

(1)⇒(2). Let $b \in A$ and $l \in I'$, and assume that $c^b_{ij} = c^b_{ji} = 0$ for all $i \in I'$, $j \in I''$. Since $c^b_{lj} = 0$ for all $j \in I''$, Lemma 4.5 implies that $c^{\rho_l(b)}_{ji} = c^b_{ji} = 0$ for all $i \in I'$ and $j \in I''$, that is, $c^{\rho_l(b)}_{ij} = c^{\rho_l(b)}_{ji} = 0$ for all $i \in I'$, $j \in I''$. Together with the analogous argument for $l \in I''$ we obtain that $c^{\rho_l(b)}_{ij} = c^{\rho_l(b)}_{ji} = 0$ for all $i \in I'$, $j \in I''$, and $l \in I$. Thus (2) follows from (1) since $\mathcal{R}$ is connected.

(3)⇒(2). Since $\hat{R}^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) = \{\alpha_i, \alpha_j\}$ for all $i \in I'$, $j \in I''$, and $a \in A$, (2) follows from Lemma 4.4 and (3).

(2)⇒(3). Since $\mathcal{R}$ is a root system of type $\mathcal{C}$, (2) and Lemma 4.4 imply that $m^a_{i,j} = 2$ for all $a \in A$, $i \in I'$, and $j \in I''$. Thus (3) is equivalent to the fact that $\sigma^a_l(\hat{R}^a) \subset \hat{R}^{\rho_l(a)}$ for all $l \in I$ and $a \in A$. Let $\alpha \in \hat{R}^a$. Then (2) implies that

$$\sigma^a_l(\alpha) \in R^{\rho_l(a)} \cap \big(\sum_{i' \in I'} \mathbb{Z}\alpha_{i'} \cup \sum_{i'' \in I''} \mathbb{Z}\alpha_{i''}\big) = \hat{R}^{\rho_l(a)},$$

and hence (3) holds.

Assume now that $\mathcal{R}$ is finite. Then, by Prop. 2.12, (2) implies that $\hat{R}^a_+ = R^a_+$ for all $a \in A$, that is, (4) holds. Obviously, (4) implies (3), and (4) is also equivalent to (5), hence the proposition is proven. □

For the equivalence $(3) \Leftrightarrow (4)$ in Prop. 4.6 the finiteness assumption on $\mathcal{R}$ is necessary, as the following example shows.

EXAMPLE 4.7. Let $I = \{1, 2, 3, 4\}$, $A = \{a\}$, and

$$C^a = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix},$$

$$\begin{aligned} R^a_+ = \{&m\alpha_1 + (m+1)\alpha_2, (m+1)\alpha_1 + m\alpha_2, \\ &m\alpha_3 + (m+1)\alpha_4, (m+1)\alpha_3 + m\alpha_4, \mid m \in \mathbb{N}_0\} \\ \cup\{&\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}. \end{aligned}$$

Then (2) holds in Prop. 4.6 for $I' = \{1, 2\}$ and $I'' = \{3, 4\}$. However, $R^a \neq R^a|_{\{1,2\}} \cup R^a|_{\{3,4\}}$ since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \notin \hat{R}^a$, and hence $\mathcal{R}$ is irreducible.

We continue with some general statements about $\mathcal{R}$.

LEMMA 4.8. *Suppose that $a \in A$ and $i, j \in I$, where $i \neq j$. The following are equivalent.*

(1) $c^a_{ij} = c^a_{ji} = -1$.
(2) $R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) = \{\alpha_i, \alpha_i + \alpha_j, \alpha_j\}$.
(3) $m^a_{i,j} = 3$.

PROOF. To (1)⇒(2). Let $\alpha = c_1\alpha_i + c_2\alpha_j$ with $c_1, c_2 \in \mathbb{N}$, and assume that $\alpha \in R^a$. Then $\sigma^a_i(\alpha) = (c_2 - c_1)\alpha_i + c_2\alpha_j$, and hence relation $\sigma^a_i(\alpha) \in R^{\rho_i(a)}_+ \cup -R^{\rho_i(a)}_+$ tells that $c_2 \ge c_1$. By symmetry one gets $c_1 = c_2$, and hence $\sigma_i^a(\alpha) = c_2\alpha_j$. By (R2) one obtains that $c_2 = 1$.

The implication (2)⇒(3) follows from the definition of $m^a_{i,j}$. We have to prove (3)⇒(1). Assume that $m^a_{i,j} = 3$. Then $c^a_{ij}, c^a_{ji} < 0$ by Lemma 4.4. Thus, $\sigma_i^{\rho_i(a)}(\alpha_j) = \alpha_j - c^{\rho_i(a)}_{ij}\alpha_i \in R^a_+ \setminus \{\alpha_i, \alpha_j\}$, and hence $\beta_1 := \alpha_j - c^a_{ij}\alpha_i \in R^a_+ \setminus \{\alpha_i, \alpha_j\}$ by (C2). Similarly, $\beta_2 := \alpha_i - c^a_{ji}\alpha_j \in R^a_+ \setminus \{\alpha_i, \alpha_j\}$, and therefore (3) implies that $\beta_1 = \beta_2$, that is, (1) holds. □

LEMMA 4.9. *Suppose that $a \in A$ and $i, j \in I$ such that $m^a_{i,j} = 3$. Then $c^{\rho_i(a)}_{il} + c^{\rho_i(a)}_{jl} = c^{\rho_i\rho_j(a)}_{il} + c^{\rho_i\rho_j(a)}_{jl}$ for all $l \in I$.*

PROOF. If $l \in \{i, j\}$, then Lemma 4.8 implies that both sides of the claimed equation are equal to 1. Assume now that $l \in I \setminus \{i, j\}$. Then

$$\sigma_i^{\rho_j\rho_i(a)}\sigma_j^{\rho_i(a)}\sigma_i^a(\alpha_l) = \sigma_j^{\rho_i\rho_j(a)}\sigma_i^{\rho_j(a)}\sigma_j^a(\alpha_l)$$

by Thm. 2.6, that is,

$$\begin{aligned}\alpha_l - (c^{\rho_i(a)}_{jl} + c^{\rho_j\rho_i(a)}_{il})\alpha_i - (c^{\rho_i(a)}_{jl} + c^a_{il})\alpha_j \\ =\alpha_l - (c^{\rho_j(a)}_{il} + c^a_{jl})\alpha_i - (c^{\rho_j(a)}_{il} + c^{\rho_i\rho_j(a)}_{jl})\alpha_j.\end{aligned}$$

One obtains the claim of the lemma by comparing the coefficients of α_j and by using (C2). □

LEMMA 4.10. *Suppose that $\mathcal{R}$ is finite. Let $a, b \in A$ and $i, j, l \in I$ such that $i \ne j$, $\rho_i(a) = \rho_j(a) = b \ne a$, and $\rho_l(a) = a$. Then the following hold.*

(1) $c^a_{ij}c^a_{il}c^a_{jl} = 0$.
(2) *If $\rho_i\rho_l(b) \ne \rho_l(b)$ and $\rho_j\rho_l(b) \ne \rho_l(b)$ then $c^a_{ln} = c^b_{ln}$ for all $n \in I$.*

PROOF. Note that $c^b_{ji} = c^a_{ji}$ and $c^b_{jl} = c^a_{jl}$ by (C2).

Let $\tilde\sigma = \sigma_l^a\sigma_j^b\sigma_i^a \in \mathrm{Hom}(a)$. Since $\mathcal{R}$ is finite, $\tilde\sigma$ must have finite order by Lemma 2.11. Let $W = \mathrm{span}_\mathbb{Z}\{\alpha_i, \alpha_j, \alpha_l\}$. Since $\tilde\sigma(W) \subset W$, $\sigma := \tilde\sigma|_W \in \mathrm{End}(W)$ has to have finite order as well. This will yield both Claim (1) and Claim (2).

To (1). One has $\sigma(\alpha_i) \in -\alpha_i - \mathbb{N}_0\alpha_j - \mathbb{N}_0\alpha_l$. If $\sigma(\alpha_i) = -\alpha_i$, then $c^a_{li} = c^a_{il} = 0$ and hence (1) holds.

It remains to consider the case when $\sigma \ne \pm\mathrm{id}$. Then finiteness of the order of σ tells that $|\mathrm{tr}\sigma| \in \{0, 1, 2\}$. Explicit calculation gives that

$$\mathrm{tr}\sigma = -c^a_{il}c^a_{ji}c^a_{lj} + c^a_{ij}c^a_{ji} + c^a_{il}c^a_{li} + c^a_{jl}c^a_{lj} - 3. \tag{4.1}$$

Exchanging i and j further implies that

$$-2 \le -c^a_{ij}c^a_{jl}c^a_{li} + c^a_{ij}c^a_{ji} + c^a_{il}c^a_{li} + c^a_{jl}c^a_{lj} - 3 \le 2.$$

If $c^a_{ij}c^a_{il}c^a_{jl} \ne 0$, then the latter relations and (M1), (M2) imply that

$$c^a_{ij} = c^a_{ji} = c^a_{il} = c^a_{li} = c^a_{jl} = c^a_{lj} = -1.$$

In this case one has

$$\sigma = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 2 \\ -2 & 1 & 2 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} -2 & 0 & 3 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$$

with respect to the basis $\{\alpha_i, \alpha_j, \alpha_l\}$. Then $\mathrm{tr}\sigma^2 = 3$, but $\sigma^2 \ne \mathrm{id}$, and hence σ does not have finite order. This is a contradiction to the assumption that $\mathcal{R}$ is finite, and hence (1) holds.

To (2). Assume that $\rho_i\rho_l(b) \ne \rho_l(b)$ and $\rho_j\rho_l(b) \ne \rho_l(b)$. If $c^a_{il}c^a_{jl} = 0$ then Claim (2) is valid by Lemma 4.5. Thus by (1) it remains to consider the setting $c^a_{ij} = 0$, $c^a_{il}c^a_{jl} \ne 0$. In this case one has

$$\sigma = \begin{pmatrix} -1 & 0 & -c^a_{il} \\ 0 & -1 & -c^a_{jl} \\ c^a_{li} & c^a_{lj} & t-1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 - c^a_{il}c^a_{li} & -c^a_{il}c^a_{lj} & * \\ * & 1 - c^a_{jl}c^a_{lj} & * \\ * & * & t^2 - 3t + 1 \end{pmatrix}$$

with respect to the basis $\{\alpha_i, \alpha_j, \alpha_l\}$, where $t = c^a_{il}c^a_{li} + c^a_{jl}c^a_{lj}$. In the next paragraph we prove that $t \ge 4$. This implies that

$$\mathrm{tr}\sigma^2 = t^2 - 4t + 3 = (t-2)^2 - 1 \ge 3.$$

Since $\sigma^2 \ne \mathrm{id}$, part (2) of the lemma is proven.

Recall that $c^a_{il} \ne 0$. Further, relation $\rho_i\rho_l(b) \ne \rho_l(b)$ implies that $\rho_i\rho_l\rho_i(a) \ne \rho_l\rho_i\rho_l(a)$, and hence $m^a_{i,l} > 3$ by Lemma 4.4 and by (R4), that is $c^a_{il}c^a_{li} \ge 2$ by Lemma 4.8. Similarly one gets $c^a_{jl}c^a_{lj} \ge 2$. Thus the assumptions in part (2) imply that $t \ge 4$. □

Now we present another technique for the analysis of the finiteness of $\mathcal{W}(\mathcal{R})$. We will use this method in Sect. 6.

PROPOSITION 4.11. *Let $m, n \in \mathbb{N}$, and define the families $G_{m,n}$ and $H_{m,n}$ of groups by generators and relations as follows.*

$$G_{m,n} = \langle s, t \rangle / (s^2, t^m, (st^{-1}st)^n),$$
$$H_{m,n} = \langle s_1, \dots, s_m, T \rangle / (s_i^2, T^m, (s_i s_{i+1})^n, T^{-1}s_i T s_{i+1} \,|\, 1 \le i \le m),$$

where the convention $s_{m+1} = s_1$ is used in the definition of $H_{m,n}$. Then there is a group isomorphism $\varphi : G_{m,n} \to H_{m,n}$ with $\varphi(s) = s_m$, $\varphi(t) = T$. Further, $G_{m,n}$ is finite if and only if $m = 1$ or $m = 2$ or $n = 1$ or $(m, n) = (3, 2)$.

PROOF. Since $T^{-1}s_mT = s_1$ and $(s_ms_1)^n = 1$ in $H_{m,n}$, there is a unique group homomorphism $\varphi : G_{m,n} \to H_{m,n}$ with $\varphi(s) = s_m$, $\varphi(t) = T$. Further, there is a group homomorphism $\psi : H_{m,n} \to G_{m,n}$ with $\psi(s_i) = t^{-i}st^i$, $\psi(T) = t$, and the identities $\varphi\psi = \mathrm{id}$ and $\psi\varphi = \mathrm{id}$ hold. Thus φ is an isomorphism.

If $m = 1$ then $H_{m,n} \simeq \mathbb{Z}/2\mathbb{Z}$. Assume now that $m \geq 2$. Let $N := \langle s_i \,|\, 1 \leq i \leq m\rangle \subset H_{m,n}$. Since $T^{-1}s_iT = s_{i+1}$ for all $i \in \{1, 2, \ldots, m\}$, N is a normal subgroup of $H_{m,n}$, and $H_{m,n}$ is the semidirect product of N and the finite abelian group $\langle T\rangle/(T^m) \simeq \mathbb{Z}/m\mathbb{Z}$. Thus $H_{m,n}$ is finite if and only if N is finite. But

$$N = \langle s_1, \ldots, s_m\rangle/(s_i^2, (s_is_{i+1})^n \,|\, 1 \leq i \leq m)$$

is a Coxeter group. It is easy to see that N is finite if and only if $n = 1$ (and $N \simeq \mathbb{Z}/2\mathbb{Z}$) or $m = 2$ (and $N \simeq \mathbb{D}_n$, the dihedral group of order $2n$) or $(m, n) = (3, 2)$ (and $N \simeq (\mathbb{Z}/2\mathbb{Z})^3$). □

PROPOSITION 4.12. *Let $m, n, p \in \mathbb{N}$, and define the families $G_{m,n,p}$ and $H_{m,n,p}$ of groups by generators and relations as follows.*

$$\begin{aligned} G_{m,n,p} =& \langle s, u, t\rangle/(s^2, u^2, t^m, (st^{-1}ut)^n, (su)^p), \\ H_{m,n,p} =& \langle s_1, \ldots, s_m, u_1, \ldots, u_m, T\rangle/(s_i^2, u_i^2, T^m, (s_iu_{i+1})^n, (s_iu_i)^p, \\ & \qquad T^{-1}s_iTs_{i+1}, T^{-1}u_iTu_{i+1} \,|\, 1 \leq i \leq m), \end{aligned}$$

where the convention $s_{m+1} = s_1$, $u_{m+1} = u_1$ is used in the definition of $H_{m,n,p}$. Then there is a group isomorphism $\varphi : G_{m,n,p} \to H_{m,n,p}$ with $\varphi(s) = s_m$, $\varphi(u) = u_m$, $\varphi(t) = T$. Further, $G_{m,n,p}$ is finite if and only if $m = 1$ or $(n, p) = (1, 1)$ or $(m, n) = (2, 1)$ or $(m, p) = (2, 1)$ or $(m, n, p) = (3, 1, 2)$ or $(m, n, p) = (3, 2, 1)$.

PROOF. Entirely similar to the proof of Prop. 4.11. □

5. Root systems of type $\mathcal{C}$ with two objects

Let $I = \{1, 2, \ldots, \theta\}$ for some $\theta \in \mathbb{N}$, and $A = \{x, y\}$ with $x \neq y$. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Without loss of generality suppose that

$$\begin{aligned} &\rho_i(x) = y, \quad \rho_i(y) = x \qquad \text{if } 1 \leq i \leq \kappa, \\ &\rho_i(x) = x, \quad \rho_i(y) = y \qquad \text{if } \kappa + 1 \leq i \leq \theta \end{aligned} \tag{5.1}$$

for some $\kappa \in I$. In this case (C2) implies that $c^x_{ij} = c^y_{ij}$ whenever $1 \leq i \leq \kappa$ and $j \in I$.

If $\theta = 1$ then $\rho_1(x) = y$, $R^x = \{\alpha_1, -\alpha_1\}$, $R^y = \{\alpha_1, -\alpha_1\}$, and $C^x = C^y = (2)$, see also [**34**, Ex. 1].

Consider now the case $\theta = 2$ and $\kappa = 1$. Then $\mathcal{W}(\mathcal{R})$ is isomorphic to the Coxeter groupoid

$$\langle s_1^x, s_1^y, s_2^x, s_2^y \rangle / (s_1^y s_1^x, s_1^x s_1^y, (s_2^x)^2, (s_2^y)^2, (s_1^y s_2^y s_1^x s_2^x)^m)$$

for some $m \in \mathbb{N}$, see Thm. 2.6. Identify σ_i^a, where $i \in \{1,2\}$ and $a \in \{x,y\}$, with its matrix with respect to the standard basis $\{\alpha_1, \alpha_2\}$ of $\mathbb{Z}^2$. Further, note that $c_{12}^x = c_{12}^y$ by (C2). One gets

$$\begin{aligned} \sigma_1\sigma_2\sigma_1\sigma_2^x &= \begin{pmatrix} -1 & -c_{12}^y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_{21}^y & -1 \end{pmatrix} \begin{pmatrix} -1 & -c_{12}^x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_{21}^x & -1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}^x c_{21}^y - 1 & c_{12}^x \\ -c_{21}^y & -1 \end{pmatrix} \begin{pmatrix} c_{12}^x c_{21}^x - 1 & c_{12}^x \\ -c_{21}^x & -1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}^x c_{21}^x c_{12}^x c_{21}^y - c_{12}^x c_{21}^y - 2c_{12}^x c_{21}^x + 1 & c_{12}^x(c_{12}^x c_{21}^y - 2) \\ -c_{12}^x c_{21}^x c_{21}^y + c_{21}^y + c_{21}^x & 1 - c_{12}^x c_{21}^y \end{pmatrix}. \end{aligned}$$

The groupoid $\mathcal{W}(\mathcal{R})$ is finite if and only if $\sigma := \sigma_1\sigma_2\sigma_1\sigma_2^x$ has finite order. Since $\sigma \in \mathrm{Aut}(\mathbb{Z}^2)$, the latter is equivalent to the condition

$$\sigma = \pm\mathrm{id} \qquad \text{or} \qquad \mathrm{tr}\sigma \in \{-1, 0, 1\}. \tag{5.2}$$

By (M2) and (C2) one has $\sigma = \mathrm{id}$ if and only if $c_{12}^x = c_{21}^x = c_{12}^y = c_{21}^y = 0$. Further, $\sigma = -\mathrm{id}$ if and only if $c_{12}^x c_{21}^y = 2$, $c_{21}^x = c_{21}^y$, that is

$$C^x = C^y = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \qquad \text{or} \qquad C^x = C^y = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

It remains to consider the second relation in Eq. (5.2). One has

$$\mathrm{tr}\sigma = (c_{12}^x c_{21}^x - 2)(c_{12}^x c_{21}^y - 2) - 2,$$

and hence relation $-1 \le \mathrm{tr}\sigma \le 1$ is equivalent to

$$1 \le (c_{12}^x c_{21}^x - 2)(c_{12}^x c_{21}^y - 2) \le 3.$$

By (M1) and (M2) one gets $c_{12}^x = c_{21}^x = c_{12}^y = c_{21}^y = -1$ or $c_{12}^x c_{21}^x \ge 3$, $c_{12}^x c_{21}^y \ge 3$. However, the first case contradicts (R4) and Lemma 4.8. Now the following statement can be obtained easily.

PROPOSITION 5.1. *Assume that $\theta = 2$ and $\kappa = 1$. Then $\mathcal{R}$ is finite if and only if the Cartan matrices C^x, C^y satisfy, up to permutation of x and y, one of the following conditions (1),(2).*

(1) *$C^x = C^y$ is of finite type $A_1 \times A_1$, B_2, or G_2, that is, $c_{12}^x = c_{21}^x = 0$ or $c_{12}^x c_{21}^x \in \{2, 3\}$.*
(2) *$c_{12}^x = c_{12}^y = -1$, $c_{21}^x = -3$, and $c_{21}^y \in \{-4, -5\}$.*

In case (1) $R^x = R^y$ is the usual set of roots corresponding to the generalized Cartan matrix $C^x = C^y$. In case (2) one has

$$R^x_+ = \{1, 2, 12, 12^2, 12^3, 1^22^3, 1^32^4, 1^32^5\},$$
$$R^y_+ = \{1, 2, 12, 12^2, 12^3, 12^4, 1^22^3, 1^22^5\}$$

if $c^y_{21} = -4$, and

$$R^x_+ = \{1, 2, 12, 12^2, 12^3, 1^22^3, 1^32^4, 1^32^5, 1^42^5, 1^42^7, 1^52^7, 1^52^8\},$$
$$R^y_+ = \{1, 2, 12, 12^2, 12^3, 12^4, 12^5, 1^22^3, 1^22^5, 1^22^7, 1^32^7, 1^32^8\}$$

if $c^y_{21} = -5$.

In the last part of the proposition the abbreviation $1^m2^n = m\alpha_1 + n\alpha_2$ was used, where exponents 1 and factors i^0 for $i \in \{1, 2\}$ are omitted. The determination of R^a_+ is straightforward, see Prop. 2.12 or, more directly, [**34**, Lemma 6].

The case $\theta = 2$, $\kappa = 2$ is even easier than the case $\kappa = 1$. Indeed, by (C2) one has $C^x = C^y$, and hence Prop. 2.12 implies that $R^x = R^y$ is the root system of rank 2 corresponding to the Cartan matrix C^x. Thus the following holds.

PROPOSITION 5.2. *Assume that $\theta = 2$ and $\kappa = 2$. Then $\mathcal{R}$ is finite if and only if $\mathcal{R}$ is standard and $C^x = C^y$ is of finite type, that is, $c^x_{12} = c^x_{21} = 0$ or $c^x_{12}c^x_{21} \in \{1, 2, 3\}$.*

Now assume that $\theta \geq 3$. As before, suppose that $1 \leq \kappa \leq \theta$ such that Eq. (5.1) holds. First we develop some general properties.

LEMMA 5.3. *Suppose that $\mathcal{R}$ is finite and that $1 \leq \kappa \leq \theta - 2$. Then $c^x_{li}c^x_{lj} = 0$ for all $l \leq \kappa$ and $i, j > \kappa$ with $i \neq j$.*

PROOF. Let $I' = \{l, i, j\}$, and consider the restriction $\mathcal{R}|_{I'}$ of $\mathcal{R}$, see Sect. 4. Since $\rho_l(x) = y$, $\mathcal{R}|_{I'}$ is connected. Thus it suffices to consider the case $I = I'$. Let $t_i := \sigma^y_l\sigma^y_i\sigma^x_l$, $t_j := \sigma^y_l\sigma^y_j\sigma^x_l$. By Prop. 2.14, the subgroup $\mathrm{Hom}(x)$ of $\mathrm{Hom}(\mathcal{W}(\mathcal{R}))$ is generated by $s_i := \sigma^x_i$, $s_j := \sigma^x_j$, t_i, and t_j. Moreover, $\mathrm{Hom}(x)$ can be presented by the relations

$$s_i^2, s_j^2, t_i^2, t_j^2, (s_is_j)^{m^x_{i,j}}, (t_it_j)^{m^y_{i,j}}, (s_it_i)^{m^x_{l,i}/2}, (s_jt_j)^{m^x_{l,j}/2}.$$

Assume now that $c^x_{li}c^x_{lj} \neq 0$. Then $m^x_{l,i} > 2$ by Lemma 4.4, that is $m^x_{l,i}/2 \geq 2$, and similarly $m^x_{l,j}/2 \geq 2$. Thus the subgroup of $\mathrm{Hom}(x)$ generated by s_i and t_j is infinite and hence $\mathrm{Hom}(x)$ is an infinite Coxeter group. This is a contradiction to the finiteness of $\mathcal{R}$, and hence the claim of the lemma is proven. □

THEOREM 5.4. *Let $\theta \in \mathbb{N}$, $I = \{1, 2, \dots, \theta\}$, $A = \{x, y\}$, and $\kappa \in I$ as in Eq.* (5.1). *If $\mathcal{R}$ is finite and irreducible, then up to permutation of $\{x, y\}$ and I, one of the following sets of conditions hold.*

(1) *$\theta \in \mathbb{N}$, $\kappa \in \{1,2,\dots,\theta\}$, and $C^x = C^y$ is an indecomposable Cartan matrix of finite type such that $c^x_{ij}c^x_{ji} \ne 1$ for all i,j with $i \le \kappa$, $j > \kappa$.*
(2) *$\theta = 2$, $\kappa = 1$, $c^x_{12} = c^y_{12} = -1$, $c^x_{21} = -3$, $c^y_{21} \in \{-4,-5\}$.*
(3) *$\theta = 3$, $\kappa = 1$,*

$$C^x = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad C^y = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

(4) *$\theta = 3$, $\kappa = 1$,*

$$C^x = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad C^y = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Conversely, if C^x, C^y satisfy one of the conditions (1)–(4), then $\mathcal{R}$ is finite and irreducible.

PROOF. If $\theta \le 2$ then the claim of the theorem holds by Props. 5.1, 5.2.

Assume that $\theta \ge 3$. Consider first the case $\kappa = 1$, that is, $\rho_1(x) = y$, $\rho_1(y) = x$, and $\rho_i(x) = x$, $\rho_i(y) = y$ if $2 \le i \le \theta$. Using a permutation of $\{2,3,\dots,\theta\}$, by Lemma 5.3 one can assume that $c^x_{1i} = 0$ for all $i \ge 3$. Then

$$c^x_{il} = c^y_{il} \qquad \text{for all } i \ge 3 \text{ and } l \in I \tag{5.3}$$

by Lemma 4.5, and hence $c^x_{i1} = c^y_{1i} = c^y_{i1} = 0$ for all $i \ge 3$ by (M2). We obtain that $\sigma^y_1\sigma^y_i\sigma^x_1 = \sigma^x_i$ for all $i \ge 3$ by Thm. 2.6. Therefore Prop. 2.14 tells that $\mathrm{Hom}(x)$ is isomorphic to the group

$$\langle s_2, s_3, \dots, s_\theta, t\rangle / (s_i^2, t^2, (s_js_k)^{m^x_{j,k}}, (ts_l)^{m^y_{2,l}}, (s_2t)^{m^x_{1,2}/2} \mid 2 \le i \le \theta, 2 \le j < k \le \theta, 2 < l \le \theta), \tag{5.4}$$

where t corresponds to the element $\sigma^y_1\sigma^y_2\sigma^x_1$. Since $\mathcal{R}$ is irreducible and $m^x_{1,i} = 2$ for $i \ge 3$, we obtain that $m^x_{1,2} \ge 3$ by Lemma 4.4 and Prop. 4.6. Thus $\mathrm{Hom}(x)$ is a Coxeter group of rank θ. Further, again since $\mathcal{R}$ is irreducible and $c^x_{1i} = 0$ for all $i \ge 3$, we have $c^x_{2r} \ne 0$ for some $r \ge 3$. Without loss of generality assume that $c^x_{23} \ne 0$. Then $c^y_{32} = c^x_{32} \ne 0$ by Eq. (5.3) and (M2), and hence $c^y_{23} \ne 0$ and $m^x_{2,3}, m^y_{2,3} \ge 3$. Since $\mathrm{Hom}(x)$ is a finite Coxeter group, this implies that $s_2t = ts_2$, that is, $m^x_{1,2} = 4$. Then Prop. 5.1 gives that $c^x_{ij} = c^y_{ij}$ for $i,j \in \{1,2\}$, and $c^x_{12}c^x_{21} = 2$.

First assume that $\theta \ge 4$ and $c^x_{2i} \ne 0$ for some $i \ge 4$. As above, we conclude that $m^x_{2,i}, m^y_{2,i} \ge 3$, which is a contradiction to the finiteness of $\mathrm{Hom}(x)$ and the relations $m^x_{2,3}, m^y_{2,3} \ge 3$. Hence, if $\theta \ge 4$, then $c^x_{2i} = c^y_{2i} = 0$ for all $i \ge 4$. Since $\mathcal{R}$ is irreducible, there exists $r \ge 4$ with $c^x_{3r} \ne 0$. Then $m^x_{2,3} = m^y_{2,3} = 3$ by the finiteness of the group in Eq. (5.4), and hence $c^x_{2i} = c^y_{2i}$ for all i. Thus, if $\theta \ge 4$, then $C^x = C^y$ as in (1). The restriction $c^x_{1j}c^x_{j1} \neq 1$ comes from Lemma 4.8 and the relation $\rho_1\rho_j\rho_1(x) = x \neq y = \rho_j\rho_1\rho_j(x)$ for $j > 1$.

If $\theta = 3$ and $\kappa = 1$, then Eq. (5.4) tells that

$$\mathrm{Hom}(x) \simeq \langle s_2, s_3, t\rangle/(s_2^2, s_3^2, t^2, (s_2s_3)^{m^x_{2,3}}, (ts_3)^{m^y_{2,3}}, s_2ts_2t). \tag{5.5}$$

Further, $m^x_{2,3}, m^y_{2,3} \geq 3$ by the above arguments. Thus, $(m^x_{2,3}, m^y_{2,3}) \in \{(3,3), (3,4), (4,3)\}$. Since $\rho_2(a) = \rho_3(a) = a$ for $a \in \{x, y\}$, Thm. 3.3 yields that $(c^x_{23}c^x_{32}, c^y_{23}c^y_{32}) \in \{(1,1), (1,2), (2,1)\}$. Recall that $c^x_{ij} = c^y_{ij}$ for all $(i,j) \neq (2,3)$. Thus, if $c^a_{23}c^a_{32} = 1$ for $a \in \{x, y\}$, then $\mathcal{R}$ satisfies the conditions in (1). Otherwise, since $c^x_{32} = c^y_{32}$, $\mathcal{R}$ satisfies up to permutation of x and y the conditions of (3) or (4).

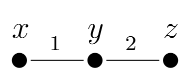
Consider now the case $\theta \geq 3$, $\kappa \geq 2$. From (C2) we obtain that $c^x_{in} = c^y_{in}$ for all $i \leq \kappa$ and $n \in I$. Further, Lemma 4.10(2) for $l > \kappa$, $i, j \leq \kappa$, $a := x$, and $b := y$ implies that $c^x_{ln} = c^y_{ln}$ for all $l > \kappa$ and $n \in I$. Thus, $\mathcal{R}$ satisfies the conditions in (1), where the restriction $c^x_{ij}c^x_{ji} \neq 1$ comes again from Lemma 4.8 and the relation $\rho_i\rho_j\rho_i(x) = x \neq y = \rho_j\rho_i\rho_j(x)$ for $i \leq \kappa$, $j > \kappa$. □

REMARK 5.5. The appearance of non-standard root systems in Thm. 5.4 is not surprizing. With an appropriate definition of the Weyl groupoid of a Nichols algebra of diagonal type, the examples in Thm. 5.4(2) can be identified with the root systems of the Nichols algebras corresponding to Row 14 and Row 17 of [**27**, Table 1], respectively. Similarly, the examples in Thm. 5.4(3),(4) can be identified with the root systems of the Nichols algebras corresponding to Row 13 and Row 18 of [**27**, Table 2], respectively.

6. Root systems of type $\mathcal{C}$ with three objects

Let $\theta \in \mathbb{N}$, $I = \{1, 2, \dots, \theta\}$, and A a set of cardinality 3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. In this case we necessarily have $\theta \geq 2$.

Let first $\theta = 2$. Then, up to enumeration of the objects and up to permutation of I, we may fix the three elements x, y, and z of A, such that the object change diagram of $\mathcal{R}$ is

(6.1)


By (C2) the generalized Cartan matrices are

$$C^x = \begin{pmatrix} 2 & -a \\ -c & 2 \end{pmatrix}, \quad C^y = \begin{pmatrix} 2 & -a \\ -d & 2 \end{pmatrix}, \quad C^z = \begin{pmatrix} 2 & -b \\ -d & 2 \end{pmatrix}, \tag{6.2}$$

where $a, b, c, d \in \mathbb{N}_0$. Further — by replacing $(1, 2)$ by $(2, 1)$ and (x, z) by (z, x), if necessary — one may assume that $a \le d$.

THEOREM 6.1. *Let $\mathcal{R}$ be a connected root system of type $\mathcal{C}$ of rank 2 with 3 objects. Assume that $A = \{x, y, z\}$ and $I = \{1, 2\}$ such that the object change diagram of $\mathcal{R}$ is as in Eq.* (6.1). *Further, let $a, b, c, d \in \mathbb{N}_0$ such that $a \le d$ and that the Cartan matrices C^x, C^y and C^z are as in Eq.* (6.2). *If $\mathcal{R}$ is finite, then (a, b, c, d) and R^x satisfy one of the following equations.*

(1) $(a, b, c, d) = (1, 1, 1, 1)$, $|R^x_+| = 3$.
(2) $(a, b, c, d) = (1, 1, 3, 3)$, $|R^x_+| = 6$.
(3) $(a, b, c, d) = (1, 2, 4, 2)$, $|R^x_+| = 6$.
(4) $(a, b, c, d) = (1, 3, 6, 2)$, $|R^x_+| = 12$.
(5) $(a, b, c, d) = (1, 4, 5, 2)$, $|R^x_+| = 12$.
(6) $(a, b, c, d) = (1, 3, 7, 2)$, $|R^x_+| = 18$.
(7) $(a, b, c, d) = (1, 5, 5, 2)$, $|R^x_+| = 18$.

Conversely, if (a, b, c, d) is one of the above 7 quadruples, then $\mathcal{R}$ is finite.

PROOF. If one of the relations $a = 0$, $b = 0$, $c = 0$, and $d = 0$ holds, then also the other three because of (M2). However, then $x = \rho_2\rho_1(y) = \rho_1\rho_2(y) = z$ by (R4) and Lemma 4.4, which is a contradiction. Thus $a, b, c, d > 0$.

Clearly, the finiteness of $\mathcal{R}$ implies that the order of the linear map $\tilde{\sigma} := \sigma_2^x\sigma_1^y\sigma_2^z\sigma_1^z\sigma_2^y\sigma_1^x \in \mathrm{Aut}(\mathbb{Z}^2)$ is finite, that is, the matrix $t = (t_{ij})_{i,j=1,2}$, where

$$\begin{aligned}
t_{11} =& -abd^2 + 2ad + bd - 1,\\
t_{12} =& a^2bd^2 - 2a^2d - 2abd + 2a + b,\\
t_{21} =& -abcd^2 + 2acd + bcd + bd^2 - c - 2d,\\
t_{22} =& a^2bcd^2 - 2a^2cd - 2abcd - abd^2 + 2ac + 2ad + bc + bd - 1,
\end{aligned}$$

has finite order. This means that either $t = \mathrm{id}$ or $t = -\mathrm{id}$ or $t_{11} + t_{22} \in \{-1, 0, 1\}$. Observe that

$$t_{11} + t_{22} - 2 = (ad - 1)(abcd - 2ac - bc - 2bd + 4), \tag{6.3}$$

$$t_{11} + t_{22} + 2 = (abd - 2a - b)(acd - c - 2d), \tag{6.4}$$

$$t_{22} = (1 - ac)t_{11} + c(-abd + a + b). \tag{6.5}$$

Thus, if $t = \mathrm{id}$, then $1 - ac - abcd + ac + bc = 1$ by Eq. (6.5), that is, $ad = 1$. Therefore $a = d = 1$, and relation $t_{21} = 0$ gives that $b + c = 2$. Since $bc \ne 0$, we obtain that $b = c = 1$. This gives solution (1).

Suppose now that $t = -\mathrm{id}$. Then $-1 + 2ac - abcd + bc = -1$ by Eq. (6.5), that is, $2a + b = abd$. Inserting this into Eq. (6.3) we get that $-4 = (ad - 1)(4 - 2bd)$. Since

$d \geq a$, we obtain the solutions $(a, b, d) \in \{(1, 2, 2), (1, 1, 3)\}$. Using equation $t_{21} = 0$ we get solutions (2) and (3).

It remains to determine all solutions with $t_{11} + t_{22} \in \{-1, 0, 1\}$. Eq. (6.3) yields that $ad \in \{2, 3, 4\}$. If $ad = 4$, then either $(a, d) = (1, 4)$ or $(a, d) = (2, 2)$. Further, by Eq. (6.4) we get that

$$(3b - 2a)(3c - 2d) \in \{1, 2, 3\}.$$

Both for $(a, d) = (1, 4)$ and for $(a, d) = (2, 2)$ there is a unique solution of this relation with $b, c \in \mathbb{N}$, namely, $(a, b, c, d) = (1, 1, 3, 4)$ and $(a, b, c, d) = (2, 1, 1, 2)$. However, in the first case one has

$$(\sigma_1\sigma_2)^4\sigma_1^x(\alpha_1) = \alpha_1 - \alpha_2,$$

and in the second case one gets $\sigma_2\sigma_1\sigma_2\sigma_1^x(\alpha_2) = \alpha_1 - \alpha_2$. Therefore (R1) gives that there is no root system of type $\mathcal{C}$ with $ad = 4$.

Assume now that $ad = 3$, that is, $a = 1$ and $d = 3$. Then Eq. (6.3) implies that $-3 \leq 2(2bc - 2c - 6b + 4) \leq -1$, which has no solution with $a, b, c, d \in \mathbb{N}$.

Finally, let $a = 1$ and $d = 2$. Then, by Eq. (6.4), $t_{11} + t_{22} \in \{-1, 0, 1\}$ if and only if

$$1 \leq (b - 2)(c - 4) \leq 3. \tag{6.6}$$

If $b = 1$ then $c \in \{1, 2, 3\}$. If $(b, c) = (1, 1)$ then $\sigma_2^x\sigma_1^y\sigma_2^z(\alpha_1) = \alpha_1 - \alpha_2$, a contradiction to (R1). If $(b, c) = (1, 2)$ then $C^x = C^y = C^z$, $|R_+^x| = 4$, but $(\rho_1\rho_2)^4(x) = y \neq x$, a contradiction to (R4). If $(b, c) = (1, 3)$ then $\sigma_1^z\sigma_2^y\sigma_1^x\sigma_2^x(\alpha_1) = \alpha_2 - \alpha_1$, a contradiction to (R1).

If $b = 3$ in Rel. (6.6), then $c \in \{5, 6, 7\}$. If $c = 5$ then direct computation shows that $(\sigma_1\sigma_2)^4 1_y(\alpha_1) = \alpha_2 - \alpha_1$, a contradiction to (R1). If $c = 6$, then we get solution (4), and if $c = 7$, then we get solution (6).

The remaining two solutions of Rel. (6.6) are $(b, c) = (4, 5)$ and $(b, c) = (5, 5)$. These correspond to solution (5) and (7), respectively.

The sets R_+^x can be calculated from Prop. 2.12. □

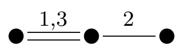
REMARK 6.2. It is interesting to note that in contrast to the case with two objects, see Rem. 5.5, not all non-standard root systems can be obtained from Nichols algebras of diagonal type. The example in Thm. 6.1(3) can be identified with the root system of the Nichols algebras corresponding to Row 10 [**27**, Table 1]. However, the only rank two Nichols algebras of diagonal type with 12 positive roots are those in Row 17 of [**27**, Table 1], and there are no such Nichols algebras with more than 12 positive roots. This can be read off from the trees in the appendix of [**28**]. The Nichols algebras corresponding to Row 17 of [**27**, Table 1] have been discussed already in Rem. 5.5: in all of the Cartan matrices one has at least one entry -1. Thus the examples in

Thm. 6.1(4)–(7) can not be obtained as the root system of a Nichols algebra of diagonal type.

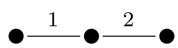
It is not clear if there are more general Nichols algebras with such root systems.

Now we assume that $\theta = 3$.

THEOREM 6.3. *Let $\mathcal{C}$ be a connected Cartan scheme with $I = \{1,2,3\}$ and with 3 objects, and let $\mathcal{R}$ be a finite irreducible root system of type $\mathcal{C}$. Then $\mathcal{R}$ is standard, and the Cartan matrices are indecomposable and of type A_3, B_3, or C_3. If $c^a_{12} = c^a_{21} = -1$ and $c^a_{13} = c^a_{31} = 0$ for all objects a, then the object change diagram of $\mathcal{R}$ is*



for type A_3 Cartan matrices, and



for type B_3 and C_3 Cartan matrices.

PROOF. Let x, y, z denote the three objects of $\mathcal{R}$. Since $\mathcal{C}$ is connected, we may assume (using permutations of I and A) that the restriction of $\mathcal{R}$ to $I = \{1,2\}$ is as for $\theta = 2$ — without supposing that $a \le d$ in Eq. (6.2). Then we have to consider three cases.

Case 1: Assume that $\rho_3 : A \to A$ is the identity. By Thm. 2.6 and Prop. 2.14 the group $\mathrm{Hom}(x)$ is isomorphic to

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, s_3, t_3, u_1, u_3\rangle/(&s_2^2, s_3^2, t_3^2, u_1^2, u_3^2, (u_1 s_2)^{m^x_{1,2}/3},\\ &(t_3 s_3)^{m^x_{1,3}/2}, (s_2 s_3)^{m^x_{2,3}}, (u_3 t_3)^{m^y_{2,3}/2}, (u_1 u_3)^{m^z_{1,3}}),\end{aligned} \tag{6.7}$$

where the isomorphism is given by $\sigma^x_2 \mapsto s_2$, $\sigma^x_3 \mapsto s_3$, $\sigma^y_1 \sigma^y_3 \sigma^x_1 \mapsto t_3$, $\sigma^y_1 \sigma^z_2 \sigma^z_1 \sigma^y_2 \sigma^x_1 \mapsto u_1$, and $\sigma^y_1 \sigma^z_2 \sigma^z_3 \sigma^y_2 \sigma^x_1 \mapsto u_3$. Thus, $\mathrm{Hom}(x)$ is a Coxeter group.

Suppose first that $m^x_{1,3} = m^y_{2,3} = 2$. Then $c^x_{13} = c^y_{23} = 0$, and hence $c^y_{13} = 0$ by (C2) and relation $\rho_1(x) = y$. Thus, $\mathcal{R}$ is not irreducible by Prop. 4.6, which is a contradiction. Note that $m^z_{2,3} = m^y_{2,3}$, since $\rho_2(y) = z$. Now, using a symmetry in the presentation of $\mathcal{R}$, we can assume that $m^y_{2,3} > 2$.

If $m^x_{1,2} > 3$, then the quotient

$$\begin{aligned}\mathrm{Hom}(x)/(t_3 s_3) \simeq \langle s_2, s_3, u_1, u_3\rangle/(&s_2^2, s_3^2, u_1^2, u_3^2, (u_1 s_2)^{m^x_{1,2}/3},\\ &(s_2 s_3)^{m^x_{2,3}}, (u_3 s_3)^{m^y_{2,3}/2}, (u_1 u_3)^{m^z_{1,3}})\end{aligned}$$

is a Coxeter group without relation between u_1 and s_3, and hence it is infinite, which is a contradiction to the finiteness of $\mathcal{R}$. If $m^x_{1,2} = 3$, then

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, s_3, t_3, u_3\rangle/(&s_2^2, s_3^2, t_3^2, u_3^2,\\ &(t_3 s_3)^{m^x_{1,3}/2}, (s_2 s_3)^{m^x_{2,3}}, (u_3 t_3)^{m^y_{2,3}/2}, (s_2 u_3)^{m^z_{1,3}}).\end{aligned}$$

In this case, if $m^x_{1,3} > 2$, then there is no Coxeter relation between s_2 and t_3, which is again a contradiction. Thus $m^x_{1,3} = 2$ and

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, s_3, u_3\rangle/(&s_2^2, s_3^2, u_3^2,\\ &(s_2s_3)^{m^x_{2,3}}, (u_3s_3)^{m^y_{2,3}/2}, (s_2u_3)^{m^z_{1,3}}).\end{aligned} \tag{6.8}$$

Since $m^x_{1,2} = 3$, $m^x_{1,3} = 2$ and $m^y_{2,3} > 2$, Thm. 6.1 yields that the Cartan matrices C^x, C^y, and C^z are

$$C^x = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -a \\ 0 & -b & 2 \end{pmatrix}, C^y = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -c \\ 0 & -d & 2 \end{pmatrix}, C^z = \begin{pmatrix} 2 & -1 & -e \\ -1 & 2 & -c \\ -f & -g & 2 \end{pmatrix},$$

where $a, b, e, f, g \in \mathbb{N}_0$ and $c, d \in \mathbb{N}$. Moreover, Lemmata 4.5, 4.9 imply that

$$d = b, \qquad a = c + e. \tag{6.9}$$

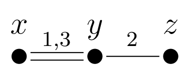
The isomorphism in Eq. (6.8) tells that σ^x_2 and σ^x_3 generate a finite Coxeter subgroup of $\mathrm{Hom}(x)$, and hence $ab \in \{0, 1, 2, 3\}$. Since $a = c+e$ and $c > 0$, we get $ab \in \{1, 2, 3\}$, and hence $m^x_{2,3} \geq 3$.

If $ab = 3$, then $m^x_{2,3} = 6$, and therefore the finiteness of $\mathrm{Hom}(x)$ and Eq. (6.8) imply that $m^y_{2,3} = 4$, $m^z_{1,3} = 2$. The former gives by Prop. 5.1 that $cd = 2$, and the latter, that $e = f = 0$. By Eq. (6.9) we get $2 = cd = ab = 3$, a contradiction.

If $ab = 1$, then $a = b = 1$, and hence Eq. (6.9) and relation $c \in \mathbb{N}$ imply that $e = 0$, $c = a = 1$, and $d = b = 1$. Thus $m^y_{2,3} = 3$ by Lemma 4.8, which is a contradiction to $\rho_2\rho_3\rho_2(y) = y \neq z = \rho_3\rho_2\rho_3(y)$ and (R4).

Finally, assume that $ab = 2$. If $e > 0$, then $a = 2$, $b = 1$, $c = 1$, $d = 1$, and $e = 1$ by Eq. (6.9). Then $m^y_{2,3} = 3$ by Lemma 4.8, but $\rho_2\rho_3\rho_2(y) \neq \rho_3\rho_2\rho_3(y)$, which contradicts (R4). Thus $e = 0$, and hence $c = a$, $d = b$, and $f = 0$ by Eq. (6.9) and (M2). Since $cd = 2$, we get $g = d$ by Prop. 5.1, and hence $C^x = C^y = C^z$ are Cartan matrices of type B_3 or C_3.

Case 2: Assume that $\rho_3(x) = y$, $\rho_3(y) = x$, and $\rho_3(z) = z$. In other words, the object change diagram of $\mathcal{R}$ is

(6.10) 

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We apply Prop. 2.14 to $\mathrm{Hom}(\mathcal{R})$. Let $X_x = 1_x$, $X_y = \sigma^x_1 \in \mathrm{Hom}(x, y)$, and $X_z = \sigma^y_2\sigma^x_1 \in \mathrm{Hom}(x, z)$. Then Thm. 2.6 and Prop. 2.14 imply that

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, t, u_1, u_3\rangle/(&s_2^2, t^{m^x_{1,3}}, u_1^2, u_3^2, (u_1s_2)^{m^x_{1,2}/3},\\ &(s_2t^{-1}u_3t)^{m^x_{2,3}/3}, (u_1u_3)^{m^z_{1,3}}),\end{aligned} \tag{6.11}$$

where the inverse of the isomorphism is given by $s_2 \mapsto \sigma^x_2$, $t \mapsto \sigma^y_1\sigma^x_3$, $u_i \mapsto \sigma^y_1\sigma^z_2\sigma^z_i\sigma^y_2\sigma^x_1$ for $i \in \{1, 3\}$.

If both $m^x_{1,2}$ and $m^x_{2,3}$ are even, then

$$\mathrm{Hom}(x)/(s_2) \simeq \langle t, u_1, u_3\rangle/(t^{m^x_{1,3}}, u_1^2, u_3^2, (u_1u_3)^{m^z_{1,3}}).$$

Since $m^x_{1,3} \geq 2$, the group $\mathrm{Hom}(x)/(s_2)$ is not finite, which is a contradiction to the finiteness of $\mathcal{R}$. By symmetry and by Thm. 6.1 we may assume that $m^x_{1,2} = 3$. Then Eq. (6.11) tells that

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, u_3, t\rangle/(s_2^2, u_3^2, t^{m^x_{1,3}},\\ (s_2t^{-1}u_3t)^{m^x_{2,3}/3}, (s_2u_3)^{m^z_{1,3}}).\end{aligned} \tag{6.12}$$

We apply Prop. 4.12 to $\mathrm{Hom}(x)$. Since $m^x_{1,3} \geq 2$ and $m^z_{1,3} \geq 2$, we conclude that $\mathrm{Hom}(x)$ is finite if and only if $(m^x_{1,3}, m^x_{2,3}) = (2,3)$ or $(m^x_{1,3}, m^x_{2,3}, m^z_{1,3}) = (3,3,2)$. Hence $m^x_{2,3} = 3$. Using relation $m^x_{1,2} = 3$, Lemma 4.8, (C2), and Thm. 6.1, we obtain that

$$C^x = \begin{pmatrix} 2 & -1 & -a\\ -1 & 2 & -1\\ -b & -1 & 2\end{pmatrix}, C^y = \begin{pmatrix} 2 & -1 & -a\\ -1 & 2 & -1\\ -b & -1 & 2\end{pmatrix}, C^z = \begin{pmatrix} 2 & -1 & -c\\ -1 & 2 & -1\\ -d & -1 & 2\end{pmatrix},$$

where $a, b, c, d \in \mathbb{N}_0$. If $m^x_{1,3} = 3$ and $m^z_{1,3} = 2$ then $c = d = 0$ and $a = b = 1$ by Lemma 4.8. Then $a + 1 \neq c + 1$, which is a contradiction to Lemma 4.9. If $m^x_{1,3} = 2$ then $a = b = 0$, and hence $c = d = 0$ by Lemma 4.9 and (M2). Then $C^x = C^y = C^z$ are Cartan matrices of type A_3. This proves the theorem in Case 2.

Case 3: Assume that $\rho_3(x) = z$, $\rho_3(y) = y$, and $\rho_3(z) = x$. For all $a \in A$ and $i, j \in \{1,2,3\}$ with $i \neq j$ the relations $(\rho_i\rho_j)^3(a) = a$ and $\rho_i\rho_j(a) \neq a$ hold, and hence $3|m^a_{i,j}$ by (R4). In particular, $c^a_{ij} < 0$ for all $a \in A$ and $i, j \in \{1,2,3\}$ with $i \neq j$.

Calculate $\sigma := \sigma^x_2\sigma^z_3\sigma^y_2\sigma^x_1 \in \mathrm{Hom}(x)$. Since $\mathcal{R}$ is finite, σ has finite order, and hence $\sigma = \mathrm{id}$ or $\mathrm{tr}\sigma < 3$. Direct calculation shows that $\sigma(\alpha_1) \in -\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$ and

$$\begin{aligned}\mathrm{tr}\sigma = c^x_{12}c^x_{21} - c^x_{12}c^x_{23}c^z_{31} + (c^x_{12}c^y_{21} - 1)(c^x_{23}c^z_{32} - 1)\\ + c^x_{13}c^z_{31} - c^x_{13}c^y_{21}c^z_{32} + c^y_{23}c^z_{32} - 2.\end{aligned}$$

By the above conclusion one obtains that $\mathrm{tr}\sigma \geq 3$ and $\sigma \neq \mathrm{id}$, which is a contradiction. Thus there are no finite root systems of type $\mathcal{C}$ in this case. This finishes the proof of the theorem. □

For the classification of root systems of type $\mathcal{C}$ of rank higher than 3 the following proposition will be useful.

PROPOSITION 6.4. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme with $I = \{1,2,3,4\}$ and three objects x, y, z, and let $\mathcal{R}$ be a finite root system of type $\mathcal{C}$. Assume that $\rho_1(x) = y$, $\rho_2(y) = z$, and $c^a_{12} = c^a_{21} = -1$ for all $a \in \{x, y, z\}$. Then $\mathcal{R}$ is standard.*

PROOF. *Step 1. Both $\mathcal{R}|_{\{1,2,3\}}$ and $\mathcal{R}|_{\{1,2,4\}}$ are standard.*

Since $\mathcal{R}|_{\{1,2\}}$ is connected and irreducible, either $\mathcal{R}|_{\{1,2,3\}} = \mathcal{R}|_{\{1,2\}} \oplus \mathcal{R}|_{\{3\}}$, or $\mathcal{R}|_{\{1,2,3\}}$ is connected and irreducible. Then $\mathcal{R}|_{\{1,2,3\}}$ is standard — in the first case, since $\mathcal{R}|_{\{1,2\}}$ is standard, and in the second case by Thm. 6.3. Similarly, $\mathcal{R}|_{\{1,2,4\}}$ is standard.

Step 2. If $\rho_3(x) \neq x$ then $\mathcal{R}$ is standard.

By Thm. 6.1, $\mathcal{R}|_{\{2,3\}}$ is connected and irreducible. Hence $\mathcal{R}|_{\{1,2,3\}}$ is connected and irreducible, and Thm. 6.3 implies that $\rho_3(x) = y$ and $c^a_{23} = c^a_{32} = -1$ for all $a \in \{x, y, z\}$. Thus Step 1 gives that $\mathcal{R}|_{\{1,2,3\}}$, $\mathcal{R}|_{\{1,2,4\}}$, and $\mathcal{R}|_{\{3,2,4\}}$ are standard, hence $\mathcal{R}$ is standard, see Rem. 4.2.

Step 3. $\mathcal{R}$ is standard.

If $\rho_3 \neq \mathrm{id}$ or $\rho_4 \neq \mathrm{id}$, then $\mathcal{R}$ is standard by Step 2. Assume now that $\rho_3(a) = \rho_4(a) = a$ for all $a \in A$. If $\mathcal{R} = \mathcal{R}|_{\{1,2\}} \oplus \mathcal{R}|_{\{3,4\}}$, then $c^y_{ij} = 0$ for all $i \in \{1, 2\}$, $j \in \{3, 4\}$. Thus $c^x_{jl} = c^y_{jl} = c^z_{jl}$ for all $j \in \{3, 4\}$ and $l \in I$ by Lemma 4.5. Since $\mathcal{R}|_{\{1,2,3\}}$ and $\mathcal{R}|_{\{1,2,4\}}$ are standard by Step 1, it follows that $c^x_{jl} = c^y_{jl} = c^z_{jl}$ for all $j \in \{1, 2\}$ and $l \in I$. Hence $\mathcal{R}$ is standard.

Assume now that $\mathcal{R}|_{\{1,2,3\}}$ is irreducible. By Thm. 6.3 we may assume that $c^a_{13} = c^a_{31} = 0$, $c^a_{23}c^a_{32} = 2$, and $m^a_{2,3} = 4$ for all $a \in \{x, y, z\}$. Then Thm. 2.6 and Prop. 2.14 give that

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, s_3, s_4, t_3, t_4, u_1, u_3, u_4\rangle/(&s_2^2, s_3^2, s_4^2, t_3^2, t_4^2, u_1^2,\\ &u_3^2, u_4^2, u_1s_2, s_3t_3, (s_4t_4)^{m^x_{1,4}/2}, (s_2s_3)^4,\\ &(s_2s_4)^{m^x_{2,4}}, (s_3s_4)^{m^x_{3,4}}, (t_3u_3)^2, (t_4u_4)^{m^y_{2,4}/2},\\ &(t_3t_4)^{m^y_{3,4}}, (u_1u_3)^2, (u_1u_4)^{m^z_{1,4}}, (u_3u_4)^{m^z_{3,4}}),\end{aligned} \tag{6.13}$$

where the inverse of the isomorphism is given by the map $s_i \mapsto \sigma^x_i$ for all $i \in \{2, 3, 4\}$, $t_i \mapsto \sigma^y_1\sigma^y_i\sigma^x_1$ for all $i \in \{3, 4\}$, and $u_i \mapsto \sigma^y_1\sigma^z_2\sigma^z_i\sigma^y_2\sigma^x_1$ for all $i \in \{1, 3, 4\}$. Therefore

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, s_3, s_4, t_4, u_3, u_4\rangle/(&s_2^2, s_3^2, s_4^2, t_4^2, u_3^2, u_4^2,\\ &(s_4t_4)^{m^x_{1,4}/2}, (s_2s_3)^4, (s_2s_4)^{m^x_{2,4}}, (s_3s_4)^{m^x_{3,4}}, (s_3u_3)^2,\\ &(t_4u_4)^{m^y_{2,4}/2}, (s_3t_4)^{m^y_{3,4}}, (s_2u_3)^2, (s_2u_4)^{m^z_{1,4}}, (u_3u_4)^{m^z_{3,4}}).\end{aligned} \tag{6.14}$$

If $m^x_{1,4} > 2$ and $m^y_{2,4} > 2$, then $\mathrm{Hom}(x)$ is a Coxeter group without Coxeter relation between s_2 and t_4, which is a contradiction to the finiteness of $\mathcal{R}$.

Assume first that $m^x_{1,4} = 2$. Since $m^x_{1,3} = 2$ as well, we obtain that $c^x_{13} = c^x_{14} = 0$, and hence $m^x_{3,4} = m^y_{3,4}$. Then

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, s_3, s_4, u_3, u_4\rangle/(&s_2^2, s_3^2, s_4^2, u_3^2, u_4^2, (s_2s_3)^4,\\ &(s_2s_4)^{m^x_{2,4}}, (s_3s_4)^{m^x_{3,4}}, (s_3u_3)^2, (s_4u_4)^{m^y_{2,4}/2},\\ &(s_2u_3)^2, (s_2u_4)^{m^z_{1,4}}, (u_3u_4)^{m^z_{3,4}}).\end{aligned} \tag{6.15}$$

If $m^y_{2,4} > 2$, then there is no Coxeter relation between s_4 and u_3, which is a contradiction, and hence $m^y_{2,4} = 2$.

Similarly, if $m^x_{1,4} > 2$ and $m^y_{2,4} = 2$, then there is no Coxeter relation between s_4 and u_3.

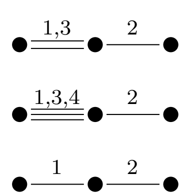
We are left with the case $m^x_{1,4} = m^y_{2,4} = 2$. Then relations $c^x_{14} = c^y_{42} = 0$ and Lemma 4.5 imply that $c^x_{42} = c^y_{42} = 0$, and hence $m^x_{2,4} = 2$. Similarly, $c^y_{24} = 0$ implies that $c^z_{41} = c^y_{41} = c^x_{41} = 0$, hence $m^z_{1,4} = 2$. Therefore

$$\begin{aligned}\mathrm{Hom}(x) \simeq \langle s_2, s_3, s_4, u_3\rangle/(&s_2^2, s_3^2, s_4^2, u_3^2, (s_2s_3)^4, (s_2s_4)^2,\\ &(s_3s_4)^{m^x_{3,4}}, (s_3u_3)^2, (s_2u_3)^2, (u_3s_4)^{m^z_{3,4}}).\end{aligned} \tag{6.16}$$

The finiteness of $\mathrm{Hom}(x)$ implies that $m^x_{3,4}, m^z_{3,4} \in \{2,3\}$. If $m^x_{3,4} = 2$, then $m^x_{1,4} = m^x_{2,4} = 0$ implies that $\mathcal{R} = \mathcal{R}|_{\{1,2,3\}} \oplus \mathcal{R}|_{\{4\}}$, and hence $\mathcal{R}$ is standard, since $\mathcal{R}|_{\{1,2,3\}}$ is standard. Further, $m^z_{3,4} = 2$. Similarly, the assumption $m^z_{3,4} = 2$ leads to the same result.

Suppose now that $m^x_{3,4} = m^z_{3,4} = 3$, that is, $c^x_{34} = c^x_{43} = c^z_{34} = c^z_{43} = -1$ by Lemma 4.8. Since $c^x_{13} = c^x_{14} = 0$, Lemma 4.5 implies that $c^y_{34} = c^y_{43} = -1$, that is, $C^x = C^y = C^z$ is a Cartan matrix of type F_4. This proves the proposition. □

THEOREM 6.5. *Let $\theta \in \mathbb{N}$ with $\theta \geq 4$, let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme with $I = \{1, 2, \dots, \theta\}$ and with 3 objects, and let $\mathcal{R}$ be a finite irreducible root system of type $\mathcal{C}$. Then $\mathcal{R}$ is standard, and the corresponding Cartan matrix C is indecomposable and of type B_4, C_4, D_4, or F_4. The object change diagram of $\mathcal{R}$ can be chosen to be*



if C is of type B_4 or C_4,

if C is of type D_4, and

if C is of type F_4.

PROOF. As explained at the beginning of this section, we can choose $x, y, z \in A$ and use a permutation of I such that the object change diagram of $\mathcal{R}|_{\{1,2\}}$ is the one in Eq. (6.1). Since $\mathcal{R}$ is irreducible, we can assume that $\mathcal{R}|_{\{1,2,3\}}$ is irreducible. Then Thm. 6.3 gives that $\mathcal{R}|_{\{1,2,3\}}$ is standard and $m^x_{1,2} \in \{2,3,4\}$. Further, since $\rho_2\rho_1(x) =$

$z \neq y = \rho_1\rho_2(x)$ and $\rho_1\rho_2\rho_1(x) = z = \rho_2\rho_1\rho_2(x)$, (R4) yields that $m^x_{1,2} = 3$. Hence $c^a_{12} = c^a_{21} = -1$ for all $a \in A$ by Lemma 4.8. By Prop. 6.4 we obtain that $\mathcal{R}|_{\{1,2,i,j\}}$ is standard for all $i, j \in I \setminus \{1, 2\}$ with $i \neq j$. Hence $\mathcal{R}$ is standard by Rem. 4.2.

Finally, Thm. 6.3 and elementary argumentations with Dynkin diagrams imply that there is no connected irreducible standard root system, where the Cartan matrices are of type A_4 or of rank bigger than 4, and that connected irreducible standard root systems of rank 4 have the given object change diagrams. The rigorous proof is left to the reader. □

7. Appendix

In the following table we list all non-standard connected irreducible finite Weyl groupoids with at most three objects. The last column contains the stabilizer group of an object. It is a Coxeter group of the indicated type.

$\mathcal{W}$	$\lvert A\rvert$	$\lvert I\rvert$	$\lvert\mathcal{W}\rvert$	$\lvert R^a_+\rvert$	$\mathrm{Hom}(a)$
Thm. 5.4(2)	2	2	32	8	B_2
Thm. 5.4(2)	2	2	48	12	G_2
Thm. 5.4(3)	2	3	192	13	B_3
Thm. 5.4(4)	2	3	192	13	B_3
Thm. 6.1(3)	3	2	36	6	$A_1 \times A_1$
Thm. 6.1(4)	3	2	72	12	B_2
Thm. 6.1(5)	3	2	72	12	B_2
Thm. 6.1(6)	3	2	108	18	G_2
Thm. 6.1(7)	3	2	108	18	G_2

CHAPTER 3

Weyl groupoids of rank two and continued fractions

M. CUNTZ AND I. HECKENBERGER
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A relationship between continued fractions and Weyl groupoids of Cartan schemes of rank two is found. This allows to decide easily if a given Cartan scheme of rank two admits a finite root system. We obtain obstructions and sharp bounds for the entries of the Cartan matrices.

1. Introduction

Root systems and crystallographic Coxeter groups appear to be main tools in the study of semisimple Lie algebras [**11**]. In the structure theory of pointed Hopf algebras [**39**] a similar role is expected to be played by Weyl groupoids and their root systems. Let us give some hints towards this claim. The most striking results on pointed Hopf algebras rely on the Lifting method of Andruskiewitsch and Schneider [**3**]. Based on it, many new examples of finite-dimensional pointed Hopf algebras have been detected, and fairly general classification results were achieved [**6**], [**30**]. The first step in the Lifting method is the determination of finite-dimensional Nichols algebras of finite group type. The upper triangular part of a small quantum group, also called Frobenius-Lusztig kernel, is a prominent example. A very natural symmetry object of Nichols algebras of finite group type is the Weyl groupoid. This was observed first in [**25**] for Nichols algebras of diagonal type, and then in [**2**] in a very general setting.

An axiomatic approach to Weyl groupoids and their root systems, without referring to Nichols algebras, was initiated in [**34**]. The theory includes and extends the theory of crystallographic Coxeter groups, but contains even such examples which do not seem to be related to Nichols algebras of diagonal type. In this paper we use the language and some structural and classification results achieved in [**17**], see Sect. 2 for the most essential definitions and facts.

For the classification of Nichols algebras of diagonal type it is crucial to be able to decide, if a given Cartan scheme (a categorical generalization of the notion of a generalized Cartan matrix, see Def. 2.1) admits a finite root system. Because of the large variety of examples, this seems to be a difficult task. In our paper, we present a very efficient method for Cartan schemes of rank two. It relies on a relationship between Cartan schemes of rank two and continued fractions [**42**]. Instead of giving a complete list of Cartan schemes of rank two admitting a finite root system (which is then unique by a result in [**17**]), we present with Thm. 6.19 an algorithm. It works with very elementary operations on sequences of positive integers, and transforms any Cartan scheme into another one, for which the answer is known. The algorithm is based on various observations: on the introduction and study of coverings of Cartan schemes in Sect. 3, on an old theorem of Stern, Pringsheim, and Tietze, and a variation of a transformation formula for continued fractions, see Sect. 4 and Lemma 5.2, on the characterization of simple connected Cartan schemes admitting a finite root system in terms of certain sequences of positive integers, see Prop. 6.5 and Thm. 6.6, and on the description of Cartan schemes with object change diagram a cycle using characteristic sequences, see Def. 6.9. As an application, in Sect. 7 we give obstructions for the entries of the Cartan matrices in a Cartan scheme admitting a finite root system. We present the power of our method on a small example at the end of Sect. 6.

We are confident that a suitable generalization of our method to Cartan schemes and Weyl groupoids of higher rank would have a deep impact on the classification of Nichols algebras, and consider it as a great challenge for the future.

2. Cartan schemes, root systems, and their Weyl groupoids

If not stated otherwise, we follow the notation in [**17**]. Let us start by recalling the main definitions.

Let I be a non-empty finite set and $\{\alpha_i \,|\, i \in I\}$ the standard basis of $\mathbb{Z}^I$. By [**37**, §1.1] a generalized Cartan matrix $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \le 0$ for all $i, j, k \in I$ with $j \ne k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

DEFINITION 2.1. Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

REMARK 2.2. The preceding definition of a Cartan scheme has the striking advantage to be very simple, but sufficiently powerful to admit the definition of a Weyl groupoid, see below. For some investigations it can be of advantage to consider more general axioms (for example by allowing the maps ρ_i to be partially defined) or to impose additional restrictions (like (C3) below, or other e. g. to exclude the existence of associated roots which are neither positive nor negative). We will mostly consider Cartan schemes admitting a root system, see below. This restriction still gives (much) more examples than those coming from contragredient Lie superalgebras and Nichols algebras of diagonal type with finite root system. Nevertheless, up to now no further axioms on Cartan schemes are known which keep this property.

Two Cartan schemes $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I', A', (\rho'_i)_{i\in I'}, (C'^a)_{a\in A'})$ are termed *equivalent*, if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1(\rho_i(a)) = \rho'_{\varphi_0(i)}(\varphi_1(a)), \qquad c^{\varphi_1(a)}_{\varphi_0(i)\varphi_0(j)} = c^a_{ij} \tag{2.1}$$

for all $i, j \in I$ and $a \in A$.

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma_i^a \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma_i^a(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{2.2}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are generated by the maps $\sigma_i^a \in \mathrm{Hom}(a, \rho_i(a))$ with $i \in I$, $a \in A$. In this paper, we will always denote the set of all morphisms of $\mathcal{W}(\mathcal{C})$ by $\mathrm{Hom}(\mathcal{W}(\mathcal{C}))$. Formally, for $a, b \in A$ the set $\mathrm{Hom}(a, b)$ consists of the triples (b, f, a), where

$$f = \sigma_{i_n}^{\rho_{i_{n-1}}\cdots\rho_{i_1}(a)} \cdots \sigma_{i_2}^{\rho_{i_1}(a)}\sigma_{i_1}^a$$

and $b = \rho_{i_n}\cdots\rho_{i_2}\rho_{i_1}(a)$ for some $n \in \mathbb{N}_0$ and $i_1, \dots, i_n \in I$. The composition is induced by the group structure of $\mathrm{Aut}(\mathbb{Z}^I)$:

$$(a_3, f_2, a_2) \circ (a_2, f_1, a_1) = (a_3, f_2 f_1, a_1)$$

for all $(a_3, f_2, a_2), (a_2, f_1, a_1) \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$. By abuse of notation we will write $f \in \mathrm{Hom}(a, b)$ instead of $(b, f, a) \in \mathrm{Hom}(a, b)$.

The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$. A Cartan scheme is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$.

In many cases it will be natural to assume that a Cartan scheme satisfies the following additional property.

(C3) If $a, b \in A$ and $(b, \mathrm{id}, a) \in \mathrm{Hom}(a, b)$, then $a = b$.

DEFINITION 2.3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subset \mathbb{Z}^I$, and define $m^a_{i,j} = |R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ for all $i, j \in I$ and $a \in A$.

We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma_i^a(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m^a_{i,j}$ is finite, then $(\rho_i\rho_j)^{m^a_{i,j}}(a) = a$.

If $\mathcal{R}$ is a root system of type $\mathcal{C}$, then we say that $\mathcal{W}(\mathcal{R}) = \mathcal{W}(\mathcal{C})$ is the *Weyl groupoid of* $\mathcal{R}$. Further, $\mathcal{R}$ is called *connected*, if $\mathcal{C}$ is a connected Cartan scheme. If $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ is a root system of type $\mathcal{C}$ and $\mathcal{R}' = \mathcal{R}'(\mathcal{C}', ({R'}^a)_{a\in A'})$ is a root system of type $\mathcal{C}'$, then we say that $\mathcal{R}$ and $\mathcal{R}'$ are *equivalent*, if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes given by maps $\varphi_0 : I \to I'$, $\varphi_1 : A \to A'$ as in Def. 2.1, and if the map $\varphi_0^* : \mathbb{Z}^I \to \mathbb{Z}^{I'}$ given by $\varphi_0^*(\alpha_i) = \alpha_{\varphi_0(i)}$ satisfies $\varphi_0^*(R^a) = {R'}^{\varphi_1(a)}$ for all $a \in A$.

There exist many interesting examples of root systems of type $\mathcal{C}$ related to semisimple Lie algebras, Lie superalgebras and Nichols algebras of diagonal type, respectively. For further details and results we refer to [**34**] and [**17**].

CONVENTION 2.4. In connection with Cartan schemes, upper indices usually refer to elements of A. Often, these indices will be omitted if they are uniquely determined by the context.

REMARK 2.5. If $\mathcal{C}$ is a Cartan scheme and there exists a root system of type $\mathcal{C}$, then $\mathcal{C}$ satisfies (C3) by [**34**, Lemma 8(iii)].

In [**17**, Def. 4.3] the concept of an *irreducible* root system of type $\mathcal{C}$ was defined. By [**17**, Prop. 4.6], if $\mathcal{C}$ is a connected Cartan scheme and $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R}$ is irreducible if and only if the generalized Cartan matrix C^a is indecomposable for one (equivalently, for all) $a \in A$.

A fundamental result about Weyl groupoids is the following theorem.

THEOREM 2.6. [**34**, Thm. 1] *Let* $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ *be a Cartan scheme and* $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ *a root system of type* $\mathcal{C}$*. Let* $\mathcal{W}$ *be the abstract groupoid with* $\mathrm{Ob}(\mathcal{W}) = A$ *such that* $\mathrm{Hom}(\mathcal{W})$ *is generated by abstract morphisms* $s_i^a \in \mathrm{Hom}(a, \rho_i(a))$*, where* $i \in I$ *and* $a \in A$*, satisfying the relations*

$$s_i s_i 1_a = 1_a, \quad (s_j s_k)^{m^a_{j,k}} 1_a = 1_a, \qquad a \in A,\ i, j, k \in I,\ j \neq k,$$

see Conv. 2.4. Here 1_a *is the identity of the object* a*, and* $(s_j s_k)^\infty 1_a$ *is understood to be* 1_a*. The functor* $\mathcal{W} \to \mathcal{W}(\mathcal{R})$*, which is the identity on the objects, and on the set of morphisms is given by* $s_i^a \mapsto \sigma_i^a$ *for all* $i \in I$*,* $a \in A$*, is an isomorphism of groupoids.*

DEFINITION 2.7. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. Let Γ be a nondirected graph, such that the vertices of Γ correspond to the elements of A. Assume that for all $i \in I$ and $a \in A$ with $\rho_i(a) \neq a$ there is precisely one edge between the vertices a and $\rho_i(a)$ with label i, and all edges of Γ are given in this way. The graph Γ is called the *object change diagram* of $\mathcal{C}$. If $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ is a root system of type $\mathcal{C}$, then we also say that Γ is the object change diagram of $\mathcal{R}$.

3. Coverings of Cartan schemes, Weyl groupoids, and root systems

Two Cartan schemes can be related to each other in different ways. In this section we analyze coverings of Cartan schemes. The definition is motivated by the corresponding notion in topology.

DEFINITION 3.1. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I, A', (\rho'_i)_{i\in I}, (C'^a)_{a\in A'})$ be connected Cartan schemes. Let $\pi : A' \to A$ be a map such that $C^{\pi(a)} = C'^a$ for all $a \in A'$ and the diagrams

$$\begin{array}{ccc} A' & \xrightarrow{\rho'_i} & A' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ A & \xrightarrow[\rho_i]{} & A \end{array} \tag{3.1}$$

commute for all $i \in I$. We say that $\pi : \mathcal{C}' \to \mathcal{C}$ is a *covering*, and that $\mathcal{C}'$ is a *covering of* $\mathcal{C}$.

The composition of two coverings is again a covering. For any covering $\pi : \mathcal{C}' \to \mathcal{C}$ of Cartan schemes $\mathcal{C}', \mathcal{C}$, the map $\pi : A' \to A$ is surjective by (3.1), since A' is nonempty and $\mathcal{C}$ is connected.

REMARK 3.2. Many of the following results can be formulated without assuming that $\mathcal{C}$ and/or $\mathcal{C}'$ in Def. 3.1 are connected Cartan schemes. In that case one should assume that π is a surjective map. However, in the applications we are interested in, all Cartan schemes are connected, and hence we prefer the above definition in order to simplify the terminology.

Any covering $\pi : \mathcal{C}' \to \mathcal{C}$ of Cartan schemes $\mathcal{C}', \mathcal{C}$ induces a covariant functor $F_\pi : \mathcal{W}(\mathcal{C}') \to \mathcal{W}(\mathcal{C})$ by letting

$$F_\pi(a') = \pi(a'), \quad F_\pi(\sigma_i^{a'}) = \sigma_i^{\pi(a')} \qquad \text{for all } i \in I,\ a' \in A'.$$

In this case the Weyl groupoid $\mathcal{W}(\mathcal{C}')$ is termed a *covering of* $\mathcal{W}(\mathcal{C})$, and the functor F_π a covering of Weyl groupoids.

First we need a technical result.

LEMMA 3.3. *Let $\pi : \mathcal{C}' \to \mathcal{C}$ be a covering, and assume that $\mathcal{C}'$ satisfies Axiom (C3). Then the following hold:*

(1) $\mathcal{C}$ satisfies (C3).

(2) Let $a \in A$ and $a', a'' \in A'$ such that $\pi(a') = \pi(a'') = a$. If there exists $w' \in \mathrm{Hom}(a', a'')$ such that $F_\pi(w') \in F_\pi(\mathrm{End}(a'))$, then $a' = a''$.

PROOF. (1) Let $a \in A$. If $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$, then Def. 3.1 gives that $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma^a_{i_k} = \sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma^{a'}_{i_k}$ in $\mathrm{Aut}(\mathbb{Z}^I)$ for all $a' \in A'$ with $\pi(a') = a$. Assume now that $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma^a_{i_k} = \mathrm{id}$. Then $\rho'_{i_1} \cdots \rho'_{i_k}(a') = a'$ for all $a' \in A'$ with $\pi(a') = a$, since $\mathcal{C}'$ satisfies (C3). Hence $\rho_{i_1} \cdots \rho_{i_k}(a) = a$ by Eq. (3.1). This yields the claim.

(2) Let $w'' \in \mathrm{End}(a')$ with $F_\pi(w'') = F_\pi(w')$. Then $F_\pi(w' w''^{-1}) = \mathrm{id}_a$, and hence $w' w''^{-1} = \mathrm{id}$ in $\mathrm{Aut}(\mathbb{Z}^I)$. Since $\mathcal{C}'$ satisfies (C3), it follows that $w' w''^{-1} = \mathrm{id}_{a'}$, and hence $a' = a''$. □

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a connected Cartan-scheme, $\mathcal{W}(\mathcal{C})$ its Weyl groupoid, and $a \in A$. Coverings of $\mathcal{C}$ can be parametrized by subgroups of $\mathrm{End}(a) \subset \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$ (up to conjugation).

PROPOSITION 3.4. *(1) Let $\mathcal{C}'$ be a connected Cartan scheme and assume that $\pi : \mathcal{C}' \to \mathcal{C}$ is a covering. Let $a' \in A'$ with $\pi(a') = a$.*

- (1.A) *The group homomorphism $F_\pi : \mathrm{End}(a') \to \mathrm{End}(a)$ is injective.*
- (1.B) *For each $b' \in A'$ with $\pi(b') = a$ the subgroup $F_\pi(\mathrm{End}(b'))$ of $\mathrm{End}(a)$ is conjugate to $F_\pi(\mathrm{End}(a'))$.*
- (1.C) *If U' is a subgroup of $\mathrm{End}(a)$ conjugate to $F_\pi(\mathrm{End}(a'))$, then there exists $b' \in A'$ with $\pi(b') = a$ and $F_\pi(\mathrm{End}(b')) = U'$.*

(2) Suppose that $U \subset \mathrm{End}(a)$ is a subgroup. Then there exists a covering $\pi : \mathcal{C}' \to \mathcal{C}$ and $b' \in A'$ such that

$$F_\pi(\mathrm{End}(b')) = U, \tag{3.2}$$

$$|\pi^{-1}(b)| = [\mathrm{End}(a) : U] \quad \text{for all } b \in A. \tag{3.3}$$

If $\mathcal{C}$ satisfies Axiom (C3), then up to equivalence there is a unique covering $\mathcal{C}'$ satisfying Eq. (3.2) *and Axiom (C3). For this covering Eq.* (3.3) *holds.*

PROOF. (1.A) Each element $w' \in \mathrm{End}(a')$ is a product of $\sigma_i^{b'}$ for some $i \in I$ and $b' \in A'$. Moreover, w' can be naturally regarded as an element in $\mathrm{Aut}(\mathbb{Z}^I)$. The same is true for $w \in \mathrm{End}(a)$. Since $C'^{b'} = C^{\pi(b')}$ for all $b' \in A'$, $F_\pi(w')$ identifies with the same element of $\mathrm{Aut}(\mathbb{Z}^I)$ as w'. This proves (1.A).

(1.B) Let $b' \in A'$. Since $\mathcal{C}'$ is connected, there exists $w' \in \mathrm{Hom}(a', b')$. Then $\mathrm{End}(b') = w' \,\mathrm{End}(a') w'^{-1}$. Since F_π is a functor, $F_\pi(\mathrm{End}(b')) = F_\pi(w') F_\pi(\mathrm{End}(a')) F_\pi(w')^{-1}$.

(1.C) Assume that $w \in \mathrm{End}(a)$ such that $U' = w F_\pi(\mathrm{End}(a')) w^{-1}$. Then $w = \sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma^a_{i_k}$ for some $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$. Let $w' = \sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma^{a'}_{i_k}$ and $b' = \rho'_{i_1} \cdots \rho'_{i_k}(a')$. Then $\mathrm{End}(b') = w' \,\mathrm{End}(a') w'^{-1}$, and hence $F_\pi(\mathrm{End}(b')) = w F_\pi(\mathrm{End}(a')) w^{-1} = U'$.

(2) We construct $\mathcal{C}'$ explicitly. Let

$$A' = \mathrm{Hom}(\mathcal{W}(\mathcal{C}))/U = \{gU \subset \mathrm{Hom}(a, b) \mid b \in A, g \in \mathrm{Hom}(a, b)\}$$

be the set of left cosets. For all $i \in I$ and $gU \in A'$ with $g \in \mathrm{Hom}(a, b)$, where $b \in A$, define $C'^{gU} = C^b$ and $\rho'_i(gU) = \sigma^b_i gU$. Then $\rho'_i : A' \to A'$ satisfies (C1) since $\sigma^{\rho_i(b)}_i \sigma^b_i = \mathrm{id}$ and $\rho^2_i = \mathrm{id}$, and $\mathcal{C}'$ fulfills (C2), since $\mathcal{C}$ does. Since $\mathcal{C}$ is connected, $\mathcal{C}' = \mathcal{C}'(I, A', (\rho'_i)_{i \in I}, (C'^{a'})_{a' \in A'})$ is a connected Cartan scheme. Define $\pi : A' \to A$ by $\pi(gU) = b$ for all $b \in A$, $g \in \mathrm{Hom}(a, b)$. Then $F_\pi(\mathrm{End}(1_a U)) = U$ and $|\pi^{-1}(a)| = [\mathrm{End}(a) : U]$. Since $\mathcal{C}'$ is connected, $|\pi^{-1}(b)| = |\pi^{-1}(a)|$ for all $b \in A$.

Assume that $\mathcal{C}$ satisfies (C3). We show that $\mathcal{C}'$ satisfies (C3). For $l \in \{1, 2\}$ let $a_l \in A$ and $g_l \in \mathrm{Hom}(a, a_l)$ such that $(g_1 U, \mathrm{id}, g_2 U) \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}'))$. Then there exist $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$ such that $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma^{a_2}_{i_k} g_2 U = g_1 U$ and that $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma^{a_2}_{i_k} = \mathrm{id}$ in $\mathrm{Aut}(\mathbb{Z}^I)$. Since $\mathcal{C}$ fulfills (C3), we obtain that $a_1 = a_2$, and hence $g_2 U = g_1 U$. Therefore $\mathcal{C}'$ satisfies (C3).

Finally, let $\pi : \mathcal{C}' \to \mathcal{C}$ and $\pi'' : \mathcal{C}'' \to \mathcal{C}$ be coverings of $\mathcal{C}$ satisfying (C3), and assume that there exist $b' \in A'$, $b'' \in A''$ such that $\pi(b') = \pi''(b'') = a$ and $F_\pi(\mathrm{End}(b')) = F_{\pi''}(\mathrm{End}(b'')) = U$. We have to show that $\mathcal{C}'$ and $\mathcal{C}''$ are equivalent Cartan schemes. Define $\phi : A' \to A''$ by

$$\phi(\rho'_{i_1} \cdots \rho'_{i_k}(b')) = \rho''_{i_1} \cdots \rho''_{i_k}(b'') \qquad \text{for all } k \in \mathbb{N}_0, i_1, \dots, i_k \in I.$$

Then ϕ is well-defined. Indeed, assume that $\rho'_{i_1} \cdots \rho'_{i_k}(b') = b'$. Then $\sigma_{i_1} \cdots \sigma^{b'}_{i_k} \in \mathrm{End}(b')$, and hence application of π resp. F_π gives that $\rho_{i_1} \cdots \rho_{i_k}(a) = a$, $\sigma_{i_1} \cdots \sigma^a_{i_k} \in U$. Thus $F_{\pi''}(\sigma_{i_1} \cdots \sigma^{b''}_{i_k}) \in U$, and hence Lemma 3.3(2) gives that $\rho''_{i_1} \cdots \rho''_{i_k}(b'') = b''$. The compatibility of ϕ with $\rho', \rho'', C'^{b'}, C''^{b''}$ is fulfilled by Def. 3.1 and by definition of ϕ. Further, $\phi : A' \to A''$ is a bijection, the construction of ϕ^{-1} being analogous. Hence ϕ gives rise to an equivalence of the Cartan schemes $\mathcal{C}'$ and $\mathcal{C}''$. □

DEFINITION 3.5. We say that a Cartan scheme $\mathcal{C}$ is *simply connected*, if $\mathrm{End}(a)$ is the trivial group for all $a \in A$.

COROLLARY 3.6. *Let $\mathcal{C}$ be a connected Cartan scheme satisfying (C3). Then up to equivalence there exists a unique covering $\mathcal{C}'$ of $\mathcal{C}$ which is simply connected and satisfies (C3).*

As usual, this simply connected covering of $\mathcal{C}$ is called the *universal covering*.

PROOF. The claim follows from Prop. 3.4(2) by setting $U = \{1\}$. □

PROPOSITION 3.7. *Let $\mathcal{C}$, $\mathcal{C}'$ be connected Cartan schemes and $\pi : \mathcal{C}' \to \mathcal{C}$ a covering.*

(1) If there exists a root system $\mathcal{R}'$ of type $\mathcal{C}'$, then the equations

$$R^a = \bigcap_{a' \in A' \,|\, \pi(a')=a} R'^{a'} \qquad \text{for all } a \in A \tag{3.4}$$

define a root system $\mathcal{R}$ of type $\mathcal{C}$.

(2) If there exists a root system $\mathcal{R}$ of type $\mathcal{C}$, and $\mathcal{C}'$ satisfies (C3), then the equations

$$R'^{a'} = R^{\pi(a')} \qquad \text{for all } a' \in A' \tag{3.5}$$

define a root system $\mathcal{R}'$ of type $\mathcal{C}'$.

PROOF. (1) By Def. 3.1 and Axioms (R1)–(R4) for $\mathcal{R}'$, the axioms (R1)–(R4) are fulfilled for $\mathcal{R}$.

(2) Since Axioms (R1)–(R3) hold for $\mathcal{R}$, they also hold for $\mathcal{R}'$. Suppose that $i, j \in I$ and $a' \in A'$ such that $i \neq j$ and that $m^{a'}_{i,j} = m^a_{i,j}$ is finite, where $a = \pi(a')$. Then $(\sigma_i\sigma_j)^{m^a_{i,j}} 1_a = \mathrm{id}_a$ by Thm. 2.6. Hence $(\sigma_i\sigma_j)^{m^a_{i,j}} 1_{a'} = \mathrm{id}$, and (C3) for $\mathcal{C}'$ implies that $(\rho'_i\rho'_j)^{m^{a'}_{i,j}}(a') = a'$. Thus (R4) holds for $\mathcal{R}'$ and hence $\mathcal{R}'$ is a root system of type $\mathcal{C}'$. □

4. Continued fractions

Continued fractions are related to Weyl groupoids of Cartan schemes of rank two. We recall some basic facts about continued fractions and formulate the facts we will use in our study.

A *continued fraction* is a sequence of indeterminates $a_1, a_2, a_3, \ldots, b_0, b_1, b_2, \ldots$ written in the form

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}$$

(see [**42**] for an introduction). Specializing the right expression to integers, the *convergents* are the numbers

$$\frac{A_n}{B_n} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots + \frac{a_n|}{|b_n},$$

also given by the recursion

$$A_{-1} = 1, \quad A_0 = b_0, \qquad B_{-1} = 0, \quad B_0 = 1,$$
$$A_\nu = b_\nu A_{\nu-1} + a_\nu A_{\nu-2}, \quad B_\nu = b_\nu B_{\nu-1} + a_\nu B_{\nu-2}$$

for all $\nu \in \mathbb{N}$, or

$$\begin{pmatrix} B_0 & A_0 \\ B_{-1} & A_{-1} \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}, \tag{4.1}$$

$$\begin{pmatrix} b_\nu & a_\nu \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_{\nu-1} & A_{\nu-1} \\ B_{\nu-2} & A_{\nu-2} \end{pmatrix} = \begin{pmatrix} B_\nu & A_\nu \\ B_{\nu-1} & A_{\nu-1} \end{pmatrix}. \tag{4.2}$$

One says that $b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \dots$ is *convergent*, if the sequence $(A_\nu/B_\nu)_{\nu \ge \nu_0}$ is well-defined and convergent (with respect to the standard topology of $\mathbb{R}$) for some $\nu_0 \in \mathbb{N}$.

The case where all a_ν are 1 is the most important one and well understood. However, we will be interested in a different case: From now on, let $a_\nu = -1$, $b_\nu \in \mathbb{N}$ for all ν and assume that the sequence $b_1, b_2, \dots$ is periodic. For any $i \in \mathbb{Z}$, let

$$\eta(i) = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{4.3}$$

We will often need the following equations, which hold for all $i, j, k \in \mathbb{Z}$.

$$\eta(i)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & i \end{pmatrix}, \tag{4.4}$$

$$\eta(i)\eta(j) = \begin{pmatrix} ij-1 & -i \\ j & -1 \end{pmatrix}, \tag{4.5}$$

$$\eta(i)\eta(j)\eta(k) = \begin{pmatrix} (ij-1)k - i & -(ij-1) \\ jk-1 & -j \end{pmatrix}, \tag{4.6}$$

$$\tau\eta(i)\tau = \eta(i)^{-1}, \quad \tau\eta(i)^{-1}\tau = \eta(i), \tag{4.7}$$

where

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.8}$$

By Eq. (4.2),

$$\begin{pmatrix} B_n \\ B_{n-1} \end{pmatrix} = \eta(b_n)\cdots\eta(b_1) \begin{pmatrix} B_0 \\ B_{-1} \end{pmatrix}.$$

The product $\eta(b_n)\cdots\eta(b_1)$ will appear in the study of Weyl groupoids of rank two. In particular, we will need to know for which sequences $b_n, \dots, b_1$ this product has finite order. If it has finite order, then, since $B_{-1} = 0$, there exists $\nu \in \mathbb{N}$ such that $B_\nu = 0$.

The following fact is well-known. Variations of it were considered for example by Stern [**42**, §51, Satz 15], Pringsheim [**42**, §53, Satz 24] and Tietze [**42**, §35, Satz 1].

THEOREM 4.1. *If $a_\nu = -1$ and $b_\nu \geq 2$ for all $\nu \in \mathbb{N}$, then the continued fraction $\frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \dots$ is convergent.*

Thus we get:

COROLLARY 4.2. *Let $n \in \mathbb{N}$ and $b_1, \dots, b_n \in \mathbb{Z}$. If $b_i \geq 2$ for all $i \in \{1, \dots, n\}$, then $\eta(b_1) \cdots \eta(b_n)$ does not have finite order.*

PROOF. Assume $b_i \geq 2$ for all $i \in \{1, \dots, n\}$. If $\eta(b_1) \cdots \eta(b_n)$ had finite order, then the periodic continued fraction

$$\frac{-1|}{|b_n} + \frac{-1|}{|b_{n-1}} + \dots + \frac{-1|}{|b_1} + \frac{-1|}{|b_n} + \frac{-1|}{|b_{n-1}} + \dots + \frac{-1|}{|b_1} + \frac{-1|}{|b_n} + \cdots$$

would have infinitely many convergents with denominator 0. This is a contradiction to Thm. 4.1. $\square$

One can also prove Cor. 4.2 without Thm. 4.1, e. g. by [**29**, Lemma 9].

5. Distinguished finite sequences of integers

We now study a special class of finite sequences of positive integers. They correspond to a class of continued fractions which are not convergent. Later we will use these sequences to classify finite root systems of type $\mathcal{C}$ and rank two. Recall the definition of the map $\eta : \mathbb{Z} \to \mathrm{SL}(2, \mathbb{Z})$ from Eq. (4.3).

DEFINITION 5.1. Let $\mathcal{A}$ denote the set of finite sequences $(c_1, \dots, c_n)$ of integers such that $n \geq 1$ and $\eta(c_1) \cdots \eta(c_n) = -\mathrm{id}$. Let $\mathcal{A}^+$ be the subset of $\mathcal{A}$ formed by those $(c_1, \dots, c_n) \in \mathcal{A}$, for which $c_i \geq 1$ for all $i \in \{1, \dots, n\}$ and the entries in the first column of $\eta(c_1) \cdots \eta(c_i)$ are nonnegative for all $i < n$.

The following lemma will be crucial for our analysis of $\mathcal{A}^+$. It is related to a well-known transformation formula for continued fractions, see [**42**, §37, Eqs. (1),(2)].

LEMMA 5.2. *Let $n \geq 3$ and $c = (c_1, 1, c_3, c_4, \dots, c_n)$ such that $c_i \in \mathbb{Z}$ for all $i \in \{1, \dots, n\}$. Let $c' = (c_1 - 1, c_3 - 1, c_4, \dots, c_n)$.*

(1) $c' \in \mathcal{A}$ if and only if $c \in \mathcal{A}$.

(2) $c' \in \mathcal{A}^+$ if and only if $c \in \mathcal{A}^+$, $c_1, c_3 \geq 2$.

(3) If $c \in \mathcal{A}^+$, then either $n = 3$, $c_1 = c_3 = 1$ or $n > 3$, $c_1, c_3 \geq 2$.

PROOF. If $i, k \in \mathbb{Z}$, then

$$\eta(i)\eta(1)\eta(k) = \begin{pmatrix} ik - i - k & 1 - i \\ k - 1 & -1 \end{pmatrix} = \eta(i-1)\eta(k-1)$$

by Eqs. (4.5), (4.6). This gives (1). By Eq. (4.5), the first column of $\eta(c_1)\eta(1)$ contains only nonnegative integers if and only if $c_1 \geq 1$. Thus (2) holds. Let $c \in \mathcal{A}^+$ such that $c_1 = 1$ or $c_3 = 1$. Then Eq. (4.6) gives that the upper left entry of $\eta(c_1)\eta(1)\eta(c_3)$ is -1, and hence $n = 3$. Then $c \in \mathcal{A}$ implies that $c_1 = c_3 = 1$. Hence (3) is proven. □

PROPOSITION 5.3. *Let* $n \in \mathbb{N}$ *and* $(c_1, \dots, c_n) \in \mathcal{A}^+$.

(1) Let $i, j \in \{1, \dots, n\}$ *with* $i \leq j$ *and* $(i, j) \neq (1, n)$. *Then* $\eta(c_i)\eta(c_{i+1}) \cdots \eta(c_j) \in \mathrm{SL}(2, \mathbb{Z})$ *such that the first column contains only nonnegative and the second only nonpositive integers.*

(2) Let $i \in \{1, \dots, n\}$. *Then* $(c_i, c_{i+1}, \dots, c_n, c_1, \dots, c_{i-1}) \in \mathcal{A}^+$.

(3) $(c_n, c_{n-1}, \dots, c_2, c_1) \in \mathcal{A}^+$.

(4) If $n \leq 3$ *then* $(c_1, \dots, c_n) = (1, 1, 1)$.

PROOF. (1) We proceed by induction on the lexicographically ordered pairs (i, j).

If $i = j$ then we are done, since the matrix $\eta(c_i)$ satisfies the claim.

Let $i, j \in \{1, \dots, n\}$ with $i < j$ and $(i, j) \neq (1, n)$. Assume that the claim holds for all pairs $(i', j') \in \{1, \dots, n\}$ such that $i' \leq j'$ and either $i' < i$ or $i' = i, j' < j$. Let

$$\eta(c_i) \cdots \eta(c_j) = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$. Clearly, $-ad + bc = 1$ since $\eta(k) \in \mathrm{SL}(2, \mathbb{Z})$ for all $k \in \mathbb{Z}$. Moreover, Eq. (4.4) gives that

$$\eta(c_i) \cdots \eta(c_{j-1}) = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & c_j \end{pmatrix} = \begin{pmatrix} b & -(bc_j - a) \\ d & -(dc_j - c) \end{pmatrix}.$$

Hence $b, d \geq 0$ by induction hypothesis.

If $i = 1$, then $a, c \geq 0$ by definition of $\mathcal{A}^+$ and the assumption $(i, j) \neq (1, n)$, and hence we are done. Otherwise

$$\eta(c_{i-1}) \cdots \eta(c_j) = \begin{pmatrix} c_{i-1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = \begin{pmatrix} c_{i-1}a - c & d - c_{i-1}b \\ a & -b \end{pmatrix},$$

and hence $a > 0$ by induction hypothesis. Since $a, b, d \geq 0$, we get $bc = 1 + ad \geq 1$, and hence $c > 0$, which proves the claim.

(2) It suffices to prove the claim for $i = 2$. If $\eta(c_1)\cdots\eta(c_n) = -\mathrm{id}$, then clearly $\eta(c_2)\cdots\eta(c_n)\eta(c_1) = -\mathrm{id}$. Let $j \in \{2,\dots,n\}$. Then the entries in the first column of $\eta(c_2)\cdots\eta(c_j)$ are nonnegative by Part (1) of the proposition. This gives (2).

(3) Recall the definition of τ in Eq. (4.8). Then Eq. (4.7) gives that

$$\eta(c_n)\eta(c_{n-1})\cdots\eta(c_1) = \tau\eta(c_n)^{-1}\eta(c_{n-1})^{-1}\cdots\eta(c_1)^{-1}\tau = -\mathrm{id}$$

since $\eta(c_1)\cdots\eta(c_n) = -\mathrm{id}$. Therefore $(c_n, c_{n-1},\dots,c_1) \in \mathcal{A}$.

Let $2 \le i \le n$ and assume that

$$\eta(c_i)\eta(c_{i+1})\cdots\eta(c_n) = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$$

for some $a,b,c,d \in \mathbb{Z}$. Then $a,b,c,d \ge 0$ and $bc - ad = 1$ by Part (1) of the proposition. We obtain that

$$\begin{aligned}\eta(c_n)\cdots\eta(c_i) =& \tau\eta(c_n)^{-1}\cdots\eta(c_i)^{-1}\tau \\ =& \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -d & b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix}.\end{aligned}$$

Thus $(c_n, c_{n-1},\dots,c_1) \in \mathcal{A}^+$.

(4) Equations $\eta(c_1) = -\mathrm{id}$, $\eta(c_1)\eta(c_2) = -\mathrm{id}$ have no solutions with $c_1, c_2 \in \mathbb{N}$ by Eqs. (4.3), (4.5). Let now $n = 3$ and $c_1, c_2, c_3 \in \mathbb{N}$. If $c_1, c_2, c_3 \ge 2$, then $(c_1, c_2, c_3) \notin \mathcal{A}$ by Cor. 4.2. Otherwise $c_1 = c_2 = c_3 = 1$ by Lemma 5.2(3) and Part (2) of the proposition. Relation $(1,1,1) \in \mathcal{A}^+$ holds by Eq. (4.5) with $i = j = 1$. This proves (4). □

By Prop. 5.3(2),(3) the dihedral group $\mathbb{D}_n$ of $2n$ elements, where $n \in \mathbb{N}$, acts on sequences of length n in $\mathcal{A}^+$ by cyclic permutation of the entries and by reflections. This action gives rise to an equivalence relation $\sim$ on $\mathcal{A}^+$ by taking the orbits of the action as equivalence classes. For brevity we will usually not distinguish between elements of $\mathcal{A}^+$ and $\mathcal{A}^+/{\sim}$. By Prop. 5.3(4) there is precisely one element of $\mathcal{A}^+/{\sim}$ of length 3.

Lemma 5.2 suggests to introduce a further equivalence relation $\approx$ on $\mathcal{A}^+$. Let $n, m \in \mathbb{N}$ with $m \ge n$, and let $c = (c_1,\dots,c_n)$, $d = (d_1,\dots,d_m) \in \mathcal{A}^+$. We write $c \approx' d$ if and only if

- $m = n$, $c \sim d$ or
- $m = n+1$, $d = (c_1+1, 1, c_2+1, c_3, c_4, \dots, c_n)$.

DEFINITION 5.4. Let $c, d \in \mathcal{A}^+$. Write $c \approx d$ if and only if there exists $k \in \mathbb{N}$ and a sequence $c = e_1, e_2, \dots, e_k = d$ of elements of $\mathcal{A}^+$, such that $e_i \approx' e_{i+1}$ or $e_{i+1} \approx' e_i$ for all $i \in \{1,2,\dots,k-1\}$.

FIGURE 1. Chain diagram

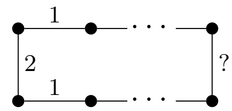

FIGURE 2. Cycle diagram

Clearly, $\approx$ is an equivalence relation on $\mathcal{A}^+$. We are interested in the equivalence classes of $\mathcal{A}^+/\approx$.

THEOREM 5.5. *The only element of* $\mathcal{A}^+/\approx$ *is* $(1,1,1)$.

PROOF. Let $n \geq 1$ and $c = (c_1, \ldots, c_n) \in \mathcal{A}^+$. By Prop. 5.3(4) it suffices to prove that if $n \geq 4$, then $c \approx c'$ for some $c' = (c'_1, c'_2, \ldots, c'_{n-1}) \in \mathcal{A}^+$.

Assume that $n \geq 4$. By Cor. 4.2 there exists $i \in \{1, \ldots, n\}$ such that $c_i = 1$. By Prop. 5.3(2) and the definition of $\approx$ we may assume that $c_2 = 1$. Now apply Lemma 5.2(2),(3) to obtain the desired $c' \in \mathcal{A}^+$. □

COROLLARY 5.6. *If* $n \in \mathbb{N}$, $(c_1, \ldots, c_n) \in \mathcal{A}^+$, *then* $\sum_{i=1}^n c_i = 3(n-2)$.

PROOF. The expression $\sum_{i=1}^n c_i - 3(n-2)$ is zero for $c = (1,1,1)$ and is an invariant of $\approx$. □

6. Connected root systems of rank two

Throughout this section let I be a set with $|I| = 2$, A a finite set, and $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ a connected Cartan scheme. Since $\rho_i^2 = \mathrm{id}$ for all $i \in I$, and $\mathcal{C}$ is connected, the object change diagram of $\mathcal{C}$ is either a chain (if ρ_i has a fixed point for some $i \in I$), see Fig. 1, or a cycle, see Fig. 2.

Recall that an element $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$ is called *even* if $\det(w) = 1$.

LEMMA 6.1. *The object change diagram of* $\mathcal{C}$ *is a cycle if and only if* $\mathrm{End}(a)$ *contains only even elements (for all* $a \in A$*).*

PROOF. If the object change diagram of $\mathcal{C}$ is a cycle, then for all $a \in A$, $\mathrm{End}(a)$ consists of the elements $(\sigma_i\sigma_j)^{k|A|/2}1_a$, where $k \in \mathbb{Z}$ and $I = \{i, j\}$. These are all even. Otherwise the object change diagram of $\mathcal{C}$ is a chain, and there exists $a \in A$ and $i \in I$ such that $\rho_i(a) = a$. Then $\mathrm{End}(a)$ is generated by σ_i^a and $(\sigma_j\sigma_i)^{|A|-1}\sigma_j^a$ which are odd. □

Assume that $\mathcal{C}$ admits a finite root system. The next proposition explains the relationship between the $m_{i,j}^a$ and the number $|A|$ of objects. For this, we need the following standard lemma.

LEMMA 6.2. *Let $M \in \mathrm{GL}(2,\mathbb{Z})$. If the order e of M is finite, then*

$$-2 \le \mathrm{tr}(M) \le 2, \qquad e \in \{1,2,3,4,6\}.$$

PROPOSITION 6.3. *Assume that $I = \{i,j\}$ and that $\mathcal{C}$ admits a finite root system. Then $m_{i,j}^a = m_{j,i}^a = |R_+^a|$ for all objects a. If the object change diagram is a cycle resp. a chain, then $m_{i,j}^a = m\frac{|A|}{2}$ resp. $m_{i,j}^a = m|A|$ for some $m \in \{1,2,3,4,6\}$.*

PROOF. We have $m_{i,j}^a = m_{j,i}^a = |R_+^a|$ by Def. 2.3 for all objects a. Axiom (R3) from the same definition implies that $m_{i,j}^a$ does not depend on a. Let $d = |A|$ if the object change diagram is a chain and $d = |A|/2$ if it is a cycle. Then $(\sigma_j\sigma_i)^k 1_a \in \mathrm{End}(a)$, $k \in \mathbb{N}_0$, if and only if $k \in \mathbb{N}_0 d$. Thm. 2.6 and Lemma 6.2 give that $md = m_{i,j}^a$ for some $m \in \{1,2,3,4,6\}$. □

We are going to give a characterization of finite connected irreducible root systems of type $\mathcal{C}$. First we analyze root systems with simply connected Cartan schemes.

LEMMA 6.4. *Assume that $\mathcal{C}$ is simply connected and that $\mathcal{R}$ is a finite root system of type $\mathcal{C}$. Then the object change diagram of $\mathcal{C}$ is a cycle with $|R^a|$ vertices, where $a \in A$.*

PROOF. Since $\mathcal{C}$ is simply connected, $\mathrm{End}(a) = \{1\}$ for all $a \in A$. By Lemma 6.1 the object change diagram of $\mathcal{C}$ is a cycle. Now

$$|\mathrm{Hom}(\mathcal{W}(\mathcal{C}))1_a| = |A| \cdot |\mathrm{End}(a)|$$

since $\mathcal{C}$ is connected. Again, $\mathcal{C}$ is simply connected, and hence $|A| = |\mathrm{Hom}(\mathcal{W}(\mathcal{C}))1_a|$. This is equal to $2|R_+^a|$ by Thm. 2.6, since $|I| = 2$. □

PROPOSITION 6.5. *Assume that $I = \{i,j\}$ and that $\mathcal{R}$ is a finite irreducible root system of type $\mathcal{C}$. Let $a \in A$ and $n = |R_+^a|$. Let $a_1, a_2, \dots, a_{2n} \in A$ and $c_1, c_2, \dots, c_{2n} \in \mathbb{Z}$ such that*

$$\begin{aligned} a_{2r-1} &= (\rho_j\rho_i)^{r-1}(a), & a_{2r} &= \rho_i(\rho_j\rho_i)^{r-1}(a), \\ c_{2r-1} &= -\, c_{ij}^{a_{2r-1}}, & c_{2r} &= -\, c_{ji}^{a_{2r}} \end{aligned} \tag{6.1}$$

for all $r \in \{1,2,\dots,n\}$. Then $(c_1, c_2, \dots, c_n) \in \mathcal{A}^+$, $c_{n+r} = c_r$ for all $r = 1,2,\dots,n$, and $\rho_j(a_{2n}) = a$.

PROOF. For all $r \in \mathbb{Z}$ let $i_r \in I$ such that $i_r = i$ for r odd and $i_r = j$ for r even. Let $\theta_{2r-1} = \sigma_i^{a_{2r-1}}\tau$, $\theta_{2r} = \tau\sigma_j^{a_{2r}} \in \mathrm{SL}(2,\mathbb{Z})$ for all $r \in \{1,\dots,n\}$. Then $\theta_r = \eta(c_r)$

for all $r \in \{1,\dots,2n\}$. Since $\mathcal{R}$ is irreducible, $c_r > 0$ for all r. By [**34**, Lemmas 4,7], $\ell(\sigma_{i_n}^{a_n}\cdots\sigma_{i_2}^{a_2}\sigma_{i_1}^{a}) = n$. Hence

$$\sigma_{i_n}^{a_n}\cdots\sigma_{i_2}^{a_2}\sigma_{i_1}^{a}(\{\alpha_1,\alpha_2\}) = \{-\alpha_1,-\alpha_2\}$$

by [**34**, Lemma 8(iii)]. Thus $\theta_n\cdots\theta_2\theta_1(\{\alpha_1,\alpha_2\}) = \{-\alpha_1,-\alpha_2\}$, and since $\det\theta_r = 1$ for all r, we conclude that $\theta_n\cdots\theta_2\theta_1 = -\mathrm{id}$. Hence $(c_n,\dots,c_2,c_1)\in\mathcal{A}$.

Clearly, if $2 \le r \le n$, then the first column of $\theta_n\cdots\theta_{r+1}\theta_r$ has nonnegative entries if and only if $\sigma_{i_n}\cdots\sigma_{i_{r+1}}\sigma_{i_r}^{a_r}(\alpha_{i_{r-1}})$ is a positive root. The latter is true by [**34**, Lemma 4], and hence $(c_n,\dots,c_2,c_1)\in\mathcal{A}^+$. Then $(c_1,c_2,\dots,c_n)\in\mathcal{A}^+$ by Prop. 5.3(3).

Replacing in the construction a by a_2 and i by j, we obtain that $(c_2,\dots,c_n,c_{n+1})\in\mathcal{A}^+$. Then $\eta(c_1)^{-1} = -\eta(c_2)\cdots\eta(c_n) = \eta(c_{n+1})^{-1}$, and hence $c_1 = c_{n+1}$. Thus $c_{n+r} = c_r$ for all $r\in\{1,2,\dots,n\}$ by induction on r. Finally, $\rho_j(a_{2n}) = (\rho_j\rho_i)^n(a) = a$ by (R4). □

The construction in Prop. 6.5 associates to any pair $(i,a)\in I\times A$ a sequence $(c_1,c_2,\dots,c_n)\in\mathcal{A}^+$. This defines a map

$$\Phi : I\times A\to\mathcal{A}^+.$$

Prop. 6.5 gives immediately, that

$$\Phi(j,a) = (c_n,c_{n-1},\dots,c_1),\quad \Phi(j,\rho_i(a)) = (c_2,c_3,\dots,c_n,c_1). \tag{6.2}$$

Thus, by definition of $\sim$, the induced map $\Phi : I\times A\to\mathcal{A}^+/\!\sim$ is constant. But we can say more.

THEOREM 6.6. *Let $n\in\mathbb{N}$ and $c = (c_1,c_2,\dots,c_n)\in\mathcal{A}^+$. Then there is a unique (up to equivalence) finite connected irreducible root system $\mathcal{R}$ with simply connected Cartan scheme of rank two such that $c\in\operatorname{Im}\Phi$.*

PROOF. Assume that $c\in\mathcal{A}^+$, $\mathcal{R}$ is a connected irreducible root system of rank two, $i\in I$, and $a\in A$ such that $\Phi(i,a) = c$. If the Cartan scheme of $\mathcal{R}$ is simply connected, then by Lemma 6.4 and Prop. 6.5 the object change diagram of $\mathcal{R}$ is a cycle and $|A| = 2n$. The Cartan matrices C^a and the sets R^a, where $a\in A$, are then uniquely determined by the construction in Prop. 6.5. Thus $\mathcal{R}$ is uniquely determined. We describe $\mathcal{R}$ explicitly.

Let $I = \{i,j\}$ and $A = \{a_1,\dots,a_{2n}\}$ a set with $2n$ elements. Define $\rho_i,\rho_j : A\to A$ such that

$$\begin{aligned} \rho_i(a_{2r-1}) &= a_{2r}, & \rho_i(a_{2r}) &= a_{2r-1},\\ \rho_j(a_{2r}) &= a_{2r+1}, & \rho_j(a_{2r+1}) &= a_{2r} \end{aligned} \tag{6.3}$$

for all $r \in \{1, 2, \dots, n\}$, where $a_{2n+1} = a_1$. Then $\rho_i^2 = \rho_j^2 = \mathrm{id}$. Let $c_{ln+r} = c_r$ for all $r \in \{1, 2, \dots, n\}$ and $l \in \mathbb{Z}$, and define

$$C^{a_{2r-1}} = \begin{pmatrix} 2 & -c_{2r-1} \\ -c_{2r-2} & 2 \end{pmatrix}, \quad C^{a_{2r}} = \begin{pmatrix} 2 & -c_{2r-1} \\ -c_{2r} & 2 \end{pmatrix} \tag{6.4}$$

for all $r \in \{1, 2, \dots, n\}$. Since $c_r \in \mathbb{N}$ for all $r \in \{1, 2, \dots, 2n\}$, the matrices C^{a_r} satisfy (M1) and (M2). Since also (C1) and (C2) hold, $\mathcal{C} = \mathcal{C}(I, A, (\rho_i, \rho_j), (C^a)_{a \in A})$ is a connected Cartan scheme.

Now define

$$R^{a_{2r-1}} = \left\{ \pm\eta(c_{2r-1})\eta(c_{2r}) \cdots \eta(c_{2r-2+l}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\Big|\, 0 \le l \le n-1 \right\},$$
$$R^{a_{2r}} = \left\{ \pm\tau\eta(c_{2r})\eta(c_{2r+1}) \cdots \eta(c_{2r+l-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\Big|\, 0 \le l \le n-1 \right\}$$

for all $r \in \{1, 2, \dots, n\}$. Note that $|R^a_+| = n$ for all $a \in A$. Indeed, otherwise $\eta(c_r)\eta(c_{r+1}) \cdots \eta(c_{r+l-1}) \binom{1}{0} = \binom{1}{0}$ for some $r \in \{1, 2, \dots, 2n\}$ and $l \in \{1, 2, \dots, n-1\}$. Then

$$\eta(c_{r+1})\eta(c_{r+2}) \cdots \eta(c_{r+l-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta(c_r)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

a contradiction to Prop. 5.3(1),(2).

Axiom (R1) is fulfilled by Prop. 5.3(2). Let $r \in \{1, 2, \dots, 2n\}$. Equation $\eta(c_r)\eta(c_{r+1}) \cdots \eta(c_{r+n-1}) = -\mathrm{id}$ implies, that

$$\eta(c_r)\eta(c_{r+1}) \cdots \eta(c_{r+n-2}) = -\eta(c_{r+n-1})^{-1},$$

and hence $\pm\alpha_1, \pm\alpha_2 \in R^{a_r}$. Since $\tau, \eta(l) \in \mathrm{SL}(2, \mathbb{Z})$ for all $l \in \mathbb{Z}$, we get (R2). (R4) holds by Eq. (6.3), since $|R^a_+| = n$ for all $a \in A$.

Now we prove (R3). Let $r \in \{1, 2, \dots, 2n\}$. Then $\sigma_i^{a_r} = \eta(-c_{ij}^{a_r})\tau = \tau\eta(-c_{ij}^{a_r})^{-1}$ by Eqs. (6.4), (4.7). If r is odd, then

$$\begin{aligned} &\sigma_i^{a_r}(R^{a_r}) \\ &= \tau\eta(c_r)^{-1} \left(\left\{ \pm\eta(c_r)\eta(c_{r+1}) \cdots \eta(c_{r+l-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\Big|\, 0 \le l \le n-1 \right\} \right) \\ &\quad \subset R^{a_{r+1}} = R^{\rho_i(a_r)}, \end{aligned}$$

and if r is even, then

$$\begin{aligned} &\sigma_i^{a_r}(R^{a_r}) \\ &= \eta(c_{r-1})\tau \left(\left\{ \pm\tau\eta(c_r)\eta(c_{r+1}) \cdots \eta(c_{r+l-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\Big|\, 0 \le l \le n-1 \right\} \right) \\ &\quad \subset R^{a_{r-1}} = R^{\rho_i(a_r)}. \end{aligned}$$

Similarly, $\sigma_j^{a_r} = \tau\eta(c_{r-1})$ for odd r and $\sigma_j^{a_r} = \eta(c_r)^{-1}\tau$ for even r. Hence $\sigma_j^{a_r}(R^{a_r}) \subset R^{\rho_j(a_r)}$, (R3) holds, and $\mathcal{R}$ is a finite irreducible root system of type $\mathcal{C}$. The Cartan scheme $\mathcal{C}$ is simply connected, since $(c_1, \dots, c_n) \in \mathcal{A}$ and $|\operatorname{Hom}(\mathcal{W}(\mathcal{C}))1_{a_1}| = 2n = |A|$. Finally, $\Phi(i, a_1) = (c_1, \dots, c_n)$ because of Eqs. (6.1), (6.3), and (6.4). □

COROLLARY 6.7. *Assume that there is a finite root system $\mathcal{R}$ of type $\mathcal{C}$. Then there are $a \in A$ and $i, j \in I$ with $i \neq j$ such that $c_{ij}^a = 0$ or $c_{ij}^a = -1$.*

PROOF. If $\mathcal{R}$ is not irreducible, then $C_{ij}^a = 0$ for all $a \in A$ and $i, j \in I$ with $i \neq j$, see the end of Sect. 2. Otherwise Prop. 6.5 gives that the negatives of the entries of the Cartan matrices of $\mathcal{C}$ give rise to a sequence $(c_1, \dots, c_n) \in \mathcal{A}^+$. By Cor. 4.2, this sequence has an entry 1, and the corollary is proven. □

REMARK 6.8. The assumption in Cor. 6.7 can be weakened for example by requiring only that $\mathcal{W}(\mathcal{C})$ is finite. We don't work out the details, since we are mainly interested in Cartan schemes admitting (finite) root systems.

We are going to give a very effective algorithm to decide if our given connected Cartan scheme $\mathcal{C}$ admits a finite irreducible root system. The central notions towards this will be the characteristic sequences and centrally symmetric Cartan schemes. Our algorithm can also be used to get a more precise classification of root systems of rank two, for example in form of explicit lists for a given number of objects.

DEFINITION 6.9. Assume that the object change diagram of $\mathcal{C}$ is a cycle. Let $i \in I$, $a \in A$, and define $a_1, \dots, a_{|A|} \in A$ and $c_1, \dots, c_{|A|} \in \mathbb{N}_0$ by

$$a_{2k-1} = (\rho_j\rho_i)^{k-1}(a), \qquad a_{2k} = (\rho_i\rho_j)^{k-1}\rho_i(a),$$
$$c_{2k-1} = -\, c_{ij}^{a_{2k-1}}, \qquad c_{2k} = -\, c_{ji}^{a_{2k}}$$

for all $k \in \{1, 2, \dots, |A|/2\}$, where $I = \{i, j\}$. Then $(c_1, c_2, \dots, c_{|A|})$ is called the *characteristic sequence of* $\mathcal{C}$ with respect to i and a. The Cartan scheme $\mathcal{C}$ is termed *centrally symmetric*, if $c_k = c_{k+|A|/2}$ for all $k \in \{1, 2, \dots, |A|/2\}$. In this case we write also $(c_1, c_2, \dots, c_{|A|/2})^2$ for $(c_1, c_2, \dots, c_{|A|})$.

REMARK 6.10. Let $(c_1, c_2, \dots, c_{|A|})$ be the characteristic sequence of $\mathcal{C}$ with respect to i and a. Then the characteristic sequences with respect to j and a and i and $\rho_i(a)$, respectively, are $(c_{|A|}, c_{|A|-1}, \dots, c_1)$ and $(c_1, c_{|A|}, c_{|A|-1}, \dots, c_3, c_2)$, respectively. Thus if $\mathcal{C}$ is centrally symmetric with respect to i and a, it is also centrally symmetric with respect to j and a and i and $\rho_i(a)$, respectively. Since $\mathcal{C}$ is connected, this means that $\mathcal{C}$ being centrally symmetric is independent of the choice of $i \in I$ and $a \in A$.

REMARK 6.11. Characteristic sequences must not be confused with elements of $\mathcal{A}$ or $\mathcal{A}^+$. Their precise relationship will not be needed in the sequel, so we don't work it out in detail.

REMARK 6.12. Let $n \in \mathbb{N}$ and let $c = (c_1, c_2, \dots, c_{2n})$ be a sequence of positive integers. By axioms (M1) and (C2) there is a unique (up to equivalence) connected Cartan scheme $\mathcal{C}$ with object change diagram a cycle, such that the characteristic sequence of $\mathcal{C}$ (with respect to some $i \in I$ and $a \in A$) is c.

REMARK 6.13. Assume that $\mathcal{C}$ is simply connected, and that there exists a finite irreducible root system of type $\mathcal{C}$. Then $\mathcal{C}$ is centrally symmetric by Lemma 6.4 and Prop. 6.5.

REMARK 6.14. Assume that the object change diagram of $\mathcal{C}$ is a cycle. By Lemma 6.1 and Prop. 3.4 the object change diagram of an n-fold covering $\mathcal{C}'$ of $\mathcal{C}$, where $n \in \mathbb{N}$, is a cycle. The characteristic sequence of $\mathcal{C}'$ is just the n-fold repetition of the characteristic sequence of $\mathcal{C}$. Thus an n-fold covering of $\mathcal{C}$ is centrally symmetric if and only if $\mathcal{C}$ is centrally symmetric or n is even.

LEMMA 6.15. *Assume that there exists a finite irreducible root system of type $\mathcal{C}$. Suppose that the object change diagram of $\mathcal{C}$ is a chain. Then there is a unique double covering $\mathcal{C}'$ of $\mathcal{C}$ and a finite irreducible root system of type $\mathcal{C}'$ such that the object change diagram of $\mathcal{C}'$ is a cycle.*

PROOF. By assumption there exists $a \in A$ and $i \in I$ such that $\rho_i(a) = a$. Then $\mathrm{End}(a)$ is generated by σ_i^a and $\tau^a = (\sigma_j\sigma_i)^{|A|-1}\sigma_j^a$, where $I = \{i, j\}$. Since σ_i^a, τ^a are reflections, for the subgroup $U = \langle \sigma_i^a \tau^a \rangle \subset \mathrm{End}(a)$ we obtain that $[\mathrm{End}(a) : U] = 2$, and U consists of even elements. By Prop. 3.4(2) there exists a unique double covering $\mathcal{C}'$ of $\mathcal{C}$ satisfying Axiom (C3) such that $\mathrm{End}(a') \simeq U$ for all $a' \in A'$. By Lemma 6.1 the object change diagram of $\mathcal{C}'$ is a cycle. The uniqueness of $\mathcal{C}'$ holds, since U is the unique subgroup of $\mathrm{End}(a)$ consisting of even elements and satisfying $[\mathrm{End}(a) : U] = 2$. The existence of a finite irreducible root system of type $\mathcal{C}'$ follows from Prop. 3.7(2). □

REMARK 6.16. If $\mathcal{C}'$ is a Cartan scheme with object change diagram a cycle, then $\mathcal{C}'$ is the double covering of a Cartan scheme with object change diagram a chain if and only if there exist $i \in I'$, $a \in A'$, such that the characteristic sequence of $\mathcal{C}'$ with respect to i and a is of the form $(c_1, \dots, c_n, c_{n+1}, c_n, c_{n-1}, \dots, c_2)$ with $n = |A'|/2$ and $c_1, \dots, c_{n+1} \in \mathbb{N}_0$.

LEMMA 6.17. *Assume that there exists a finite irreducible root system of type $\mathcal{C}$. Suppose that the object change diagram of $\mathcal{C}$ is a cycle, and that $\mathcal{C}$ is not centrally symmetric. Then there is a unique double covering $\mathcal{C}'$ of $\mathcal{C}$ which admits a (finite irreducible) root system. The Cartan scheme $\mathcal{C}'$ is centrally symmetric.*

PROOF. Since the object change diagram of $\mathcal{C}$ is a cycle, $\mathrm{End}(a)$ is cyclic for all $a \in A$. The universal covering of $\mathcal{C}$ is centrally symmetric by Rem. 6.13. Since $\mathcal{C}$ is not centrally symmetric, $|\mathrm{End}(a)|$ is even by Rem. 6.14 and Prop. 3.4(2). By Prop. 3.4(2) there is a unique double covering $\mathcal{C}'$ of $\mathcal{C}$ satisfying (C3). It admits a finite irreducible root system of type $\mathcal{C}'$ by Prop. 3.7(2). All coverings of $\mathcal{C}$ admitting a root system fulfill (C3). Hence $\mathcal{C}'$ is the only double covering of $\mathcal{C}$ admitting a root system. This $\mathcal{C}'$ is centrally symmetric by Rem. 6.14. □

REMARK 6.18. Let $\mathcal{C}'$ be a Cartan scheme with object change diagram a centrally symmetric cycle, and $n = |A'|$. Then $\mathcal{C}'$ is the double covering of a Cartan scheme with object change diagram a not centrally symmetric cycle if and only if $n \in 4\mathbb{N}$, and with respect to one (equivalently, all) pair $(i', a') \in I' \times A'$ the characteristic sequence of $\mathcal{C}'$ is not of the form

$$(c_1, c_2, \dots, c_{n/4}, c_1, c_2, \dots, c_{n/4})^2,$$

where $c_1, \dots, c_{n/4} \in \mathbb{N}_0$.

In order to decide if a given connected Cartan scheme admits a finite root system, Lemmas 6.15 and 6.17 allow to concentrate on centrally symmetric Cartan schemes. Further, since the classification of finite root systems with at most three objects is known, see [**17**], we may assume that the Cartan scheme has at least 4 objects.

For any matrix C, let C^{t} denote the transpose of C.

THEOREM 6.19. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected centrally symmetric Cartan scheme with $|A| \geq 4$.*

(1) Assume that the characteristic sequence of $\mathcal{C}$ contains 0. Then $c^a_{ij} = 0$ for all $a \in A$ and $i, j \in I$ with $i \neq j$. Moreover, $\mathcal{C}$ admits a finite root system if and only if $|A| = 4$.

(2) If all entries of the characteristic sequence of $\mathcal{C}$ are at least two, then $\mathcal{C}$ does not admit a finite root system.

(3) Assume that the characteristic sequence of $\mathcal{C}$ is of the form

$$c = (c_1, 1, c_3, c_4, \dots, c_{|A|/2})^2.$$

(Thus $0 \notin \{c_1, \dots, c_{|A|/2}\}$ by (1).)
If $c_1 = 1$ or $c_3 = 1$, then there is a finite root system of type $\mathcal{C}$ if and only if $|A| = 6$ and $c_1 = c_3 = 1$. If $c_1 > 1$ and $|A| = 4$, then there is a finite root system of type $\mathcal{C}$ if and only if $c_1 \in \{2, 3\}$. If $c_1 > 1$, $c_3 > 1$, and $|A| \geq 6$, then there is a finite root system of type $\mathcal{C}$ if and only if the Cartan scheme with object change diagram a cycle with $|A| - 2$ edges and with characteristic sequence

$$(c_1 - 1, c_3 - 1, c_4, \dots, c_{|A|/2})^2 \tag{6.5}$$

admits a finite root system.

PROOF. (1) follows from (M2), (C2), and (R4), and (2) from Cor. 6.7.

(3) If $c_1 = 1$ or $c_3 = 1$, then there exists $a \in A$ such that $c^a_{ij} = c^a_{ji} = -1$, where $I = \{i, j\}$. Then [**17**, Lemma 4.8] gives that $m^a_{i,j} = 3$ and $c_r = 1$ for all $r \in \{1, 3, 4, \dots, |A|/2\}$. By (R4) we get $|A| = 6$.

Assume next that $c_1 > 1$ and $|A| = 4$. Then $C^a = C^b$ for all $a, b \in A$, and hence $\mathcal{C}$ admits a finite root system if and only if C^a is of finite type and (R4) holds (cf. [**17**, Thm. 3.3]), that is, $c_1 \in \{2, 3\}$.

Finally, assume that $c_1 > 1$, $c_3 > 1$, $|A| \geq 6$, and $\mathcal{C}$ admits a finite root system. By Prop. 3.7, the universal covering $\mathcal{C}'$ of $\mathcal{C}$ admits a finite root system. Hence A' is finite by (C1) and (R4). Therefore $\mathrm{End}(a) \subset \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$ is finite for all $a \in A$ by Eq. (3.3). Let $m = |\mathrm{End}(a)|$. Rem. 6.14 and Lemma 6.4 tell that the object change diagram of $\mathcal{C}'$ is a centrally symmetric cycle, and the characteristic sequence of $\mathcal{C}'$ is an m-fold repetition of c. Let

$$\tilde{c} = (c_1, 1, c_3, c_4, \dots, c_{|A|/2}).$$

By Prop. 6.5 the m-fold repetition of $\tilde{c}$ is an element of $\mathcal{A}^+$. Since $|A| \geq 6$, Lemma 5.2(2) gives that the m-fold repetition of

$$\tilde{c}' = (c_1 - 1, c_3 - 1, c_4, \dots, c_{|A|/2})$$

is in $\mathcal{A}^+$. Let $\mathcal{C}''$ be the connected simply connected Cartan scheme which corresponds to the m-fold repetition of $\tilde{c}'$ via Thm. 6.6. It admits a finite root system. Now $\mathcal{C}''$ is the m-fold covering of a Cartan scheme $\mathcal{C}'''$ with characteristic sequence given in Eq. (6.5). Hence Prop. 3.7 gives that $\mathcal{C}'''$ admits a finite root system.

We have shown that if $\mathcal{C}$ admits a finite root system, then also $\mathcal{C}'''$. The proof of the converse goes in the same way, and we are done. □

EXAMPLE 6.20. Consider the connected Cartan scheme $\mathcal{C}$ of rank two with 4 objects, object change diagram a cycle and characteristic sequence $(5, 1, 2, 2)$. To check that $\mathcal{C}$ admits a finite root system, consider the double covering $\mathcal{C}'$ corresponding to the characteristic sequence $(5, 1, 2, 2)^2$. By Prop. 3.7, $\mathcal{C}$ admits a finite root system if and only if $\mathcal{C}'$ does. Thm. 6.19(3) allows to replace $\mathcal{C}'$ by the Cartan scheme with characteristic sequence $(4, 1, 2)^2$ respectively $(3, 1)^2$. Thus $\mathcal{C}$ admits a finite root system.

If we start with the characteristic sequence $(5, 1, 2, 3)$ for $\mathcal{C}$, then the analogous arguments produce the characteristic sequences $(5, 1, 2, 3)^2$, $(4, 1, 3)^2$ and $(3, 2)^2$, and then $\mathcal{C}$ does not admit a finite root system by Thm. 6.19(2).

EXAMPLE 6.21. Thm. 6.19 also enables us to list all connected centrally symmetric Cartan schemes which admit a finite root system to a fixed number of objects. For example if $|A| = 4$, then there are 3 such schemes and they belong to the characteristic sequences $(0, 0)^2$, $(1, 2)^2$, $(1, 3)^2$. Therefore by Thm. 6.19(2) and (3), the only connected centrally symmetric Cartan schemes (up to equivalence) which have

6 objects and admit a finite root system are those, which correspond to the characteristic sequences $(1,1,1)^2$, $(2,1,3)^2$ and $(2,1,4)^2$, respectively; if $|A| = 8$, then we obtain $(2,1,2,1)^2$, $(3,1,2,3)^2$, $(2,2,1,4)^2$, $(3,1,4,1)^2$, $(3,1,2,4)^2$, $(2,2,1,5)^2$ and $(3,1,5,1)^2$. Similarly, we have 15, 47 resp. 136 connected centrally symmetric Cartan schemes up to equivalence with 10, 12 resp. 14 objects which admit a finite root system.

According to Lemma 6.17 and the above lists for $|A| = 4$ and $|A| = 8$, the complete list of all characteristic sequences to irreducible Cartan schemes which admit a finite root system, with object change diagram a cycle and 4 objects is thus: $(1,2,1,2)$, $(1,3,1,3)$, $(3,1,2,3)$, $(2,2,1,4)$, $(3,1,4,1)$, $(3,1,2,4)$, $(2,2,1,5)$, $(3,1,5,1)$.

Rem. 6.16 and the list for $|A| = 8$ also supports us with Cartan schemes with 4 objects which admit a finite root system and have a chain as object change diagram. The symmetry property mentioned in Rem. 6.16 is fulfilled for the sequences $(2,1,2,1)^2$ (also in the form $(1,2,1,2)^2$), $(3,1,4,1)^2$ (also in the form $(4,1,3,1)^2$), and $(3,1,5,1)^2$ (also in the form $(5,1,3,1)^2$). This yields the following 6 Cartan schemes with 4 objects (the Cartan matrices represent the objects).

$$\begin{pmatrix}2&-2\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-2\\-1&2\end{pmatrix}\overset{1}{\text{—}}\begin{pmatrix}2&-2\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-2\\-1&2\end{pmatrix}$$
$$\begin{pmatrix}2&-1\\-2&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-1\\-2&2\end{pmatrix}\overset{1}{\text{—}}\begin{pmatrix}2&-1\\-2&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-1\\-2&2\end{pmatrix}$$
$$\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-4\\-1&2\end{pmatrix}\overset{1}{\text{—}}\begin{pmatrix}2&-4\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-3\\-1&2\end{pmatrix}$$
$$\begin{pmatrix}2&-4\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\overset{1}{\text{—}}\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-4\\-1&2\end{pmatrix}$$
$$\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-5\\-1&2\end{pmatrix}\overset{1}{\text{—}}\begin{pmatrix}2&-5\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-3\\-1&2\end{pmatrix}$$
$$\begin{pmatrix}2&-5\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\overset{1}{\text{—}}\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\overset{2}{\text{—}}\begin{pmatrix}2&-5\\-1&2\end{pmatrix}$$

In order to complete the classification of all connected Cartan schemes with finite root system and 4 objects, it remains to calculate all connected Cartan schemes with finite root system and 8 objects with object change diagram a not centrally symmetric cycle, and then to apply Rem. 6.16 to them, as indicated above, to get all chains with 4 objects. This is certainly an easy task for a computer but there are too many such Cartan schemes to list them here.

7. Bounds

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme of rank two admitting a finite irreducible root system of type $\mathcal{C}$. Then A is finite by (C1) and (R4). Let $-q = -q(\mathcal{C})$ denote the sum of all non-diagonal entries of the Cartan matrices of $\mathcal{C}$, and $h = |\operatorname{End}(a)|$ for an $a \in A$. Then $|\operatorname{End}(b)| = h$ for all $b \in A$, since $\mathcal{C}$ is connected.

THEOREM 7.1. *We have* $h(6|A| - q) = 24$ *and*

$$|R^a_+| = \frac{h|A|}{2} = \frac{12|A|}{6|A| - q}.$$

PROOF. The universal covering $\mathcal{C}'$ of $\mathcal{C}$ has $h|A|$ objects by Eq. (3.3), and $q(\mathcal{C}')/4 = 3(h|A|/2 - 2)$ by Prop. 6.5 and Cor. 5.6. Since $q(\mathcal{C}') = hq(\mathcal{C})$, we obtain that $hq = 6(h|A| - 4)$. Hence $h(6|A| - q) = 24$. Lemma 6.4 tells that $|R^a_+| = h|A|/2$. This yields the claim. □

REMARK 7.2. Prop. 6.3 and Thm. 7.1 give that $h \in \{1, 2, 3, 4, 6\}$ if the object change diagram of $\mathcal{C}$ is a cycle, and $h/2 \in \{1, 2, 3, 4, 6\}$ if it is a chain. But this result could have been obtained much easier. Nevertheless, Thm. 7.1 gives a restriction for $q = 6|A| - 24/h$ for given number $|A|$ of objects in a finite irreducible root system.

Next we give sharp bounds for the entries of the Cartan matrices.

PROPOSITION 7.3. *Assume that* $|A| \geq 2$. *Let* $c \leq 0$ *be an entry of* C^a *for some* $a \in A$. *If the object change diagram is a cycle resp. a chain, then* $|c| \leq |A| + 1$ *resp.* $|c| \leq 2|A| + 1$.

PROOF. Assume first that the object change diagram of $\mathcal{C}$ is a cycle. If $|A| \geq 4$ and $\mathcal{C}$ is centrally symmetric, then Thm. 6.19(2),(3) yields by induction on $|A|$, that $|c| \leq |A|/2 + 1$. If $\mathcal{C}$ is not centrally symmetric, then by Lemma 6.17 there exists a double covering of $\mathcal{C}$ which is centrally symmetric. Hence $|c| \leq |A| + 1$.

If the object change diagram of $\mathcal{C}$ is a chain, then by Lemma 6.15 there exists a double covering of $\mathcal{C}$ which has a cycle as object change diagram. Hence $|c| \leq 2|A| + 1$. □

PROPOSITION 7.4. *For all* $n \geq 1$ *there exist finite connected irreducible root systems* $\mathcal{R}$ *of rank two with* $|A| = 2n$ *and object change diagram a cycle resp.* $|A| = n$ *and object change diagram a chain such that* $-(2n+1)$ *is an entry in a Cartan matrix* C^a, $a \in A$.

PROOF. For $n = 1$ the claim follows from [**17**, Prop. 5.2].

Thm. 6.19 tells that for all $n \geq 2$ the Cartan scheme $\mathcal{C}_n$ with $4n$ objects, object change diagram a cycle, and characteristic sequence

$$(3, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}}, 1, 2n+1, 1, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}})^2 \tag{7.1}$$

admits a finite irreducible root system with $|A| = 4n$. Indeed, if $n = 2$, then using Thm. 6.19(3) we can transform the sequence $(3, 1, 5, 1)^2$ first to $(2, 4, 1)^2$. By changing the reference object, the latter is equivalent to $(4, 1, 2)^2$, and using Thm. 6.19(3) we may reduce it to $(3, 1)^2$. If $n > 2$, then using Thm. 6.19(3) we may transform the sequence in (7.1) in two steps, first to

$$(3, \underbrace{2, 2, \dots, 2}_{n-3 \text{ times}}, 1, 2n, 1, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}})^2,$$

and then to

$$(3, \underbrace{2, 2, \dots, 2}_{n-3 \text{ times}}, 1, 2n-1, 1, \underbrace{2, 2, \dots, 2}_{n-3 \text{ times}})^2.$$

By induction on n we obtain that $\mathcal{C}_n$ admits a finite irreducible root system. By Rem. 6.18, $\mathcal{C}_n$ is the double covering of a Cartan scheme $\mathcal{C}'_n$ with $2n$ objects, object change diagram a cycle, and characteristic sequence

$$(3, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}}, 1, 2n+1, 1, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}}).$$

By Prop. 3.7, $\mathcal{C}'_n$ admits a finite irreducible root system $\mathcal{R}'$, and $\mathcal{R}'$ is such a root system we are looking for. By Rem. 6.16, $\mathcal{C}'_n$ is the double covering of a Cartan scheme $\mathcal{C}''_n$ with n objects and object change diagram a chain. By Prop. 3.7, $\mathcal{C}''_n$ admits a finite irreducible root system $\mathcal{R}''$, and the proposition is proven. □

COROLLARY 7.5. *Any $c \in \mathbb{N}$ occurs as the negative of an entry of a Cartan matrix of a finite connected irreducible root system of rank two.*

PROOF. For even c use the appropriate intermediate step in the proof of Prop. 7.4. □

COROLLARY 7.6. *For $r, n \in \mathbb{N}$, there are only finitely many finite root systems $\mathcal{R}$ of rank r with n objects.*

PROOF. Let I, A be finite sets with $|I| = r$ and $|A| = n$, and let $\mathcal{R}$ be a finite root system of rank r with object set A. For all $i, j \in I$ with $i \neq j$ the restriction $\mathcal{R}|_{\{i,j\}}$, see [**17**, Def. 4.1], is a finite root system of rank two. Hence the entries of the Cartan matrices of $\mathcal{R}$ are bounded by $2|A| + 1$ by Prop. 7.3. Since for all $i \in I$, ρ_i is one of finitely many permutations of A, and since finite root systems are uniquely determined by their Cartan scheme, the claim is proven. □

CHAPTER 4

Reflection groupoids of rank two and Cluster algebras of type A

M. CUNTZ AND I. HECKENBERGER
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We extend the classification of finite Weyl groupoids of rank two. Then we generalize these Weyl groupoids to 'reflection groupoids' by admitting non-integral entries of the Cartan matrices. This leads to the unexpected observation that the spectrum of the cluster algebra of type A_{n-3} completely describes the set of finite reflection groupoids of rank two with $2n$ objects.

1. Introduction

In the last years, the classification of pointed Hopf algebras has grown to a very fruitful subject. Amongst others, the Weyl groupoid was invented [**26**] [**34**], a structure which plays a similar role as the Weyl group for semisimple Lie algebras. A Weyl groupoid is a groupoid which is defined using a family of generalized Cartan matrices. The abstract notion of the Weyl groupoid is perhaps somewhat too general from the viewpoint of pointed Hopf algebras, but nevertheless its system of axioms is good enough to provide a rich general theory and to admit a classification at least of the finite case (see [**16**], [**19**]), thus it has proven to be very useful anyway.

Recent work on finite rank three Weyl groupoids has revealed a connection to combinatorics, because a Weyl groupoid yields a simplicial arrangement in the same manner a Coxeter group does. Since the classification of simplicial arrangements in the real projective plane is still an open problem - Grünbaum conjectures that he has a complete list [**23**] - it appears now to be a natural question to generalize Weyl groupoids to the case in which the Cartan entries are not necessarily integers: This would at least yield one more case, the Coxeter group of type H_3.

However, the classification algorithm for rank three in [**19**] requires a good knowledge on finite rank two Weyl groupoids. So to find an explanation of simplicial arrangements in terms of Weyl groupoids, it is necessary to understand at least finite reflection groupoids of rank two first. These are certain groupoids generated by reflections given by a vast generalization of Cartan matrices, see Section 2 for a precise definition.

In general, we are mostly interested in Weyl groupoids which admit a finite root system in the sense of [**17**, Def. 2.2]. A Weyl groupoid is called *universal* if for each object a the group $\mathrm{End}(a)$ is trivial. It is *irreducible* if none of the generalized Cartan matrices are decomposable. In [**16**], a construction is given to obtain all finite universal Weyl groupoids of rank two which admit a finite root system. We refine this result by explaining the combinatorics:

THEOREM 1.1 (Theorem 3.4). *There is a natural bijection between the set of isomorphism classes of connected irreducible universal Weyl groupoids of rank two with $2n$ objects which admit a finite root system, and the triangulations of a convex n-gon by non-intersecting diagonals, up to the symmetry of the dihedral group $\mathbb{D}_n$.*

Further, we explicitly give the corresponding root systems in Proposition 3.7 in terms of sequences closely related to the well-known Farey series. We conclude in Corollary 3.8 that any positive root is either simple or the sum of two positive roots (at the same object). We also describe the quotients of universal coverings in Proposition 3.12.

For the classification of connected finite reflection groupoids of rank two we proceed in the most natural way. We view the entries of the defining 'Cartan matrices' as indeterminates, translate the axioms for finiteness into polynomials and consider the resulting variety. Using an induction on the number of objects with a similar rule as in [**16**], we obtain a surprising explanation for the theory in rank two, in which matrix mutation appears in a completely natural way.

THEOREM 1.2 (Theorem 4.3). *Let $n \in \mathbb{N}_{\geq 3}$ and let $\mathcal{W}$ be a connected finite reflection groupoid of rank two with $2n$ objects. The set of connected finite reflection groupoids of rank two with the same objects and the same object change diagram as $\mathcal{W}$ is a variety isomorphic to the spectrum of the cluster algebra of type A_{n-3}, where the edges of the n-gon $q_1, \dots, q_n$ are specialized to* 1.

For the definition of the object change diagram see Section 2.

This note is organized as follows. In Section 2 we recall the definition of the main structures as in [**17**], except that the Cartan entries may come from an arbitrary ring here. In Section 3 we explain the combinatorics of finite Weyl groupoids of rank two. In the last section, we shortly describe the Grassmannian $\mathrm{Gr}(2, n)$ following [**21**, Lecture 3] and exhibit the connection to the reflection groupoids of rank two.

2. Reflection groupoids of rank two

Let K be a ring. We first introduce and recall some definitions and notations needed to formulate the definition of a finite reflection groupoid of rank two. Remark that the only reason why we restrict to the case of rank two is that we have no canonical choice for a system of axioms in the higher rank yet.

DEFINITION 2.1. Let I be a set with $|I| = 2$. Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a matrix in $K^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a K-*Cartan scheme* if

(M1) $c^a_{ii} = 2$ for all $a \in A$, $i \in I$,
(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

The notion K-*Cartan scheme* has its origins in [**17**] where Cartan schemes are defined using a family of generalized Cartan matrices. More precisely, a Cartan scheme is a $\mathbb{Z}$-Cartan scheme for which all matrices C^a, where $a \in A$, are generalized Cartan matrices in the sense of [**37**, Sect. 1.1]. In Definition 2.1 the matrices C^a are not necessarily generalized Cartan matrices, even if $K = \mathbb{Z}$. The reason for the generality in our definition is twofold. First, if we considered additional axioms on the matrices C^a then the interpretation of Theorem 4.3 would be more complicated than Theorem 1.2. Second, let $K = \mathbb{Z}$ and assume that any real root (see below) associated to any $a \in A$ is either positive or negative. Then for all $a \in A$ the matrix C^a is a generalized Cartan matrix, see [**17**, Lemma 2.5].

Two K-Cartan schemes are termed *equivalent* if there exist bijections between their sets I respectively A which satisfy the natural compatibility conditions, see [**17**, Def. 2.1] for details.

For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(K^I)$ by

$$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{2.1}$$

Then for all $i \in I$ and $a \in A$ the linear map σ^a_i is a reflection in the sense of [**11**, Ch. V, §2.2].

To $\mathcal{C}$ belongs a category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are generated by the maps σ^a_i which are by definition in $\mathrm{Hom}(a, \rho_i(a))$ with $i \in I$, $a \in A$.

To each object $a \in A$, one can associate a set of *real roots*

$$R^a = \{\varphi(\alpha_i) \mid i \in I,\ \ \varphi \in \mathrm{Hom}(b, a),\ \ b \in A\}.$$

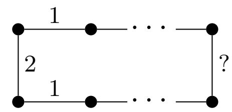


FIGURE 1. Cycle diagram

In the special case $K = \mathbb{Z}$ we have a set $R^a_+ = R^a \cap \mathbb{N}_0^I$ of *positive roots*. We write $\mathcal{R}^{re}(\mathcal{C}) = (\mathcal{C}, (R^a)_{a\in A})$.

REMARK 2.2. Let $\mathcal{C}$ be a $\mathbb{Z}$-Cartan scheme and assume that for all $a \in A$ the matrix C^a is a generalized Cartan matrix. Then $\mathcal{C}$ is a *Cartan scheme*. Although for the purposes of this paper we assumed that $|I| = 2$, in the general theory [**17**] this assumption is not necessary.

We say that $\mathcal{C}$ is *irreducible* if the generalized Cartan matrix C^a is indecomposable for any $a \in A$. The groupoid $\mathcal{W}(\mathcal{C})$ is called the *Weyl groupoid of* $\mathcal{C}$ (compare [**17**]). The Weyl groupoid $\mathcal{W}(\mathcal{C})$ is *finite* if the number of objects in each connected component of $\mathcal{W}(\mathcal{C})$ is finite and if $|\operatorname{Hom}(a, b)| < \infty$ for all $a, b \in A$. We say that $\mathcal{C}$ is *connected* if $\mathcal{W}(\mathcal{C})$ is connected. Further, $\mathcal{C}$ is *simply connected* if $|\operatorname{End}(a)| = 1$ for all $a \in A$.

For the existence of a root system of type $\mathcal{C}$ in the sense of [**17**, Def. 2.2] it is necessary that the real roots are contained in $\mathbb{N}_0^I \cup -\mathbb{N}_0^I$. This is not always the case even if the rank of $\mathcal{C}$ is two: A counterexample can be found in the proof of [**17**, Thm. 6.1], see [**31**, Rem. 1.1] for details. By [**17**, Prop. 2.12], if there is a finite root system of type $\mathcal{C}$ then all roots are real. Therefore there exists a finite root system of type $\mathcal{C}$ if and only if $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$.

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a K-Cartan scheme. We will omit the source of a morphism in the notation if it is clear from the context. The number $|I|$ is called the *rank* of $\mathcal{C}$. It is always two in this article.

Let Γ be a non directed graph, such that the vertices of Γ correspond to the elements of A. Assume that for all $i \in I$ and $a \in A$ with $\rho_i(a) \neq a$ there is precisely one edge between the vertices a and $\rho_i(a)$ with label i, and all edges of Γ are given in this way. The graph Γ is called the *object change diagram* of $\mathcal{C}$.

Assume that $\mathcal{C}$ is connected. Then the object change diagram of $\mathcal{C}$ is either a chain or a cycle. In the second case the cardinality of A is even, see Figure 1. Let now $n \in \mathbb{N}$ and assume that the object change diagram of $\mathcal{C}$ is a cycle with $2n$ vertices.

Let $i, j \in I$ and $a \in A$. In analogy to [**22**, 1.1.6], for all $m \in \mathbb{N}$ let

$$\operatorname{Prod}(m; a, i, j) = \ldots \sigma_i \sigma_j \sigma_i^a$$

where the number of factors is m. Let $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

DEFINITION 2.3. For a K-Cartan scheme $\mathcal{C}$ (of rank two, i.e. $|I| = 2$) we call $\mathcal{W}(\mathcal{C})$ a *finite reflection groupoid* if the object change diagram of $\mathcal{C}$ is a cycle and

$$\operatorname{Prod}(n; a, i, j) = -\tau^n \tag{2.2}$$

for all $a \in A$ and $i, j \in I$ with $i \neq j$, where $n = |A|/2$.

Although we have no roots here, axiom (2.2) is reasonable since it generalizes the fact that a "longest word" should map all positive roots to negative ones.

For the remaining part of this section assume that $\mathcal{C}$ is a K-Cartan scheme such that $\mathcal{W}(\mathcal{C})$ is a finite reflection groupoid with $|A| = 2n$ objects. For simplicity assume that $I = \{1, 2\}$.

By (C2), the K-Cartan scheme $\mathcal{C}$ is locally of the form

$$\cdots \overset{1}{\longrightarrow} \begin{pmatrix} 2 & -c_1 \\ -c_2 & 2 \end{pmatrix} \overset{2}{\longrightarrow} \begin{pmatrix} 2 & -c_3 \\ -c_2 & 2 \end{pmatrix} \overset{1}{\longrightarrow} \begin{pmatrix} 2 & -c_3 \\ -c_4 & 2 \end{pmatrix} \overset{2}{\longrightarrow} \cdots$$

for some $c_1, c_2, \ldots \in K$. Hence giving the sequence of Cartan entries $c_1, c_2, \ldots$ and the label of the first morphism in this sequence completely determines the K-Cartan scheme (and the reflection groupoid). More precisely (compare [**16**, Prop. 6.5]):

LEMMA 2.4. *Let $a \in A$ and $i, j \in I$ with $I = \{i, j\}$. Let $a_1, a_2, \ldots, a_{2n} \in A$ and $c_1, c_2, \ldots, c_{2n} \in K$ such that*

$$\begin{array}{llllll} a_1 = a, & a_2 = \rho_i(a_1), & a_3 = \rho_j(a_2), & a_4 = \rho_i(a_3), & a_5 = \rho_j(a_4), & \ldots \\ c_1 = -c_{ij}^{a_1}, & c_2 = -c_{ji}^{a_2}, & c_3 = -c_{ij}^{a_3}, & c_4 = -c_{ji}^{a_4}, & c_5 = -c_{ij}^{a_5}, & \ldots \end{array}$$

Then $c_{n+r} = c_r$ for all $r \in \{1, 2, \ldots, n\}$.

PROOF. By Equation (2.2),

$$\begin{aligned} (\sigma \operatorname{Prod}(n-1; \rho_i(a), j, i)) &= \operatorname{Prod}(n; \rho_i(a), j, i) \\ &= -\tau^n \\ &= \operatorname{Prod}(n; a, i, j) \\ &= (\operatorname{Prod}(n-1; \rho_i(a), j, i)\sigma_i^a) \end{aligned}$$

in $\operatorname{Aut}(K^I)$, where σ is the first map in $\operatorname{Prod}(n; \rho_i(a), j, i)$. Hence $\sigma = \sigma_i^a$ or $\tau\sigma\tau = \sigma_i^a$ depending on whether n is even or odd. But in both cases the c_1 and c_{n+1} defining the reflections are equal. Starting at the other objects gives $c_{n+r} = c_r$ for all $r \in \{1, 2, \ldots, n\}$. □

Let

$$\Phi : I \times A \to K^n, \quad (i, a) \mapsto (c_1, c_2, \ldots, c_n), \tag{2.3}$$

where $c_1, c_2, \dots, c_n \in K$ are depending on $i \in I$ and $a \in A$ as in Lemma 2.4.

3. Classification of finite rank two Weyl groupoids

In [**16**] we gave an inductive method to construct all finite Weyl groupoids of rank two which admit a root system. Here, we extend these results and give a natural bijection to triangulations of convex polygons. We describe in Proposition 3.7 the set of positive roots in terms of ordered sequences of vectors in $\mathbb{N}_0^I$. This sequences are closely related to the well-known Farey series, see the discussion in Remark 3.6. As a corollary we obtain that any non-simple positive root is a sum of two positive roots.

3.1. Connected simply connected Cartan schemes. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected simply connected Cartan scheme (see Rem. 2.2) with $I = \{1, 2\}$ and assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system (of type $\mathcal{C}$). By [**16**, Section 6] the object change diagram of $\mathcal{C}$ is a cycle with an even number of vertices. For all $x \in \mathbb{Z}$ let

$$\eta(x) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

Following [**16**, Def. 5.1] let $\mathcal{A}$ denote the set of finite sequences $(c_1, \dots, c_n)$ of integers such that $n \geq 1$ and $\eta(c_1)\cdots\eta(c_n) = -\mathrm{id}$. Let $\mathcal{A}^+$ be the subset of $\mathcal{A}$ formed by those $(c_1, \dots, c_n) \in \mathcal{A}$, for which $c_i \geq 1$ for all $i \in \{1, \dots, n\}$ and the entries in the first column of $\eta(c_1)\cdots\eta(c_i)$ are non negative for all $i < n$.

In [**16**, Prop. 6.5] it was shown that Lemma 2.4 holds for $\mathcal{C}$, and that $(c_1, \dots, c_n) \in \mathcal{A}^+$ (using the notation of the lemma).

By [**16**, Prop. 5.3] (2),(3) the dihedral group $\mathbb{D}_n$ of $2n$ elements, where $n \in \mathbb{N}$, acts on sequences of length n in $\mathcal{A}^+$ by cyclic permutation of the entries and by reflections. This action gives rise to an equivalence relation $\sim$ on $\mathcal{A}^+$ by taking the orbits of the action as equivalence classes.

Let $n, m \in \mathbb{N}$ with $m \geq n$, and let $c = (c_1, \dots, c_n)$, $d = (d_1, \dots, d_m) \in \mathcal{A}^+$. We write $c \approx' d$ if and only if

- $m = n$, $c \sim d$ or
- $m = n + 1$, $d = (c_1 + 1, 1, c_2 + 1, c_3, c_4, \dots, c_n)$.

The following is [**16**, Def. 5.4]: Let $c, d \in \mathcal{A}^+$. Write $c \approx d$ if and only if there exists $k \in \mathbb{N}$ and a sequence $c = e_1, e_2, \dots, e_k = d$ of elements of $\mathcal{A}^+$, such that $e_i \approx' e_{i+1}$ or $e_{i+1} \approx' e_i$ for all $i \in \{1, 2, \dots, k-1\}$.

THEOREM 3.1. [**16**, Thm. 5.5] *The only element of* $\mathcal{A}^+/{\approx}$ *is* $(1, 1, 1)$.

Summarizing this, the set $\mathcal{A}^+$ may also be defined as follows:

DEFINITION 3.2. Define η-*sequences* recursively in the following way:

(1) $(1,1,1)$ is an η-sequence.
(2) If $(c_1,\dots,c_n)$ is an η-sequence, then $(c_2,c_3,\dots,c_{n-1},c_n,c_1)$ and $(c_n,c_{n-1},\dots,c_2,c_1)$ are η-sequences.
(3) If $(c_1,\dots,c_n)$ is an η-sequence, then $(c_1+1,1,c_2+1,c_3,\dots,c_n)$ is an η-sequence.
(4) Every η-sequence is obtained recursively by (1),(2),(3).

Then $\mathcal{A}^+$ is the set of η-sequences.

Recall the map Φ from Equation (2.3).

THEOREM 3.3. [**16**, Thm. 6.6] *Let $n \in \mathbb{N}$ and $c = (c_1,c_2,\dots,c_n) \in \mathcal{A}^+$. Then there is a unique (up to equivalence) connected simply connected Cartan scheme of rank two such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is an irreducible root system and $c \in \operatorname{Im}\Phi$.*

Together with [**16**, Prop. 6.5] this gives that the connected simply connected Cartan schemes of rank two, for which $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is an irreducible root system, are those with cycle diagram as object change diagram, and where the Cartan entries yield an η-sequence. Since the symmetry group of such object change diagrams is the dihedral group, we obtain exactly one equivalence class of connected simply connected Cartan schemes for each element of $\mathcal{A}^+/\sim$.

Let $\mathcal{T}_n$ be the set of triangulations of a convex n-gon by non-intersecting diagonals for a fixed labeling of the vertices by $1,\dots,n$, and let $\mathcal{T} = \bigcup_{n=3}^{\infty} \mathcal{T}_n$.

Let $\Psi : \mathcal{A}^+ \to \mathcal{T}$ be the map that maps an η-sequence $(c_1,\dots,c_n)$ to the triangulation of the n-gon which has c_i triangles attached to the vertex i: The sequence $(1,1,1)$ maps to the triangle. Inserting a '1' in the η-sequence as in Definition 3.2(3) corresponds to attaching a new triangle at the edge $(1,2)$. Definition 3.2(2) just corresponds to the natural action of the dihedral group on the polygon. Thus Ψ is well-defined.

We can now deduce the result of this subsection:

THEOREM 3.4. *The map Ψ is bijective.*

PROOF. Since an element of $\mathcal{T}$ uniquely defines an η-sequence, Ψ is injective. On the other hand, each triangulation of the n-gon by non-intersecting diagonals may be obtained by starting with a triangle and by successively attaching triangles at the edges (this is sometimes called a flexagon or a planar 2-tree), so Ψ is surjective. □

We end this subsection by determining the roots of the Cartan schemes of rank two for which $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system.

DEFINITION 3.5. Define $\mathcal{F}$*-sequences* as finite sequences of length ≥ 3 with entries in $\mathbb{N}_0^2$ given by the following recursion.

(1) $((0,1),(1,1),(1,0))$ is an $\mathcal{F}$-sequence.
(2) If $(v_1,\dots,v_n)$ is an $\mathcal{F}$-sequence, then
$$(v_1,\dots,v_i,v_i+v_{i+1},v_{i+1},\dots,v_n)$$
are $\mathcal{F}$-sequences for $i=1,\dots,n-1$.
(3) Every $\mathcal{F}$-sequence is obtained recursively by (1) and (2).

REMARK 3.6. Notice the resemblance to Farey series (see [**24**, III]): The Farey series $\mathcal{F}_n$ of order n is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n. A characteristic property is: If $\frac{h}{k}$, $\frac{h''}{k''}$, and $\frac{h'}{k'}$ are three successive terms of $\mathcal{F}_n$, then $\frac{h''}{k''}=\frac{h+h'}{k+k'}$.

Except $(1,0)$, we may view an entry (a,b) of an $\mathcal{F}$-sequence as the rational number $\frac{a}{b}$. This maps an $\mathcal{F}$-sequence to an ascending sequence of rational numbers. Axiom (2) corresponds to the above characteristic property of a Farey series. On positive roots (including $(1,0)$), we define
$$(a,b)\leq_{\mathbb{Q}}(c,d) \quad\Longleftrightarrow\quad ad\leq cb.$$

PROPOSITION 3.7. *To each $\mathcal{F}$-sequence f there is up to equivalence a unique connected simply connected Cartan scheme $\mathcal{C}$ such that $\mathcal{R}^{re}(\mathcal{C})$ is a finite root system and the set of entries of f is R^a_+ for some $a\in A$.*

Conversely, let $\mathcal{C}=\mathcal{C}(I,A,(\rho_i)_{i\in I},(C^a)_{a\in A})$ be a Cartan scheme of rank two. Assume that $\mathcal{R}^{re}(C)$ is a finite irreducible root system. Let $a\in A$. Then sorting the elements of $R^a\cap\mathbb{N}_0^2$ with respect to $\leq_{\mathbb{Q}}$ in ascending order yields an $\mathcal{F}$-sequence.

PROOF. For $c\in\mathcal{A}^+$ of length n, consider the Cartan scheme $\mathcal{C}$ with objects $a_1,\dots,a_{2n}$ such that $\Phi(2,a_1)=c$ and $\rho_2(a_1)=a_2$. The numbers $c_1,\dots,c_n$ are the Cartan entries used for certain reflections $s_i\in\mathrm{Hom}(a_{i+1},a_i)$, $1\leq i\leq 2n$, where $a_{2n+1}=a_1$. By the proof of [**16**, Thm. 6.6], the real positive roots at a_1 are
$$\{\beta_\nu:=s_1s_2\cdots s_\nu(\alpha_{2-(\nu \bmod 2)})\mid \nu=0,\dots,n-1\},$$
or equivalently
$$\{\tilde\beta_\nu:=s_{2n}^{-1}s_{2n-1}^{-1}\cdots s_{2n+1-\nu}^{-1}(\alpha_{(\nu \bmod 2)+1})\mid \nu=0,\dots,n-1\}.$$
For $0\leq\mu<n$, the equation $s_1\dots s_{\mu+1}(-\tau^n)=s_{2n}^{-1}\dots s_{n+\mu+2}^{-1}$ implies that $\beta_\mu=\tilde\beta_{n-1-\mu}$, thus the set of real positive roots is also
$$\{\beta_0,\dots,\beta_\nu,\tilde\beta_{n-2-\nu},\dots,\tilde\beta_0\}$$

for any fixed $0 \le \nu < n$. If using Definition 3.2(2),(3) we include a '1' in the sequence at position ν (and of course also at position $\nu + n$), i.e.

$$(\dots, c_\nu, c_{\nu+1}, \dots) \mapsto (\dots, c_\nu + 1, 1, c_{\nu+1} + 1, \dots),$$

this will not affect the roots $\tilde\beta_{n-2-\nu}, \dots, \tilde\beta_0, \beta_0, \dots, \beta_{\nu-1}$ coming from $a_{n+3+\nu}, \dots, a_{2n}, a_1, a_1, \dots, a_\nu$. Let $i, j \in I$ be such that $\rho_i(a_\nu) = a_{\nu+1}$ and $i \ne j$. If $\tilde s_\nu$, t, $\tilde s_{\nu+1}$ are the new morphisms corresponding to $c_\nu + 1$, 1, $c_{\nu+1} + 1$ respectively, then

$$s_1 \cdots s_{\nu-1} \tilde s_\nu(\alpha_j) = s_1 \cdots s_\nu(\alpha_j) + s_1 \cdots s_{\nu-1}(\alpha_i) = \beta_\nu + \beta_{\nu-1},$$

$$s_1 \cdots s_{\nu-1} \tilde s_\nu t(\alpha_i) = \beta_\nu.$$

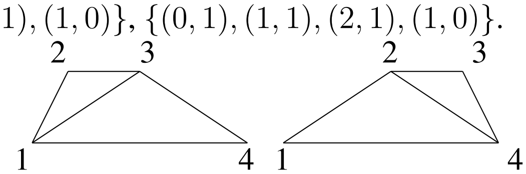
These will be the positive roots coming from the two objects replacing $a_{\nu+1}$. Hence $\beta_\nu + \beta_{\nu-1}$ is the new root after the transformation. Since the positive roots for the η-sequence $(1, 1, 1)$ are $((0, 1), (1, 1), (1, 0))$, this proves that $\mathcal{F}$-sequences are sets of positive roots.

Conversely, a set of positive roots uniquely determines an η-sequence by [**16**, Prop. 6.5], and this η-sequence corresponds to an $\mathcal{F}$-sequence by the above construction. □

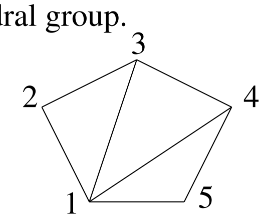
COROLLARY 3.8. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme of rank two. If $\mathcal{R}^{re}(\mathcal{C}) = (\mathcal{C}, (R^a)_{a\in A})$ is a finite root system of type $\mathcal{C}$ then, for all $a \in A$ and $\alpha \in R^a_+$, either α is simple or it is the sum of two positive roots in R^a_+.*

EXAMPLE 3.9. We determine the sets R^a_+ for all Cartan schemes $\mathcal{C}$ where $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system with at most 5 positive roots.

(1) With three positive roots: The η-sequence is $(1, 1, 1)$, it corresponds to the universal covering of the Weyl group of type A_2 (see Definition 3.10). The set of positive roots is $\{(0, 1), (1, 1), (1, 0)\}$.
(2) With four positive roots: There are two positions for including a '1', one obtains $(2, 1, 2, 1)$ or $(1, 2, 1, 2)$ which are equivalent with respect to $\sim$. The sets of positive roots are (type B_2 and C_2):
$\{(0, 1), (1, 2), (1, 1), (1, 0)\}$, $\{(0, 1), (1, 1), (2, 1), (1, 0)\}$.



(3) With five positive roots: There is only one triangulation of a convex pentagon up to operation of the dihedral group.



The corresponding sets of positive roots are

$\{(0,1),(1,3),(1,2),(1,1),(1,0)\}$ (assoc. to the given triangulation),
$\{(0,1),(1,1),(2,1),(3,1),(1,0)\}, \{(0,1),(1,2),(2,3),(1,1),(1,0)\}$,
$\{(0,1),(1,2),(1,1),(2,1),(1,0)\}, \{(0,1),(1,1),(3,2),(2,1),(1,0)\}$.

3.2. Finite rank two Weyl groupoids. After considering simply connected Cartan schemes, we may now investigate the quotients of these universal coverings. Coverings of Cartan schemes have been treated in [**16**, Sect. 3]. We recall the concept from [**16**, Def. 3.1]:

DEFINITION 3.10. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I, A', (\rho'_i)_{i\in I}, (C'^a)_{a\in A'})$ be connected Cartan schemes. Let $\pi : A' \to A$ be a map such that $C^{\pi(a)} = C'^a$ for all $a \in A'$ and the diagrams

(3.2)
$$\begin{array}{ccc} A' & \xrightarrow{\rho'_i} & A' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ A & \xrightarrow[\rho_i]{} & A \end{array}$$

commute for all $i \in I$. We say that $\pi : \mathcal{C}' \to \mathcal{C}$ is a *covering*, and that $\mathcal{C}'$ is a *covering of* $\mathcal{C}$.

By [**16**, Prop. 3.4] and [**16**, Cor. 3.6], coverings behave similarly as for example in topology: If $\mathcal{C}$ is a Cartan scheme and a an object, then for each subgroup $U \le \mathrm{End}(a)$ there exists up to equivalence a unique Cartan scheme $\mathcal{C}'$, a covering $\pi : \mathcal{C}' \to \mathcal{C}$ and an object b of $\mathcal{C}'$ such that $\mathrm{End}(b)$ and U are isomorphic via π (see the definition of the covariant functor F_π in [**16**, Sect. 3]). In particular if U is the trivial subgroup, then we obtain the universal covering.

Let $n \in \mathbb{N}$ and let $\mathcal{C}$ be a connected simply connected Cartan scheme of rank two with finite root system and $2n$ objects. By Definition 3.10, to obtain all coverings $\mathcal{C} \to \mathcal{C}'$, we have to determine the pairs of objects a, b of $\mathcal{C}$ for which $\Phi(i,a) = \Phi(i,b)$ for some (equivalently all) $i \in I$.

Consider the action of the dihedral group $\mathbb{D}_{2n}$ on the object change diagram of $\mathcal{C}$ and write $\mathbb{D}_{2n} = Z_{2n} \rtimes Z_2$, where Z_k is the cyclic group with k elements; let τ be the generator of the Z_2 part on the right and μ a generator of Z_{2n} rotating the object change diagram by one generating morphism. Let H be the following subgroup of $\mathbb{D}_{2n}$:

(3.3)
$$H = \{\mu^{2k}, \tau\mu^{2k+1} \in \mathbb{D}_{2n} \mid k \in \mathbb{Z}\}.$$

The group $\mathrm{End}(a')$ for an object a' of $\mathcal{C}'$ will at most be a subgroup of H. Indeed, let $i \in I$ and $a, b \in A$ with $\Phi(i,a) = \Phi(i,b)$. Then either the distance between a and

b in the object change diagram is even, then we just rotate the object change diagram; or the distance is odd, then we rotate and change direction.

We determine first the periodicity of an η-sequence. If c is a sequence, let c^r denote the r times repeated sequence c.

PROPOSITION 3.11. *Let c be an η-sequence. Write $c = (a_1, \dots, a_n)^r$ for suitable $r, n, a_1, \dots, a_n \in \mathbb{N}$. Then $r \in \{1, 2, 3\}$.*

PROOF. Observe first that by Definition 3.2 an η-sequence always contains a 1. If $n = 1$, then $c = (1, 1, 1)$ again by Definition 3.2, so $r = 3$.

Assume that $n = 2$. Then $c = (a, 1)^r$ for some $a \in \mathbb{N}$ and $a \neq 1$ since two consecutive 1's may never appear by construction unless the sequence is $(1, 1, 1)$. If $r = 2k$ then

$$c = (a, 1, a, 1)^k \approx (a-1, a-1, 1)^k \sim (a-1, 1, a-1)^k.$$

Thus if $a - 1 = 1$ then $r = 2$ and $c = (2, 1, 2, 1)$. Otherwise $a > 2$ and $(a-1, 1, a-1)^k \approx (a-2, a-2)^k$ which is a contradiction. If $r = 2k + 1$ for some $k \geq 1$, then $c = (a, 1, a, 1)^k(a, 1) \approx (a-1, a-1, 1)^k(a, 1)$ and therefore $a > 2$. The sequence $(a-1, a-1, 1)^k(a, 1)$ is equivalent to

$$(a-1, 1, a-1)^{k-1}(a-1, 1, a, 1, a-1) \approx (a-2)^{2k+1}$$

and hence $a = 3$, $k = 1$, i.e. $c = (3, 1, 3, 1, 3, 1)$ and $r = 3$.

Assume now that $n > 2$. Then without loss of generality $a_2 = 1$, $a_1, a_3 > 1$, and $c \approx (a_1 - 1, a_3 - 1, a_4, \dots, a_n)^r$ has length $r(n-1)$. Hence we conclude by induction on n that $r \in \{1, 2, 3\}$. $\square$

Call an η-sequence $c = (a_1, \dots, a_n)$ of "*type 1*" if there exists k such that

$$(a_k, \dots, a_n, a_1, \dots, a_{k-1}) = (a_k, a_{k-1}, \dots, a_1, a_n, \dots, a_{k+1}).$$

Otherwise, we say that c is of "*type 2*". We obtain:

PROPOSITION 3.12. *Let $\mathcal{C}$ be a connected simply connected Cartan scheme of rank two. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system. Let $a \in A$ and $c = \Phi(1, a)$. Let $\pi : \mathcal{C} \to \mathcal{C}'$ be the covering for which $\mathrm{End}(\pi(a))$ is maximal. Write $c = d^r$ for a sequence d and $r \in \mathbb{N}$ such that r is maximal. Let $m = |A|/2r$ be the length of d. Then $\mathrm{End}(\pi(a))$ is given by the following subgroup of H (see Equation* (3.3)*):*

m even: c	$r=1$	$r=2$	$r=3$
type 1	$Z_2 \times Z_2$	$\mathbb{D}_4$	$\mathbb{D}_6$
type 2	$Z_2 = \langle \mu^n \rangle$	$Z_4 = \langle \mu^{\frac{n}{2}} \rangle$	$Z_6 = \langle \mu^{\frac{n}{3}} \rangle$

m odd: c	$r=1$	$r=2$	$r=3$
type 1	Z_2	$Z_2 \times Z_2$	$Z_3 \rtimes Z_2$
type 2	1	$Z_2 = \langle \mu^n \rangle$	$Z_3 = \langle \mu^{\frac{2n}{3}} \rangle$

PROOF. Remember that if the η-sequence c is of length n then we have $2n$ objects and the Cartan entries used for the $2n$ generating morphisms are the entries of c^2. So by Proposition 3.11 and the discussion above it, we are looking for the elements σ of H fixing the sequence $c^2 = d^{2r}$, where $r \in \{1, 2, 3\}$. The result is obtained via case by case considerations. □

REMARK 3.13. To construct all Cartan schemes for which $\mathcal{C}$ is a covering, proceed in the following way: Choose a subgroup U of the group given in Proposition 3.12, and view it as a subgroup of $\mathbb{D}_{2n}$ as explained above Proposition 3.11, so U acts on the set A of objects of $\mathcal{C}$. Define A' to be the set of orbits of U on A, and $\pi : A \to A'$ maps an object to its orbit. Then set $C'^{\pi(a)} = C^a$ and ρ'_i are given by the commutative diagram in Definition 3.10. The quadruple $\mathcal{C}' = \mathcal{C}'(I, A', (\rho'_i)_{i\in I}, (C'^a)_{a\in A'})$ is then a Cartan scheme with finite root system and $\mathcal{C} \to \mathcal{C}'$ is a covering as defined in Definition 3.10.

4. Finite reflection groupoids and the Grassmannians

Let $n \in \mathbb{N}_{\geq 3}$. Recall from [**21**, Sects. 3.4, 4.1] that the cluster algebra of type A_n is constructed in the following way: Start with a convex n-gon and choose a triangulation by non-intersecting diagonals. Label these diagonals by $x_1, \dots, x_{n-3}$, the edges by $q_1, \dots, q_n$ and the vertices of the n-gon by $1, \dots, n$. The labels for the remaining diagonals are obtained by flipping diagonals; two adjacent triangles (x, a, b) and (x, c, d) yield a relation for the other diagonal y (see Figure 2):

$$xy = ac + bd.$$

The cluster algebra of type A_{n-3} is the subalgebra of the rational function field in $x_1, \dots, x_{n-3}, q_1, \dots, q_n$ generated by all the labels in the n-gon. In a simplified version, which will be our point of view in this text, the variables $q_1, \dots, q_n$ are specialized to 1.

Another description is by the Grassmannian $\mathrm{Gr}(2, n)$ (see [**21**, Lecture 3]): Take a matrix $z = (z_{ij}) \in K^{2\times n}$. For $1 \leq k < l \leq n$, let

$$P_{k,l} = \det \begin{pmatrix} z_{1k} & z_{1l} \\ z_{2k} & z_{2l} \end{pmatrix}.$$

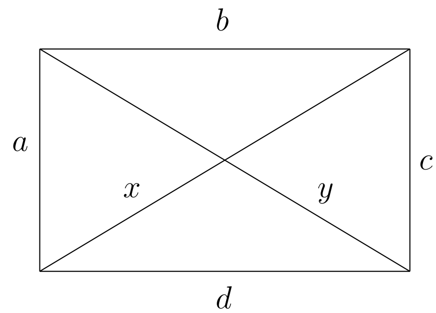


FIGURE 2. Diagonal flip

These are the Plücker coordinates of the row span of z as an element of the Grassmannian $\mathrm{Gr}(2,n)$. They satisfy the relations

$$P_{i,k}P_{j,l} = P_{i,j}P_{k,l} + P_{i,l}P_{j,k} \tag{4.1}$$

for all $i,j,k,l \in \{1,\dots,n\}$ with $i<j<k<l$. By assigning the value of $P_{i,j}$ to the variable x_k associated to the segment (i,j) of the n-gon, it follows by induction that the rational function associated to every diagonal is equal to the corresponding $P_{k,l}$.

Let $\mathcal{O}(\mathrm{Gr}(2,n))$ be the homogeneous coordinate ring of $\mathrm{Gr}(2,n)$. Equivalently, it is the commutative K-algebra generated by elements $P_{i,j}$ with $1\le i<j\le n$ and the Plücker relations above. Define

$$P_{i,j} = P_{j,i}$$

whenever $1\le j<i\le n$, and consider the indices of the Plücker coordinates as elements of the cyclic group Z_n. Let

$$\mathcal{G}(2,n) = \mathcal{O}(\mathrm{Gr}(2,n))/I, \quad I = (P_{i,i+1}-1 \mid i=1,\dots,n).$$

LEMMA 4.1. *The algebra $\mathcal{G}(2,n)$ is generated by the elements*

$$P_{1,3}, P_{2,4}, \dots, P_{n-2,n}.$$

PROOF. Let $\mathcal{G}$ be the subalgebra of $\mathcal{G}(2,n)$ generated by the elements $P_{i,i+2}$ with $1\le i\le n-2$. Let $i,j\in\{1,\dots,n\}$ with $i+2<j$. Then $P_{i,i+1}P_{i+2,j}+P_{i,j}P_{i+1,i+2} = P_{i,i+2}P_{i+1,j}$, that is

$$P_{i,j} = P_{i,i+2}P_{i+1,j} - P_{i+2,j}.$$

Hence $P_{i,j}\in\mathcal{G}$ for all $i,j\in\{1,\dots,n\}$ with $i+1<j$ by induction on $j-i$. This proves the claim. □

Let $\mathcal{C}$ be a connected simply connected Cartan scheme of rank two such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system. Assume that $n=|A|/2$ and that $I=\{1,2\}$. Recall that

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta(x) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$$

for all $x \in \mathbb{Z}$. Then

$$\sigma_1^a = \begin{pmatrix} -1 & -c_{12}^a \\ 0 & 1 \end{pmatrix} = \eta(-c_{12}^a)\tau,$$
$$\sigma_2^a = \begin{pmatrix} 1 & 0 \\ -c_{21}^a & -1 \end{pmatrix} = \tau\eta(-c_{21}^a)$$

for all $a \in A$, where the matrices are given with respect to the standard basis of $\mathbb{Z}^2$. Since $\tau^2 = \mathrm{id}$,

$$\begin{aligned}\mathrm{Prod}(n; a, 1, 2)\tau =& \ldots \sigma_1\sigma_2\sigma_1^a\tau \\ =& \ldots \sigma_1\tau\tau\sigma_2\sigma_1^a\tau \\ =&\tau^{n-1}\eta(c_n)\ldots\eta(c_2)\eta(c_1) = -\tau^{n-1}\end{aligned}$$

for all $a \in A$, where the last equation follows from [**16**, Prop. 6.5]. This argument shows that Axiom (2.2) is valid for $\mathcal{C}$:

COROLLARY 4.2. *Let $\mathcal{C}$ be a connected simply connected Cartan scheme of rank two. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. Then $\mathcal{W}(\mathcal{C})$ is a finite reflection groupoid.*

We now formulate our main result on reflection groupoids in purely algebraic terms. Recall that $n \in \mathbb{N}_{\geq 3}$.

THEOREM 4.3. *Let J be the ideal of $R = K[c_1, \ldots, c_n]$ generated by the coefficients of the 2×2-matrix*

$$\eta(c_n)\ldots\eta(c_2)\eta(c_1) + \mathrm{id}. \tag{4.2}$$

There is a unique algebra isomorphism

$$\bar{\varphi}:\ R/J \to \mathcal{G}(2, n), \quad c_i \mapsto P_{i,i+2}.$$

The inverse of $\bar{\varphi}$ can be defined by

$$\bar{\varphi}^{-1}(P_{i,j}) = (\eta(c_{j-2})\cdots\eta(c_{i+1})\eta(c_i))_{11} \quad \textit{for all } 1 \leq i < j \leq n,$$

where $(\cdot)_{11}$ denotes the entry in the first row and first column.

The proof of Theorem 4.3 uses several observations and lemmata which are collected here.

LEMMA 4.4. *Let J be the ideal of $R = K[c_1, \ldots, c_n]$ generated by the coefficients of the 2×2-matrix in Equation* (4.2). *The algebra R/J is generated by the elements $c_1, \ldots, c_{n-2}$.*

PROOF. Since $\eta(c)$ is invertible for all $c \in R$, Equation (4.2) implies that

$$\eta(c_{n-2})\cdots\eta(c_2)\eta(c_1) = -(\eta(c_n)\eta(c_{n-1}))^{-1}.$$

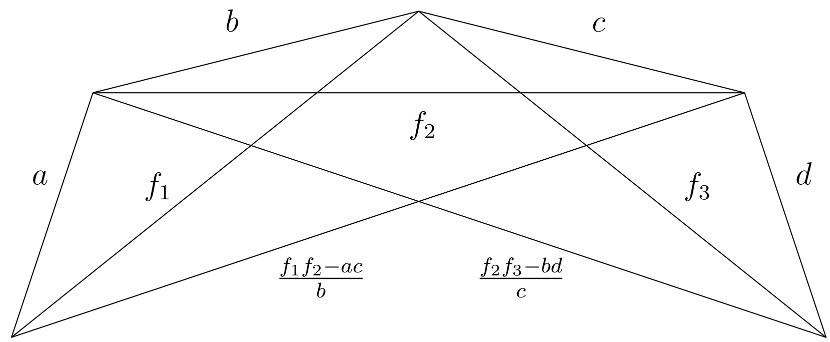


FIGURE 3. Equation (4.3)

Looking at the non-diagonal entries yields that c_{n-1} and c_n are polynomials in $c_1, \dots,$ c_{n-2}. □

For all $a, b, c \in K$ let

$$\mu(a,b,c) = \begin{pmatrix} a & -b \\ c & 0 \end{pmatrix} \in K^{2\times 2}.$$

Explicit calculation shows that (see Figure 3 for an illustration)

$$\begin{aligned} &\mu(f_1,a,b)\mu(f_2,b,c)\mu(f_3,c,d)f_2 = \\ &\qquad \mu(f_1f_2-ac,ab,bf_2)\mu(f_2f_3-bd,cf_2,cd) \end{aligned} \tag{4.3}$$

for all $f_1, f_2, f_3, a, b, c, d \in K$. Equation (4.3) is a generalization of [**16**, 5.2].

LEMMA 4.5. *Equation*

$$\prod_{i=1}^{n} \mu(P_{i,i+2}, P_{i,i+1}, P_{i+1,i+2}) = -\prod_{i=1}^{n} P_{i,i+1}\,\mathrm{id} \tag{4.4}$$

holds in $\mathcal{O}(\mathrm{Gr}(2,n))^{2\times 2}$.

PROOF. Recall that $P_{i,j} = P_{j,i}$ for all $i \neq j$. We proceed by induction on n. If $n = 3$, then Equation (4.4) becomes

$$\mu(P_{1,3}, P_{1,2}, P_{2,3})\mu(P_{1,2}, P_{2,3}, P_{1,3})\mu(P_{2,3}, P_{1,3}, P_{1,2}) = -P_{1,2}P_{2,3}P_{1,3}\mathrm{id},$$

which can be checked by explicit calculation. Similarly, Eq. (4.4) holds for $n = 4$.

Assume that $n > 4$. The subalgebra of $\mathcal{O}(\mathrm{Gr}(2,n))$ generated by all $P_{i,j}$ with $i, j \neq 3$ is isomorphic to $\mathcal{O}(\mathrm{Gr}(2,n-1))$, hence by induction hypothesis we have

$$\begin{aligned} \mu(P_{1,4}, P_{1,2}, P_{2,4})\mu(P_{2,5}, P_{2,4}, P_{4,5}) &\prod_{i=4}^{n} \mu(P_{i,i+2}, P_{i,i+1}, P_{i+1,i+2}) \\ &= -P_{1,2}P_{2,4}P_{4,5}\cdots P_{n,n+1}\mathrm{id}. \end{aligned} \tag{4.5}$$

By Equation (4.3) and the Grassmann-Plücker relations (4.1) we have

$$\begin{aligned}\mu(P_{1,3},P_{1,2},P_{2,3})\mu(P_{2,4},P_{2,3},P_{3,4})\mu(P_{3,5},P_{3,4},P_{4,5})P_{2,4}&\\ =\mu(P_{1,4},P_{1,2},P_{2,4})\mu(P_{2,5},P_{2,4},P_{4,5})P_{2,3}P_{3,4}.&\end{aligned}\tag{4.6}$$

By multiplying Equation (4.5) with $P_{2,3}P_{3,4}$, inserting Equation (4.6), and dividing by $P_{2,4}$ ($\mathcal{O}(\mathrm{Gr}(2,n))$ is an integral domain) we obtain Equation (4.4) which completes the induction. □

LEMMA 4.6. *Let S be a ring with 1. For all $x \in S$ define $\eta(x) \in S^{2\times 2}$ similarly to Equation (3.1). Let $x, y \in S$, $A \in S^{2\times 2}$, and $a = A_{11}$. Then $(\eta(x)A)_{21} = a$ and $(A\eta(y))_{12} = (\eta(x)A\eta(y))_{22} = -a$.*

PROOF OF THEOREM 4.3. There exists a unique algebra map $\varphi : R \to \mathcal{G}(2,n)$ such that $\varphi(c_i) = P_{i,i+2}$ for all i. By Lemma 4.1 this map is surjective. It remains to show that $\ker\varphi = J$. First we conclude that $\bar{\varphi}$ is well-defined, that is, $J \subseteq \ker\varphi$. Indeed, specializing $P_{i,i+1}$ to 1 for all $i = 1,\dots,n$ allows us to replace the factors $\mu(P_{i,i+2}, P_{i,i+1}, P_{i+1,i+2})$ in Equation (4.4) by $\eta(P_{i,i+2})$ for all $i \in \{1,2,\dots,n\}$. Thus $\varphi(J) = 0$, and hence $\bar{\varphi}$ is well-defined.

We prove the injectivity of $\bar{\varphi}$ by determining the inverse map. First let

$$\psi(P_{i,j}) = (\eta(c_{j-2})\cdots\eta(c_{i+1})\eta(c_i))_{11} \in R/J \tag{4.7}$$

for all $1 \le i < j \le n$. Then $\psi(P_{i,i+1}) = 1$ for all $i \in \{1,2,\dots,n-1\}$ and $\psi(P_{i,i+2}) = c_i$ for all $i \in \{1,2,\dots,n-2\}$. Lemma 4.6 implies that

$$\begin{aligned}\psi(P_{j,k+1}) &= \sum_{r=1}^{2}\eta(c_{k-1})_{1r}(\eta(c_{k-2})\cdots\eta(c_{j+1})\eta(c_j))_{r1}\\ &= c_{k-1}\psi(P_{j,k}) - \psi(P_{j,k-1})\end{aligned}\tag{4.8}$$

for all $1 \le j < k < n$, where the potential summand $\psi(P_{j,j})$ is considered as 0. Using Equation (4.8), one shows by induction on k that

$$\psi(P_{i,j}) = \psi(P_{i,k})\psi(P_{j,k+1}) - \psi(P_{j,k})\psi(P_{i,k+1}) \tag{4.9}$$

for all $1 \le i < j \le k < n$.

Let $\psi : \mathcal{O}(\mathrm{Gr}(2,n)) \to R/J$ be the unique algebra map where $\psi(P_{i,j})$ is as in Equation (4.7). We prove that ψ is well-defined. Let $i,j,k,l \in \{1,2,\dots,n\}$ with $i < j < k < l$. Then

$$\begin{aligned}\psi(P_{i,l})\psi(P_{j,k}) &= (\eta(c_{l-2})\cdots\eta(c_i))_{11}\psi(P_{j,k})\\ &= \sum_{r=1}^{2}(\eta(c_{l-2})\cdots\eta(c_k))_{1r}(\eta(c_{k-1})\cdots\eta(c_i))_{r1}\psi(P_{j,k})\\ &= (\psi(P_{k,l})\psi(P_{i,k+1}) - \psi(P_{k+1,l})\psi(P_{i,k}))\psi(P_{j,k}),\end{aligned}$$

where the last equation follows from Lemma 4.6. Apply Equation (4.9) to the first summand of the last expression. Then

$$\begin{aligned}\psi(P_{i,l})\psi(P_{j,k}) &= \psi(P_{k,l})\psi(P_{i,k})\psi(P_{j,k+1}) - \psi(P_{k,l})\psi(P_{i,j}) \\ &\quad -\psi(P_{k+1,l})\psi(P_{i,k})\psi(P_{j,k}) \\ &= \psi(P_{i,k})\psi(P_{j,l}) - \psi(P_{k,l})\psi(P_{i,j}),\end{aligned}$$

where the second equation is obtained from the definition of $\psi(P_{j,l})$ and Lemma 4.6. The obtained formula proves that ψ is well-defined. Since $\psi(P_{i,i+1}) = 1$ for all $i \in \{1, 2, \ldots, n-1\}$, it induces an algebra map $\bar{\psi} : \mathcal{G}(2, n) \to R/J$. Further, Lemma 4.4 tells that R/J is generated by the elements $c_1, \ldots, c_{n-2}$. Since $\bar{\psi}\bar{\varphi}(c_i) = \bar{\psi}(P_{i,i+2}) = c_i$ for all $i \in \{1, 2, \ldots, n-2\}$, we conclude that $\bar{\psi}\bar{\varphi} = \mathrm{id}_{R/J}$, and hence $\bar{\varphi}$ is injective. Clearly, $\bar{\psi}$ is the inverse of $\bar{\varphi}$. □

REMARK 4.7. Alternatively, we can prove that $J = \ker \varphi$ by constructing a matrix $z = (z_{ij})_{i,j} \in (R/J)^{2\times n}$ such that if

$$\mathrm{det}_z(i, j) := \det \begin{pmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{pmatrix} \quad \text{for} \quad 1 \le i, j \le n,$$

then $\mathrm{det}_z(i, i+1) = 1$ and $\mathrm{det}_z(i, i+2) = c_i$ for all $i = 1, \ldots, n$, where all indices are to be read modulo n and $\mathrm{det}_z(n-1, 1) = -c_{n-1}$, $\mathrm{det}_z(n, 2) = -c_n$, $\mathrm{det}_z(n, 1) = -1$, in the following way:

Let $\xi^{(i)} = -\eta(c_i)^{-1} \cdots \eta(c_2)^{-1}\eta(c_1)^{-1}$ and write $\bar{\xi}^{(i)} = (\xi^{(i)})^{-1}$; this is well-defined over R/J since $\det \eta(x) = 1$. For $i \equiv 1 \pmod 4$ set

$$\begin{aligned} z_{1i} &= \xi_{11}^{(i-1)}, & z_{1\,i+1} &= \xi_{11}^{(i)}, & z_{1\,i+2} &= -\bar{\xi}_{21}^{(i)}, & z_{1\,i+3} &= -\bar{\xi}_{21}^{(i+1)}, \\ z_{2i} &= \xi_{12}^{(i-1)}, & z_{2\,i+1} &= \xi_{12}^{(i)}, & z_{2\,i+2} &= \bar{\xi}_{11}^{(i)}, & z_{2\,i+3} &= \bar{\xi}_{11}^{(i+1)}. \end{aligned}$$

(Ignore the cases for which i, $i+1$, $i+2$ or $i+3$ are greater than n.) Several explicit calculations give that $\mathrm{det}_z(i, i+1) = 1$ for $i = 1, \ldots, n-1$ and $\mathrm{det}_z(i, i+2) = c_i$ for $i = 1, \ldots, n-2$ hold even in R. For $\mathrm{det}_z(n-1, 1) = -c_{n-1}$, $\mathrm{det}_z(n, 2) = -c_n$, and $\mathrm{det}_z(n, 1) = -1$ we use Equation (4.2).

Recall that the determinants satisfy the Plücker relations. Hence, once the matrix z is constructed, it follows that there is an algebra map from $\mathcal{G}(2, n)$ to R/J which maps $P_{i,i+2}$ to c_i.

CHAPTER 5

Finite Weyl groupoids of rank three

M. CUNTZ AND I. HECKENBERGER
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We continue our study of Cartan schemes and their Weyl groupoids and obtain a complete list of all connected simply connected Cartan schemes of rank three for which the real roots form a finite irreducible root system. We achieve this result by providing an algorithm which determines all the root systems and eventually terminates: Up to equivalence there are exactly 55 such Cartan schemes, and the number of corresponding real roots varies between 6 and 37. We identify those Weyl groupoids which appear in the classification of Nichols algebras of diagonal type.

1. Introduction

Root systems associated with Cartan matrices are widely studied structures in many areas of mathematics, see [**11**] for the fundaments. The origins of the theory of root systems go back at least to the study of Lie groups by Lie, Killing and Cartan. The symmetry of the root system is commonly known as its Weyl group. Root systems associated with a family of Cartan matrices appeared first in connection with Lie superalgebras [**36**, Prop. 2.5.6] and with Nichols algebras [**25**], [**29**]. The corresponding symmetry is not a group but a groupoid, and is called the Weyl groupoid of the root system.

Weyl groupoids of root systems properly generalize Weyl groups. The nice properties of this more general structure have been the main motivation to develop an axiomatic approach to the theory, see [**34**], [**17**]. In particular, Weyl groupoids are generated by reflections and Coxeter relations, and they satisfy a Matsumoto type theorem [**34**]. To see more clearly the extent of generality it would be desirable to have a classification of finite Weyl groupoids.[1] However, already the appearance of a large family

[1] In this introduction by a Weyl groupoid we will mean the Weyl groupoid of a connected Cartan scheme, and we assume that the real roots associated to the Weyl groupoid form an irreducible root system in the sense of [**17**]. Further, our sloppy but meaningful terminology of classification of finite Weyl groupoids should be understood as the classification of connected simply connected Cartan schemes such that this root system is finite.

of examples of Lie superalgebras and Nichols algebras of diagonal type indicated that a classification of finite Weyl groupoids is probably much more complicated than the classification of finite Weyl groups. Additionally, many of the usual classification tools are not available in this context because of the lack of the adjoint action and a positive definite bilinear form.

In previous work, see [**16**] and [**15**], we have been able to determine all finite Weyl groupoids of rank two. The result of this classification is surprisingly nice: We found a close relationship to the theory of continued fractions and to cluster algebras of type A. The structure of finite rank two Weyl groupoids and the associated root systems has a natural characterization in terms of triangulations of convex polygons by non-intersecting diagonals. In particular, there are infinitely many such groupoids.

At first view there is no reason to assume that the situation for finite Weyl groupoids of rank three would be much different from the rank two case. In this paper we first give some theoretical indications which strengthen the opposite point of view. For example in Theorem 3.13 we show that the entries of the Cartan matrices in a finite Weyl groupoid cannot be smaller than -7. Recall that for Weyl groupoids there is no lower bound for the possible entries of generalized Cartan matrices. Our main achievement in this paper is then to provide an algorithm to classify finite Weyl groupoids of rank three. Our algorithm terminates within a short time, and produces a complete list of representatives of root systems. In the appendix we list the root systems characterizing the Weyl groupoids of the classification: There are 55 of them which correspond to pairwise non-isomorphic Weyl groupoids. The number of positive roots in these root systems varies between 6 and 37. Among our root systems are the usual root systems of type A_3, B_3, and C_3, but for most of the other examples we don't have yet an explanation.

It is remarkable that the number 37 has a particular meaning for simplicial arrangements in the real projective plane. An arrangement is the complex generated by a family of straight lines not forming a pencil. The vertices of the complex are the intersection points of the lines, the edges are the segments of the lines between two vertices, and the faces are the connected components of the complement of the set of lines generating the arrangement. An arrangement is called simplicial, if all faces are triangles. Simplicial arrangements have been introduced in [**38**]. The classification of simplicial arrangements in the real projective plane is an open problem. The largest known exceptional example is generated by 37 lines. Grünbaum conjectures that the list given in [**23**] is complete. In our appendix we provide some data of our root systems which can be used to compare Grünbaum's list with Weyl groupoids. There is an astonishing analogy between the two lists, but more work has to be done to be able to explain the precise relationship. This would be desirable in particular since our classification of finite Weyl groupoids of rank three does not give any evidence for the range of solutions besides the explicit computer calculation.

In order to ensure the termination of our algorithm, besides Theorem 3.13 we use a weak convexity property of certain affine hyperplanes, see Theorem 3.11: We can show that any positive root in an affine hyperplane "next to the origin" is either simple or can can be written as the sum of a simple root and another positive root. Our algorithm finally becomes practicable by the use of Proposition 3.6, which can be interpreted as another weak convexity property for affine hyperplanes. It is hard to say which of these theorems are the most valuable because avoiding any of them makes the algorithm impracticable (unless one has some replacement).

It should also be mentioned that using the knowledge on finite Weyl groupoids of rank three and an enhanced version of the algorithm, it is now possible to push the classification up to rank seven. It is thus conceivable that with some additional theory a complete classification will be within reach.

The paper is organized as follows. We start with two sections proving the necessary theorems to formulate the algorithm: The results which do not require that the rank is three are in Section 2, the obstructions for rank three in Section 3. We then describe the algorithm in the next section. Finally we summarize the resulting data and make some observations in the last section.

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2. Cartan schemes and Weyl groupoids

We mainly follow the notation in [**17, 16**]. The fundaments of the general theory have been developed in [**34**] using a somewhat different terminology. Let us start by recalling the main definitions.

Let I be a non-empty finite set and $\{\alpha_i \,|\, i \in I\}$ the standard basis of $\mathbb{Z}^I$. By [**37**, §1.1] a generalized Cartan matrix $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \le 0$ for all $i, j, k \in I$ with $j \ne k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c_{ij}^a = c_{ij}^{\rho_i(a)}$ for all $a \in A$ and $i, j \in I$.

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma_i^a \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma_i^a(\alpha_j) = \alpha_j - c_{ij}^a \alpha_i \qquad \text{for all } j \in I. \tag{2.1}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are compositions of maps σ_i^a with $i \in I$ and $a \in A$, where σ_i^a is considered as an element in $\mathrm{Hom}(a, \rho_i(a))$. The category $\mathcal{W}(\mathcal{C})$ is a groupoid in the sense that all morphisms are isomorphisms. The set of morphisms of $\mathcal{W}(\mathcal{C})$ is denoted by $\mathrm{Hom}(\mathcal{W}(\mathcal{C}))$, and we use the notation

$$\mathrm{Hom}(a, \mathcal{W}(\mathcal{C})) = \mathop{\cup}_{b\in A} \mathrm{Hom}(a, b) \quad \text{(disjoint union)}.$$

For notational convenience we will often neglect upper indices referring to elements of A if they are uniquely determined by the context. For example, the morphism $\sigma_{i_1}^{\rho_{i_2}\cdots\rho_{i_k}(a)} \cdots \sigma_{i_{k-1}}^{\rho_{i_k}(a)} \sigma_{i_k}^a \in \mathrm{Hom}(a, b)$, where $k \in \mathbb{N}$, $i_1, \dots, i_k \in I$, and $b = \rho_{i_1} \cdots \rho_{i_k}(a)$, will be denoted by $\sigma_{i_1} \cdots \sigma_{i_k}^a$ or by $\mathrm{id}_b \sigma_{i_1} \cdots \sigma_{i_k}$. The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$. A Cartan scheme is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$. The Cartan scheme is called *simply connected*, if $\mathrm{Hom}(a, a) = \{\mathrm{id}_a\}$ for all $a \in A$.

Let $\mathcal{C}$ be a Cartan scheme. For all $a \in A$ let

$$(R^{\mathrm{re}})^a = \{\mathrm{id}_a \sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \,|\, k \in \mathbb{N}_0,\, i_1, \dots, i_k, j \in I\} \subset \mathbb{Z}^I.$$

The elements of the set $(R^{\mathrm{re}})^a$ are called *real roots* (at a). The pair $(\mathcal{C}, ((R^{\mathrm{re}})^a)_{a\in A})$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. A real root $\alpha \in (R^{\mathrm{re}})^a$, where $a \in A$, is called positive (resp. negative) if $\alpha \in \mathbb{N}_0^I$ (resp. $\alpha \in -\mathbb{N}_0^I$). In contrast to real roots associated to a single generalized Cartan matrix, $(R^{\mathrm{re}})^a$ may contain elements which are neither positive nor negative. A good general theory, which is relevant for example for the study of Nichols algebras, can be obtained if $(R^{\mathrm{re}})^a$ satisfies additional properties.

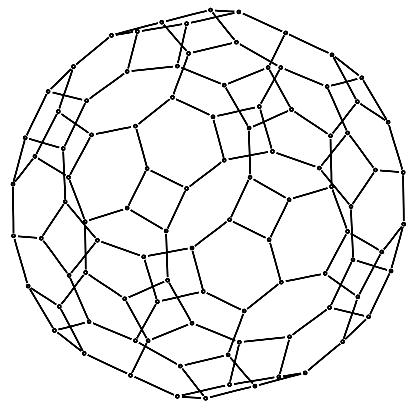
Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subset \mathbb{Z}^I$, and define $m_{i,j}^a = |R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R_+^a \cup -R_+^a$, where $R_+^a = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma_i^a(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m_{i,j}^a$ is finite, then $(\rho_i\rho_j)^{m_{i,j}^a}(a) = a$.

FIGURE 1. The object change diagram of a Cartan scheme of rank three (nr. 15 in Table 1)



The axioms (R2) and (R3) are always fulfilled for $\mathcal{R}^{\mathrm{re}}$. The root system $\mathcal{R}$ is called *finite* if for all $a \in A$ the set R^a is finite. By [**17**, Prop. 2.12], if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R} = \mathcal{R}^{\mathrm{re}}$, and hence $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$ in that case.

In [**17**, Def. 4.3] the concept of an *irreducible* root system of type $\mathcal{C}$ was defined. By [**17**, Prop. 4.6], if $\mathcal{C}$ is a Cartan scheme and $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R}$ is irreducible if and only if for all $a \in A$ the generalized Cartan matrix C^a is indecomposable. If $\mathcal{C}$ is also connected, then it suffices to require that there exists $a \in A$ such that C^a is indecomposable.

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. Let Γ be a nondirected graph, such that the vertices of Γ correspond to the elements of A. Assume that for all $i \in I$ and $a \in A$ with $\rho_i(a) \neq a$ there is precisely one edge between the vertices a and $\rho_i(a)$ with label i, and all edges of Γ are given in this way. The graph Γ is called the *object change diagram* of $\mathcal{C}$.

In the rest of this section let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system. For brevity we will write R^a instead of $(R^{\mathrm{re}})^a$ for all $a \in A$. We say that a subgroup $H \subset \mathbb{Z}^I$ is a *hyperplane* if $\mathbb{Z}^I/H \cong \mathbb{Z}$. Then $\mathrm{rk}\, H = \#I - 1$ is the rank of H. Sometimes we will identify $\mathbb{Z}^I$ with its image under the canonical embedding $\mathbb{Z}^I \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^I \cong \mathbb{Q}^I$.

LEMMA 2.1. *Let $a \in A$ and let $H \subset \mathbb{Z}^I$ be a hyperplane. Suppose that H contains $\mathrm{rk}\, H$ linearly independent elements of R^a. Let $\mathfrak{n}_H$ be a normal vector of H in $\mathbb{Q}^I$ with respect to a scalar product $(\cdot,\cdot)$ on $\mathbb{Q}^I$. If $(\mathfrak{n}_H, \alpha) \geq 0$ for all $\alpha \in R^a_+$, then H contains $\mathrm{rk}\, H$ simple roots, and all roots contained in H are linear combinations of these simple roots.*

PROOF. The assumptions imply that any positive root in H is a linear combination of simple roots contained in H. Since $R^a = R^a_+ \cup -R^a_+$, this implies the claim. $\square$

LEMMA 2.2. *Let $a \in A$ and let $H \subset \mathbb{Z}^I$ be a hyperplane. Suppose that H contains $\operatorname{rk} H$ linearly independent elements of R^a. Then there exist $b \in A$ and $w \in \operatorname{Hom}(a, b)$ such that $w(H)$ contains $\operatorname{rk} H$ simple roots.*

PROOF. Let $(\cdot,\cdot)$ be a scalar product on $\mathbb{Q}^I$. Choose a normal vector $\mathfrak{n}_H$ of H in $\mathbb{Q}^I$ with respect to $(\cdot,\cdot)$. Let $m = \#\{\alpha \in R^a_+ \,|\, (\mathfrak{n}_H, \alpha) < 0\}$. Since $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is finite, m is a nonnegative integer. We proceed by induction on m. If $m = 0$, then H contains $\operatorname{rk} H$ simple roots by Lemma 2.1. Otherwise let $j \in I$ with $(\mathfrak{n}_H, \alpha_j) < 0$. Let $a' = \rho_j(a)$ and $H' = \sigma^a_j(H)$. Then $\sigma^a_j(\mathfrak{n}_H)$ is a normal vector of H' with respect to the scalar product $(\cdot,\cdot)' = (\sigma^{\rho_j(a)}_j(\cdot), \sigma^{\rho_j(a)}_j(\cdot))$. Since $\sigma^a_j : R^a_+ \setminus \{\alpha_j\} \to R^{a'}_+ \setminus \{\alpha_j\}$ is a bijection and $\sigma^a_j(\alpha_j) = -\alpha_j$, we conclude that

$$\#\{\beta \in R^{a'}_+ \,|\, (\sigma^a_j(\mathfrak{n}_H), \beta)' < 0\} = \#\{\alpha \in R^a_+ \,|\, (\mathfrak{n}_H, \alpha) < 0\} - 1.$$

By induction hypothesis there exists $b \in A$ and $w' \in \operatorname{Hom}(a', b)$ such that $w'(H')$ contains $\operatorname{rk} H' = \operatorname{rk} H$ simple roots. Then the claim of the lemma holds for $w = w'\sigma^a_j$. $\square$

The following "volume" functions will be useful for our analysis. Let $k \in \mathbb{N}$ with $k \le \#I$. By the Smith normal form there is a unique left $\mathrm{GL}(\mathbb{Z}^I)$-invariant right $\mathrm{GL}(\mathbb{Z}^k)$-invariant function $\mathrm{Vol}_k : (\mathbb{Z}^I)^k \to \mathbb{Z}$ such that

$$\mathrm{Vol}_k(a_1\alpha_1, \dots, a_k\alpha_k) = |a_1 \cdots a_k| \quad \text{for all } a_1, \dots, a_k \in \mathbb{Z}, \tag{2.2}$$

where $|\cdot|$ denotes absolute value. In particular, if $k = 1$ and $\beta \in \mathbb{Z}^I \setminus \{0\}$, then $\mathrm{Vol}_1(\beta)$ is the largest integer v such that $\beta = v\beta'$ for some $\beta' \in \mathbb{Z}^I$. Further, if $k = \#I$ and $\beta_1, \dots, \beta_k \in \mathbb{Z}^I$, then $\mathrm{Vol}_k(\beta_1, \dots, \beta_k)$ is the absolute value of the determinant of the matrix with columns $\beta_1, \dots, \beta_k$.

Let $a \in A$, $k \in \{1, 2, \dots, \#I\}$, and let $\beta_1, \dots, \beta_k \in R^a$ be linearly independent elements. We write $V^a(\beta_1, \dots, \beta_k)$ for the unique maximal subgroup $V \subseteq \mathbb{Z}^I$ of rank k which contains $\beta_1, \dots, \beta_k$. Then $\mathbb{Z}^I / V^a(\beta_1, \dots, \beta_k)$ is free. In particular, $V^a(\beta_1, \dots, \beta_{\#I}) = \mathbb{Z}^I$ for all $a \in A$ and any linearly independent subset $\{\beta_1, \dots, \beta_{\#I}\}$ of R^a.

DEFINITION 2.3. Let $W \subseteq \mathbb{Z}^I$ be a cofree subgroup (that is, $\mathbb{Z}^I/W$ is free) of rank k. We say that $\{\beta_1, \dots, \beta_k\}$ is a *base for W at a*, if $\beta_i \in W$ for all $i \in \{1, \dots, k\}$ and $W \cap R^a \subseteq \sum_{i=1}^k \mathbb{N}_0\beta_i \cup -\sum_{i=1}^k \mathbb{N}_0\beta_i$.

Now we discuss the relationship of linearly independent roots in a root system. Recall that $\mathcal{C}$ is a Cartan scheme such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$.

THEOREM 2.4. *Let $a \in A$, $k \in \{1, \dots, \#I\}$, and let $\beta_1, \dots, \beta_k \in R^a$ be linearly independent roots. Then there exist $b \in A$, $w \in \mathrm{Hom}(a, b)$, and a permutation τ of I such that*

$$w(\beta_i) \in \mathrm{span}_{\mathbb{Z}}\{\alpha_{\tau(1)}, \dots, \alpha_{\tau(i)}\} \cap R^b_+$$

for all $i \in \{1, \dots, k\}$.

PROOF. Let $r = \#I$. Since R^a contains r simple roots, any linearly independent subset of R^a can be enlarged to a linearly independent subset of r elements. Hence it suffices to prove the theorem for $k = r$. We proceed by induction on r. If $r = 1$, then the claim holds.

Assume that $r > 1$. Lemma 2.2 with $H = V^a(\beta_1, \dots, \beta_{r-1})$ tells that there exist $d \in A$ and $v \in \mathrm{Hom}(a, d)$ such that $v(H)$ is spanned by simple roots. By multiplying v from the left with the longest element of $\mathcal{W}(\mathcal{C})$ in the case that $v(\beta_r) \in -\mathbb{N}_0^I$, we may even assume that $v(\beta_r) \in \mathbb{N}_0^I$. Now let J be the subset of I such that $\#J = r - 1$ and $\alpha_i \in v(H)$ for all $i \in J$. Consider the restriction $\mathcal{R}^{\mathrm{re}}(\mathcal{C})|_J$ of $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ to the index set J, see [**17**, Def. 4.1]. Since $v(\beta_i) \in H$ for all $i \in \{1, \dots, r-1\}$, induction hypothesis provides us with $b \in A$, $u \in \mathrm{Hom}(d, b)$, and a permutation τ' of J such that u is a product of simple reflections σ_i, where $i \in J$, and

$$uv(\beta_n) \in \mathrm{span}_{\mathbb{Z}}\{\alpha_{\tau'(j_1)}, \dots, \alpha_{\tau'(j_n)}\} \cap R^b_+$$

for all $n \in \{1, 2, \dots, r-1\}$, where $J = \{j_1, \dots, j_{r-1}\}$. Since $v(\beta_r) \notin v(H)$ and $v(\beta_r) \in \mathbb{N}_0^I$, the i-th entry of $v(\beta_r)$, where $i \in I \setminus J$, is positive. This entry does not change if we apply u. Therefore $uv(\beta_r) \in \mathbb{N}_0^I$, and hence the theorem holds with $w = uv \in \mathrm{Hom}(a, b)$ and with τ the unique permutation with $\tau(n) = \tau'(j_n)$ for all $n \in \{1, \dots, r-1\}$. □

COROLLARY 2.5. *Let $a \in A$, $k \in \{1, \dots, \#I\}$, and let $\beta_1, \dots, \beta_k \in R^a$ be linearly independent elements. Then $\{\beta_1, \dots, \beta_k\}$ is a base for $V^a(\beta_1, \dots, \beta_k)$ at a if and only if there exist $b \in A$, $w \in \mathrm{Hom}(a, b)$, and a permutation τ of I such that $w(\beta_i) = \alpha_{\tau(i)}$ for all $i \in \{1, \dots, k\}$. In this case $\mathrm{Vol}_k(\beta_1, \dots, \beta_k) = 1$.*

PROOF. The if part of the claim holds by definition of a base and by the axioms for root systems.

Let b, w and τ be as in Theorem 2.4. Let $i \in \{1, \dots, k\}$. The elements $w(\beta_1), \dots, w(\beta_i)$ are linearly independent and are contained in $V^b(\alpha_{\tau(1)}, \dots, \alpha_{\tau(i)})$. Thus $\alpha_{\tau(i)}$ is a rational linear combination of $w(\beta_1), \dots, w(\beta_i)$. Now by assumption, $\{w(\beta_1), \dots, w(\beta_k)\}$ is a base for $V^b(w(\beta_1), \dots, w(\beta_k))$ at b. Hence $\alpha_{\tau(i)}$ is a linear combination of the positive roots $w(\beta_1), \dots, w(\beta_i)$ with nonnegative integer coefficients. This is possible only if $\{w(\beta_1), \dots, w(\beta_i)\}$ contains $\alpha_{\tau(i)}$. By induction on i we obtain that $\alpha_{\tau(i)} = w(\beta_i)$. □

In the special case $k = \#I$ the above corollary tells that the bases of $\mathbb{Z}^I$ at an object $a \in A$ are precisely those subsets which can be obtained as the image, up to a permutation, of the standard basis of $\mathbb{Z}^I$ under the action of an element of $\mathcal{W}(\mathcal{C})$.

In [**15**] the notion of an $\mathcal{F}$-sequence was given, and it was used to explain the structure of root systems of rank two. Consider on $\mathbb{N}_0^2$ the total ordering $\leq_\mathbb{Q}$, where $(a_1, a_2) \leq_\mathbb{Q} (b_1, b_2)$ if and only if $a_1 b_2 \leq a_2 b_1$. A finite sequence $(v_1, \dots, v_n)$ of vectors in $\mathbb{N}_0^2$ is an $\mathcal{F}$-sequence if and only if $v_1 <_\mathbb{Q} v_2 <_\mathbb{Q} \cdots <_\mathbb{Q} v_n$ and one of the following holds.

- $n = 2$, $v_1 = (0, 1)$, and $v_2 = (1, 0)$.
- $n > 2$ and there exists $i \in \{2, 3, \dots, n-1\}$ such that $v_i = v_{i-1} + v_{i+1}$ and $(v_1, \dots, v_{i-1}.v_{i+1}, \dots, v_n)$ is an $\mathcal{F}$-sequence.

In particular, any $\mathcal{F}$-sequence of length ≥ 3 contains $(1, 1)$.

PROPOSITION 2.6. [**15**, Prop. 3.7] *Let $\mathcal{C}$ be a Cartan scheme of rank two. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system. Then for any $a \in A$ the set R^a_+ ordered by $\leq_\mathbb{Q}$ is an $\mathcal{F}$-sequence.*

PROPOSITION 2.7. [**15**, Cor. 3.8] *Let $\mathcal{C}$ be a Cartan scheme of rank two. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system. Let $a \in A$ and let $\beta \in R^a_+$. Then either β is simple or it is the sum of two positive roots.*

COROLLARY 2.8. *Let $a \in A$, $n \in \mathbb{N}$, and let $\alpha, \beta \in R^a$ such that $\beta - n\alpha \in R^a$. Assume that $\{\alpha, \beta - n\alpha\}$ is a base for $V^a(\alpha, \beta)$ at a. Then $\beta - k\alpha \in R^a$ for all $k \in \{1, 2, \dots, n\}$.*

PROOF. By Corollary 2.5 there exist $b \in A$, $w \in \mathrm{Hom}(a, b)$, and $i, j \in I$ such that $w(\alpha) = \alpha_i$, $w(\beta - n\alpha) = \alpha_j$. Then $n\alpha_i + \alpha_j = w(\beta) \in R^b_+$. Hence $(n-k)\alpha_i + \alpha_j \in R^b$ for all $k \in \{1, 2, \dots, n\}$ by Proposition 2.7 and (R2). This yields the claim of the corollary. □

COROLLARY 2.9. *Let $a \in A$, $k \in \mathbb{Z}$, and $i, j \in I$ such that $i \neq j$. Then $\alpha_j + k\alpha_i \in R^a$ if and only if $0 \leq k \leq -c^a_{ij}$,*

PROOF. Axiom (R1) tells that $\alpha_j + k\alpha_i \notin R^a$ if $k < 0$. Since $c^{\rho_i(a)}_{ij} = c^a_{ij}$ by (C2), Axiom (R3) gives that $\alpha_j - c^a_{ij}\alpha_i = \sigma^{\rho_i(a)}_i(\alpha_j) \in R^a$ and that $\alpha_j + k\alpha_i \notin R^a$ if $k > -c^a_{ij}$. Finally, if $0 < k < -c^a_{ij}$ then $\alpha_j + k\alpha_i \in R^a$ by Corollary 2.8 for $\alpha = \alpha_i$, $\beta = \alpha_j - c^a_{ij}\alpha_i$, and $n = -c^a_{ij}$. □

Proposition 2.7 implies another important fact.

THEOREM 2.10. *Let $\mathcal{C}$ be a Cartan scheme. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$. Let $a \in A$ and $\alpha \in R^a_+$. Then either α is simple, or it is the sum of two positive roots.*

PROOF. Assume that α is not simple. Let $i \in I$, $b \in A$, and $w \in \mathrm{Hom}(b,a)$ such that $\alpha = w(\alpha_i)$. Then $\ell(w) > 0$. We may assume that for all $j \in I$, $b' \in A$, and $w' \in \mathrm{Hom}(b',a)$ with $w'(\alpha_j) = \alpha$ we have $\ell(w') \geq \ell(w)$. Since $w(\alpha_i) \in \mathbb{N}_0^I$, we obtain that $\ell(w\sigma_i) > \ell(w)$ [**34**, Cor. 3]. Therefore, there is a $j \in I \setminus \{i\}$ with $\ell(w\sigma_j) < \ell(w)$. Let $w = w_1 w_2$ such that $\ell(w) = \ell(w_1) + \ell(w_2)$, $\ell(w_1)$ minimal and $w_2 = \ldots \sigma_i \sigma_j \sigma_i \sigma_j^b$. Assume that $w_2 = \sigma_i \cdots \sigma_i \sigma_j^b$ — the case $w_2 = \sigma_j \cdots \sigma_i \sigma_j^b$ can be treated similarly. The length of w_1 is minimal, thus $\ell(w_1\sigma_j) > \ell(w_1)$, and $\ell(w) = \ell(w_1) + \ell(w_2)$ yields that $\ell(w_1\sigma_i) > \ell(w_1)$. Using once more [**34**, Cor. 3] we conclude that

$$w_1(\alpha_i) \in \mathbb{N}_0^I, \quad w_1(\alpha_j) \in \mathbb{N}_0^I. \tag{2.3}$$

Let $\beta = w_2(\alpha_i)$. Then $\beta \in \mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j$, since $\ell(w_2\sigma_i) > \ell(w_2)$. Moreover, β is not simple. Indeed, $\alpha = w(\alpha_i) = w_1(\beta)$, so β is not simple, since $\ell(w_1) < \ell(w)$ and $\ell(w)$ was chosen of minimal length. By Proposition 2.7 we conclude that β is the sum of two positive roots $\beta_1, \beta_2 \in \mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j$. It remains to check that $w_1(\beta_1)$, $w_1(\beta_2)$ are positive. But this follows from (2.3). □

3. Obstructions for Weyl groupoids of rank three

In this section we analyze the structure of finite Weyl groupoids of rank three. Let $\mathcal{C}$ be a Cartan scheme of rank three, and assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. In this case a hyperplane in $\mathbb{Z}^I$ is the same as a cofree subgroup of rank two, which will be called a *plane* in the sequel. For simplicity we will take $I = \{1,2,3\}$, and we write R^a_+ for the set of positive real roots at $a \in A$.

Recall the definition of the functions Vol_k, where $k \in \{1,2,3\}$, from the previous section. As noted, for three elements $\alpha, \beta, \gamma \in \mathbb{Z}^3$ we have $\mathrm{Vol}_3(\alpha,\beta,\gamma) = 1$ if and only if $\{\alpha,\beta,\gamma\}$ is a basis of $\mathbb{Z}^3$. Also, we will heavily use the notion of a base, see Definition 2.3.

LEMMA 3.1. *Let $a \in A$ and $\alpha, \beta \in R^a$. Assume that $\alpha \neq \pm\beta$ and that $\{\alpha,\beta\}$ is not a base for $V^a(\alpha,\beta)$ at a. Then there exist $k, l \in \mathbb{N}$ and $\delta \in R^a$ such that $\beta - k\alpha = l\delta$ and $\{\alpha,\delta\}$ is a base for $V^a(\alpha,\beta)$ at a.*

PROOF. By Theorem 2.4 there exists an object b and $w \in \mathrm{Hom}(a,b)$ such that $w(\alpha)$ is simple and $w(\beta) = kw(\alpha) + lw(\delta)$ for some simple root $w(\delta)$ and $k, l \in \mathbb{Z}_{\geq 0}$. Thus $\beta = k\alpha + l\delta$. The relation $\beta \neq \delta$ follows from the assumption that $\{\alpha,\beta\}$ is not a base for $V^a(\alpha,\beta)$ at a. □

LEMMA 3.2. *Let $a \in A$ and $\alpha, \beta \in R^a$ such that $\alpha \neq \pm\beta$. Then $\{\alpha, \beta\}$ is a base for $V^a(\alpha, \beta)$ if and only if $\mathrm{Vol}_2(\alpha, \beta) = 1$ and $\alpha - \beta \notin R^a$.*

PROOF. Assume first that $\{\alpha, \beta\}$ is a base for $V^a(\alpha, \beta)$ at a. By Corollary 2.5 we may assume that α and β are simple roots. Therefore $\mathrm{Vol}_2(\alpha, \beta) = 1$ and $\alpha - \beta \notin R^a$.

Conversely, assume that $\mathrm{Vol}_2(\alpha, \beta) = 1$, $\alpha - \beta \notin R^a$, and that $\{\alpha, \beta\}$ is not a base for $V^a(\alpha, \beta)$ at a. Let k, l, δ as in Lemma 3.1. Then

$$1 = \mathrm{Vol}_2(\alpha, \beta) = \mathrm{Vol}_2(\alpha, \beta - k\alpha) = l\mathrm{Vol}_2(\alpha, \delta).$$

Hence $l = 1$, and $\{\alpha, \delta\} = \{\alpha, \beta - k\alpha\}$ is a base for $V^a(\alpha, \beta)$ at a. Then $\beta - \alpha \in R^a$ by Corollary 2.8 and since $k > 0$. This gives the desired contradiction to the assumption $\alpha - \beta \notin R^a$. □

Recall that a semigroup ordering $<$ on a commutative semigroup $(S, +)$ is a total ordering such that for all $s, t, u \in S$ with $s < t$ the relations $s + u < t + u$ hold. For example, the lexicographic ordering on $\mathbb{Z}^I$ induced by any total ordering on I is a semigroup ordering.

LEMMA 3.3. *Let $a \in A$, and let $V \subset \mathbb{Z}^I$ be a plane containing at least two positive roots of R^a. Let $<$ be a semigroup ordering on $\mathbb{Z}^I$ such that $0 < \gamma$ for all $\gamma \in R^a_+$, and let α, β denote the two smallest elements in $V \cap R^a_+$ with respect to $<$. Then $\{\alpha, \beta\}$ is a base for V at a.*

PROOF. Let α be the smallest element of $V \cap R^a_+$ with respect to $<$, and let β be the smallest element of $V \cap (R^a_+ \setminus \{\alpha\})$. Then $V = V^a(\alpha, \beta)$ by (R2). By Lemma 3.1 there exists $\delta \in V \cap R^a$ such that $\{\alpha, \delta\}$ is a base for V at a. First suppose that $\delta < 0$. Let $m \in \mathbb{N}_0$ be the smallest integer with $\delta + (m+1)\alpha \notin R^a$. Then $\delta + n\alpha < 0$ for all $n \in \mathbb{N}_0$ with $n \le m$. Indeed, this holds for $n = 0$ by assumption. By induction on n we obtain from $\delta + n\alpha < 0$ and the choice of α that $\delta + n\alpha < -\alpha$, since δ and α are not collinear. Hence $\delta + (n+1)\alpha < 0$. We conclude that $-(\delta + m\alpha) > 0$. Moreover, $\{\alpha, -(\delta + m\alpha)\}$ is a base for V at a by Lemma 3.2 and the choice of m. Therefore, by replacing $\{\alpha, \delta\}$ by $\{\alpha, -(\delta + m\alpha)\}$, we may assume that $\delta > 0$. Since $\beta > 0$, we conclude that $\beta = k\alpha + l\delta$ for some $k, l \in \mathbb{N}_0$. Since β is not a multiple of α, this implies that $\beta = \delta$ or $\beta > \delta$. Then the choice of β and the positivity of δ yield that $\delta = \beta$, that is, $\{\alpha, \beta\}$ is a base for V at a. □

LEMMA 3.4. *Let $k \in \mathbb{N}_{\ge 2}$, $a \in A$, $\alpha \in R^a_+$, and $\beta \in \mathbb{Z}^I$ such that α and β are not collinear and $\alpha + k\beta \in R^a$. Assume that $\mathrm{Vol}_2(\alpha, \beta) = 1$ and that $(-\mathbb{N}\alpha + \mathbb{Z}\beta) \cap \mathbb{N}_0^I = \emptyset$. Then $\beta \in R^a$ and $\alpha + l\beta \in R^a$ for all $l \in \{1, 2, \dots, k\}$.*

PROOF. We prove the claim indirectly. Assume that $\beta \notin R^a$. By Lemma 3.3 there exists a base $\{\gamma_1, \gamma_2\}$ for $V^a(\alpha, \beta)$ at a such that $\gamma_1, \gamma_2 \in R^a_+$. The assumptions of the lemma imply that there exist $m_1, l_1 \in \mathbb{N}_0$ and $m_2, l_2 \in \mathbb{Z}$ such that $\gamma_1 = m_1\alpha + m_2\beta$, $\gamma_2 = l_1\alpha + l_2\beta$. Since $\beta \notin R^a$, we obtain that $m_1 \geq 1$ and $m_2 \geq 1$. Therefore relations $\alpha, \alpha + k\beta \in R^a_+$ imply that $\{\alpha, \alpha + k\beta\} = \{\gamma_1, \gamma_2\}$. The latter is a contradiction to $\mathrm{Vol}_2(\gamma_1, \gamma_2) = 1$ and $\mathrm{Vol}_2(\alpha, \alpha + k\beta) = k > 1$. Thus $\beta \in R^a$. By Lemma 3.1 we obtain that $\{\beta, \alpha - m\beta\}$ is a base for $V^a(\alpha, \beta)$ at a for some $m \in \mathbb{N}_0$. Then Corollary 2.8 and the assumption that $\alpha + k\beta \in R^a$ imply the last claim of the lemma. □

We say that a subset S of $\mathbb{Z}^3$ is *convex*, if any rational convex linear combination of elements of S is either in S or not in $\mathbb{Z}^3$. We start with a simple example.

LEMMA 3.5. *Let $a \in A$. Assume that $c^a_{12} = 0$.*

(1) Let $k_1, k_2 \in \mathbb{Z}$. Then $\alpha_3 + k_1\alpha_1 + k_2\alpha_2 \in R^a$ if and only if $0 \leq k_1 \leq -c^a_{13}$ and $0 \leq k_2 \leq -c^a_{23}$.

(2) Let $\gamma \in (\alpha_3 + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2) \cap R^a$. Then $\gamma - \alpha_1 \in R^a$ or $\gamma + \alpha_1 \in R^a$. Similarly $\gamma - \alpha_2 \in R^a$ or $\gamma + \alpha_2 \in R^a$.

PROOF. (1) The assumption $c^a_{12} = 0$ implies that $c^{\rho_1(a)}_{23} = c^a_{23}$, see [**17**, Lemma 4.5]. Applying $\sigma_1^{\rho_1(a)}$, $\sigma_2^{\rho_2(a)}$, and $\sigma_1\sigma_2^{\rho_2\rho_1(a)}$ to α_3 we conclude that $\alpha_3 - c^a_{13}\alpha_1$, $\alpha_3 - c^a_{23}\alpha_2$, $\alpha_3 - c^a_{13}\alpha_1 - c^a_{23}\alpha_2 \in R^a_+$. Thus Lemma 3.4 implies that $\alpha_3 + m_1\alpha_1 + m_2\alpha_2 \in R^a$ for all $m_1, m_2 \in \mathbb{Z}$ with $0 \leq m_1 \leq -c^a_{13}$ and $0 \leq m_2 \leq -c^a_{23}$. Further, (R1) gives that $\alpha_3 + k_1\alpha_1 + k_2\alpha_2 \notin R^a$ if $k_1 < 0$ or $k_2 < 0$. Applying again the simple reflections σ_1 and σ_2, a similar argument proves the remaining part of the claim. Observe that the proof does not use the fact that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is irreducible.

(2) Since $c^a_{12} = 0$, the irreducibility of $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ yields that $c^a_{13}, c^a_{23} < 0$ by [**17**, Def. 4.5, Prop. 4.6]. Hence the claim follows from (1). □

PROPOSITION 3.6. *Let $a \in A$ and let $\gamma_1, \gamma_2, \gamma_3 \in R^a$. Assume that $\mathrm{Vol}_3(\gamma_1, \gamma_2, \gamma_3) = 1$ and that $\gamma_1 - \gamma_2, \gamma_1 - \gamma_3 \notin R^a$. Then $\gamma_1 + \gamma_2 \in R^a$ or $\gamma_1 + \gamma_3 \in R^a$.*

PROOF. Since $\gamma_1 - \gamma_2 \notin R^a$ and $\mathrm{Vol}_3(\gamma_1, \gamma_2, \gamma_3) = 1$, Theorem 2.4 and Lemma 3.2 imply that there exists $b \in A$, $w \in \mathrm{Hom}(a, b)$ and $i_1, i_2, i_3 \in I$ such that $w(\gamma_1) = \alpha_{i_1}$, $w(\gamma_2) = \alpha_{i_2}$, and $w(\gamma_3) = \alpha_{i_3} + k_1\alpha_{i_1} + k_2\alpha_{i_2}$ for some $k_1, k_2 \in \mathbb{N}_0$. Assume that $\gamma_1 + \gamma_2 \notin R^a$. Then $c^b_{i_1 i_2} = 0$. Since $\gamma_3 - \gamma_1 \notin R^a$, Lemma 3.5(2) with $\gamma = w(\gamma_3)$ gives that $\gamma_3 + \gamma_1 \in R^a$. This proves the claim. □

LEMMA 3.7. *Assume that $R^a \cap (\mathbb{N}_0\alpha_1 + \mathbb{N}_0\alpha_2)$ contains at most 4 positive roots.*

(1) The set $S_3 := (\alpha_3 + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2) \cap R^a$ is convex.

(2) Let $\gamma \in S_3$. Then $\gamma = \alpha_3$ or $\gamma - \alpha_1 \in R^a$ or $\gamma - \alpha_2 \in R^a$.

PROOF. Consider the roots of the form $w^{-1}(\alpha_3) \in R^a$, where $w \in \mathrm{Hom}(a, \mathcal{W}(\mathcal{C}))$ is a product of reflections σ_1^b, σ_2^b with $b \in A$. All of these roots belong to S_3. Using Lemma 3.4 the claims of the lemma can be checked case by case, similarly to the proof of Lemma 3.5. □

REMARK 3.8. The lemma can be proven by elementary calculations, since all non-simple positive roots in $(\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2) \cap R^a$ are of the form say $\alpha_1 + k\alpha_2$, $k \in \mathbb{N}$ when $\#(R^a \cap (\mathbb{N}_0\alpha_1 + \mathbb{N}_0\alpha_2)) \leq 4$. We will see in Theorem 4.1 that the classification of connected Cartan schemes of rank three admitting a finite irreducible root system has a finite set of solutions. Thus it is possible to check the claim of the lemma for any such Cartan scheme. Using computer calculations one obtains that the lemma holds without any restriction on the (finite) cardinality of $R^a \cap (\mathbb{N}_0\alpha_1 + \mathbb{N}_0\alpha_2)$.

LEMMA 3.9. *Let $\alpha, \beta, \gamma \in R^a$ such that $\mathrm{Vol}_3(\alpha, \beta, \gamma) = 1$. Assume that $\alpha - \beta$, $\beta - \gamma$, $\alpha - \gamma \notin R^a$ and that $\{\alpha, \beta, \gamma\}$ is not a base for $\mathbb{Z}^I$ at a. Then the following hold.*

(1) There exist $w \in \mathrm{Hom}(a, \mathcal{W}(\mathcal{C}))$ and $n_1, n_2 \in \mathbb{N}$ such that $w(\alpha)$, $w(\beta)$, and $w(\gamma - n_1\alpha - n_2\beta)$ are simple roots.

(2) None of the vectors $\alpha - k\beta$, $\alpha - k\gamma$, $\beta - k\alpha$, $\beta - k\gamma$, $\gamma - k\alpha$, $\gamma - k\beta$, where $k \in \mathbb{N}$, is contained in R^a.

(3) $\alpha + \beta$, $\alpha + \gamma$, $\beta + \gamma \in R^a$.

(4) One of the sets $\{\alpha + 2\beta, \beta + 2\gamma, \gamma + 2\alpha\}$ and $\{2\alpha + \beta, 2\beta + \gamma, 2\gamma + \alpha\}$ is contained in R^a, the other one has trivial intersection with R^a.

(5) None of the vectors $\gamma - \alpha - k\beta$, $\gamma - k\alpha - \beta$, $\beta - \gamma - k\alpha$, $\beta - k\gamma - \alpha$, $\alpha - \beta - k\gamma$, $\alpha - k\beta - \gamma$, where $k \in \mathbb{N}_0$, is contained in R^a.

(6) Assume that $\alpha + 2\beta \in R^a$. Let $k \in \mathbb{N}$ such that $\alpha + k\beta \in R^a$, $\alpha + (k+1)\beta \notin R^a$. Let $\alpha' = \alpha + k\beta$, $\beta' = -\beta$, $\gamma' = \gamma + \beta$. Then $\mathrm{Vol}_3(\alpha', \beta', \gamma') = 1$, $\{\alpha', \beta', \gamma'\}$ is not a base for $\mathbb{Z}^I$ at a, and none of $\alpha' - \beta'$, $\alpha' - \gamma'$, $\beta' - \gamma'$ is contained in R^a.

(7) None of the vectors $\alpha + 3\beta$, $\beta + 3\gamma$, $\gamma + 3\alpha$, $3\alpha + \beta$, $3\beta + \gamma$, $3\gamma + \alpha$ is contained in R^a. In particular, $k = 2$ holds in (6).

PROOF. (1) By Theorem 2.4 there exist $m_1, m_2, n_1, n_2, n_3 \in \mathbb{N}_0$, $i_1, i_2, i_3 \in I$, and $w \in \mathrm{Hom}(a, \mathcal{W}(\mathcal{C}))$, such that $w(\alpha) = \alpha_{i_1}$, $w(\beta) = m_1\alpha_{i_1} + m_2\alpha_{i_2}$, and $w(\gamma) = n_1\alpha_{i_1} + n_2\alpha_{i_2} + n_3\alpha_{i_3}$. Since $\det w \in \{\pm 1\}$ and $\mathrm{Vol}_3(\alpha, \beta, \gamma) = 1$, this implies that $m_2 = n_3 = 1$. Further, $\beta - \alpha \notin R^a$, and hence $w(\beta) = \alpha_{i_2}$ by Corollary 2.9. Since $\{\alpha, \beta, \gamma\}$ is not a base for $\mathbb{Z}^I$ at a, we conclude that $w(\gamma) \neq \alpha_{i_3}$. Then Corollary 2.9 and the assumptions $\gamma - \alpha$, $\gamma - \beta \notin R^a$ imply that $w(\gamma) \notin \alpha_{i_3} + \mathbb{N}_0\alpha_{i_1}$ and $w(\gamma) \notin \alpha_{i_3} + \mathbb{N}_0\alpha_{i_2}$. Thus the claim is proven.

(2) By (1), $\{\alpha,\beta\}$ is a base for $V^a(\alpha,\beta)$ at a. Thus $\alpha-k\beta\notin R^a$ for all $k\in\mathbb{N}$. The remaining claims follow by symmetry.

(3) Suppose that $\alpha+\beta\notin R^a$. By (1) there exist $b\in A$, $w\in\mathrm{Hom}(a,b)$, $i_1,i_2,i_3\in I$ and $n_1,n_2\in\mathbb{N}$ such that $w(\alpha)=\alpha_{i_1}$, $w(\beta)=\alpha_{i_2}$, and $w(\gamma)=\alpha_{i_3}+n_1\alpha_{i_1}+n_2\alpha_{i_2}\in R^b_+$. By Theorem 2.10 there exist $n_1',n_2'\in\mathbb{N}_0$ such that $n_1'\le n_1$, $n_2'\le n_2$, $n_1'+n_2'<n_1+n_2$, and

$$\alpha_{i_3}+n_1'\alpha_{i_1}+n_2'\alpha_{i_2}\in R^b_+,\quad (n_1-n_1')\alpha_{i_1}+(n_2-n_2')\alpha_{i_2}\in R^b_+.$$

Since $\alpha+\beta\notin R^a$, Proposition 2.6 yields that $R^b_+\cap\mathrm{span}_\mathbb{Z}\{\alpha_{i_1},\alpha_{i_2}\}=\{\alpha_{i_1},\alpha_{i_2}\}$. Thus $\gamma-\alpha\in R^a$ or $\gamma-\beta\in R^a$. This is a contradiction to the assumption of the lemma. Hence $\alpha+\beta\in R^a$. By symmetry we obtain that $\alpha+\gamma,\beta+\gamma\in R^a$.

(4) Suppose that $\alpha+2\beta, 2\alpha+\beta\notin R^a$. By (1) the set $\{\alpha,\beta\}$ is a base for $V^a(\alpha,\beta)$ at a, and $\alpha+\beta\in R^a$ by (3). Then Proposition 2.6 implies that $R^a\cap\mathrm{span}_\mathbb{Z}\{\alpha,\beta\}=\{\pm\alpha,\pm\beta,\pm(\alpha+\beta)\}$. Thus (1) and Lemma 3.7(2) give that $\gamma-\alpha\in R^a$ or $\gamma-\beta\in R^a$, a contradiction to the initial assumption of the lemma. Hence by symmetry each of the sets $\{\alpha+2\beta,2\alpha+\beta\}$, $\{\alpha+2\gamma,2\alpha+\gamma\}$, $\{\beta+2\gamma,2\beta+\gamma\}$ contains at least one element of R^a.

Assume now that $\gamma+2\alpha$, $\gamma+2\beta\in R^a$. By changing the object via (1) we may assume that α, β, and $\gamma-n_1\alpha-n_2\beta$ are simple roots for some $n_1,n_2\in\mathbb{N}$. Then Lemma 3.4 applies to $\gamma+2\alpha\in R^a_+$ and $\beta-\alpha$, and tells that $\beta-\alpha\in R^a$. This gives a contradiction.

By the previous two paragraphs we conclude that if $\gamma+2\alpha\in R^a$, then $\gamma+2\beta\notin R^a$, and hence $\beta+2\gamma\in R^a$. Similarly, we also obtain that $\alpha+2\beta\in R^a$. By symmetry this implies (4).

(5) By symmetry it suffices to prove that $\gamma-(\alpha+k\beta)\notin R^a$ for all $k\in\mathbb{N}_0$. For $k=0$ the claim holds by assumption.

First we prove that $\gamma-(\alpha+2\beta)\notin R^a$. By (3) we know that $\gamma+\alpha$, $\alpha+\beta\in R^a$, and $\gamma-\beta\notin R^a$ by assumption. Since $\mathrm{Vol}_2(\gamma+\alpha,\alpha+\beta)=1$, Lemma 3.2 gives that $\{\gamma+\alpha,\alpha+\beta\}$ is a base for $V^a(\gamma+\alpha,\alpha+\beta)$ at a. Since $\gamma-(\alpha+2\beta)=(\gamma+\alpha)-2(\alpha+\beta)$, we conclude that $\gamma-(\alpha+2\beta)\notin R^a$.

Now let $k\in\mathbb{N}$. Assume that $\gamma-(\alpha+k\beta)\in R^a$ and that k is minimal with this property. Let $\alpha'=-\alpha$, $\beta'=-\beta$, $\gamma'=\gamma-(\alpha+k\beta)$. Then $\alpha',\beta',\gamma'\in R^a$ with $\mathrm{Vol}_3(\alpha',\beta',\gamma')=1$. Moreover, $\alpha'-\beta'\notin R^a$ by assumption, $\alpha'-\gamma'=-(\gamma-k\beta)\notin R^a$ by (2), and $\beta'-\gamma'=-(\gamma-\alpha-(k-1)\beta)\notin R^a$ by the minimality of k. Further, $\{\alpha',\beta',\gamma'\}$ is not a base for R^a, since $\gamma=\gamma'-\alpha'-k\beta'$. Hence Claim (3) holds for α',β',γ'. In particular,

$$\gamma'+\beta'=\gamma-(\alpha+(k+1)\beta)\in R^a.$$

This and the previous paragraph imply that $k \geq 3$.

We distinguish two cases depending on the parity of k. First assume that k is even. Let $\alpha' = \gamma + \alpha$ and $\beta' = -(\alpha + k/2\beta)$. Then $\mathrm{Vol}_2(\alpha', \beta') = 1$ and $\alpha' + 2\beta' = \gamma - (\alpha + k\beta) \in R^a$. Lemma 3.4 applied to α', β' gives that $\gamma - k/2\beta = \alpha' + \beta' \in R^a$, which contradicts (2).

Finally, the case of odd k can be excluded similarly by considering $V^a(\gamma + \alpha, \gamma - (\alpha + (k+1)\beta))$.

(6) We get $\mathrm{Vol}_3(\alpha', \beta', \gamma') = 1$ since $\mathrm{Vol}_3(\alpha, \beta, \gamma) = 1$ and Vol_3 is invariant under the right action of $\mathrm{GL}(\mathbb{Z}^3)$. Further, $\beta' - \gamma' = -(2\beta + \gamma) \notin R^a$ by (4), and $\alpha' - \gamma' \notin R^a$ by (5). Finally, $(\alpha', \beta', \gamma')$ is not a base for $\mathbb{Z}^I$ at a, since $R^a \ni \gamma - n_1\alpha - n_2\beta = \gamma' - n_1\alpha' + (1 + n_2 - kn_1)\beta'$, where $n_1, n_2 \in \mathbb{N}$ are as in (1).

(7) We prove that $\gamma + 3\alpha \notin R^a$. The rest follows by symmetry. If $2\alpha + \beta \in R^a$, then $\gamma + 2\alpha \notin R^a$ by (4), and hence $\gamma + 3\alpha \notin R^a$. Otherwise $\alpha + 2\beta, \gamma + 2\alpha \in R^a$ by (4). Let k, α', β', γ' be as in (6). Then (6) and (3) give that $R^a \ni \gamma' + \alpha' = \gamma + \alpha + (k+1)\beta$. Since $\gamma + \alpha \in R^a$, Lemma 3.4 implies that $\gamma + \alpha + 2\beta \in R^a$. Let w be as in (1). If $\gamma + 3\alpha \in R^a$, then Lemma 3.4 for the vectors $w(\gamma + \alpha + 2\beta)$ and $w(\alpha - \beta)$ implies that $w(\alpha - \beta) \in R^a$, a contradiction. Thus $\gamma + 3\alpha \notin R^a$. □

Recall that $\mathcal{C}$ is a Cartan scheme of rank three and $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$.

THEOREM 3.10. *Let $a \in A$ and $\alpha, \beta, \gamma \in R^a$. If $\mathrm{Vol}_3(\alpha, \beta, \gamma) = 1$ and none of $\alpha - \beta$, $\alpha - \gamma$, $\beta - \gamma$ are contained in R^a, then $\{\alpha, \beta, \gamma\}$ is a base for $\mathbb{Z}^I$ at a.*

PROOF. Assume to the contrary that $\{\alpha, \beta, \gamma\}$ is not a base for $\mathbb{Z}^I$ at a. Exchanging α and β if necessary, by Lemma 3.9(4) we may assume that $\alpha + 2\beta \in R^a$. By Lemma 3.9(6),(7) the triple $(\alpha + 2\beta, -\beta, \gamma + \beta)$ satisfies the assumptions of Lemma 3.9, and $(\alpha + 2\beta) + 2(-\beta) = \alpha \in R^a$. Hence $2\alpha + 3\beta = 2(\alpha + 2\beta) + (-\beta) \notin R^a$ by Lemma 3.9(4). Thus $V^a(\alpha, \beta) \cap R^a = \{\pm\alpha, \pm(\alpha + \beta), \pm(\alpha + 2\beta), \pm\beta\}$ by Proposition 2.6, and hence, using Lemma 3.9(1), we obtain from Lemma 3.7(2) that $\gamma - \alpha \in R^a$ or $\gamma - \beta \in R^a$. This is a contradiction to our initial assumption, and hence $\{\alpha, \beta, \gamma\}$ is a base for $\mathbb{Z}^I$ at a. □

COROLLARY 3.11. *Let $a \in A$ and $\gamma_1, \gamma_2, \alpha \in R^a$. Assume that $\{\gamma_1, \gamma_2\}$ is a base for $V^a(\gamma_1, \gamma_2)$ at a and that $\mathrm{Vol}_3(\gamma_1, \gamma_2, \alpha) = 1$. Then either $\{\gamma_1, \gamma_2, \alpha\}$ is a base for $\mathbb{Z}^I$ at a, or one of $\alpha - \gamma_1$, $\alpha - \gamma_2$ is contained in R^a.*

For the proof of Theorem 4.1 we need a bound for the entries of the Cartan matrices of $\mathcal{C}$. To get this bound we use the following.

LEMMA 3.12. *Let $a \in A$.*

(1) At most one of c^a_{12}, c^a_{13}, c^a_{23} is zero.

(2) $\alpha_1 + \alpha_2 + \alpha_3 \in R^a$.

(3) Let $k \in \mathbb{Z}$. Then $k\alpha_1 + \alpha_2 + \alpha_3 \in R^a$ if and only if $k_1 \le k \le k_2$, where

$$k_1 = \begin{cases} 0 & \text{if } c^a_{23} < 0, \\ 1 & \text{if } c^a_{23} = 0, \end{cases} \qquad k_2 = \begin{cases} -c^a_{12} - c^a_{13} & \text{if } c^{\rho_1(a)}_{23} < 0, \\ -c^a_{12} - c^a_{13} - 1 & \text{if } c^{\rho_1(a)}_{23} = 0. \end{cases}$$

(4) We have $2\alpha_1 + \alpha_2 + \alpha_3 \in R^a$ if and only if either $c^a_{12} + c^a_{13} \le -3$ or $c^a_{12} + c^a_{13} = -2$, $c^{\rho_1(a)}_{23} < 0$.

(5) Assume that

$$\#(R^a_+ \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2)) \ge 5. \tag{3.1}$$

Then there exist $k \in \mathbb{N}_0$ such that $k\alpha_1 + 2\alpha_2 + \alpha_3 \in R^a$. Let k_0 be the smallest among all such k. Then k_0 is given by the following.

$$\begin{cases} 0 & \text{if } c^a_{23} \le -2, \\ 1 & \text{if } -1 \le c^a_{23} \le 0,\ c^a_{21} + c^a_{23} \le -2,\ c^{\rho_2(a)}_{13} < 0, \\ 1 & \text{if } -1 \le c^a_{23} \le 0,\ c^a_{21} + c^a_{23} \le -3,\ c^{\rho_2(a)}_{13} = 0, \\ 2 & \text{if } c^a_{21} = c^a_{23} = -1,\ c^{\rho_2(a)}_{13} = 0, \\ 2 & \text{if } c^a_{21} = -1,\ c^a_{23} = 0,\ c^{\rho_2(a)}_{13} \le -2, \\ 3 & \text{if } c^a_{21} = -1,\ c^a_{23} = 0,\ c^{\rho_2(a)}_{13} = -1,\ c^{\rho_2(a)}_{12} \le -3, \\ 3 & \text{if } c^a_{21} = -1,\ c^a_{23} = 0,\ c^{\rho_2(a)}_{13} = -1,\ c^{\rho_2(a)}_{12} = -2,\ c^{\rho_1\rho_2(a)}_{23} < 0, \\ 4 & \text{otherwise.} \end{cases}$$

Further, if $c^a_{13} = 0$ then $k_0 \le 2$.

PROOF. We may assume that $\mathcal{C}$ is connected. Then, since $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is irreducible, Claim (1) holds by [**17**, Def. 4.5, Prop. 4.6].

(2) The claim is invariant under permutation of I. Thus by (1) we may assume that $c^a_{23} \ne 0$. Hence $\alpha_2 + \alpha_3 \in R^a$. Assume first that $c^a_{13} = 0$. Then $c^{\rho_1(a)}_{13} = 0$ by (C2), $c^{\rho_1(a)}_{23} \ne 0$ by (1), and $\alpha_2 + \alpha_3 \in R^{\rho_1(a)}_+$. Hence $\sigma^{\rho_1(a)}_1(\alpha_2 + \alpha_3) = -c^a_{12}\alpha_1 + \alpha_2 + \alpha_3 \in R^a$. Therefore (2) holds by Lemma 3.4 for $\alpha = \alpha_2 + \alpha_3$ and $\beta = \alpha_1$.

Assume now that $c^a_{13} \ne 0$. By symmetry and the previous paragraph we may also assume that $c^a_{12}, c^a_{23} \ne 0$. Let $b = \rho_1(a)$. If $c^b_{23} = 0$ then $\alpha_1 + \alpha_2 + \alpha_3 \in R^b$ by the previous paragraph. Then

$$R^a \ni \sigma^b_1(\alpha_1 + \alpha_2 + \alpha_3) = (-c^a_{12} - c^a_{13} - 1)\alpha_1 + \alpha_2 + \alpha_3,$$

and the coefficient of α_1 is positive. Further, $\alpha_2+\alpha_3 \in R^a$, and hence (2) holds in this case by Lemma 3.4. Finally, if $c^b_{23} \neq 0$, then $\alpha_2+\alpha_3 \in R^b_+$, and hence $(-c^a_{12}-c^a_{13})\alpha_1+\alpha_2+\alpha_3 \in R^a$. Since $-c^a_{12}-c^a_{13} > 0$, (2) follows again from Lemma 3.4.

(3) If $c^a_{23} < 0$, then $\alpha_2+\alpha_3 \in R^a$ and $-k\alpha_1+\alpha_2+\alpha_3 \notin R^a$ for all $k \in \mathbb{N}$. If $c^a_{23} = 0$, then $\alpha_1+\alpha_2+\alpha_3 \in R^a$ by (2), and $-k\alpha_1+\alpha_2+\alpha_3 \notin R^a$ for all $k \in \mathbb{N}_0$. Applying the same argument to $R^{\rho_1(a)}$ and using the reflection $\sigma_1^{\rho_1(a)}$ and Lemma 3.4 gives the claim.

(4) This follows immediately from (3).

(5) The first case follows from Corollary 2.9 and the second and third cases are obtained from (4) by interchanging the elements 1 and 2 of I. We also obtain that if k_0 exists then $k_0 \geq 2$ in all other cases. By (3.1) and Proposition 2.6 we conclude that $\alpha_1+\alpha_2 \in R^a$. Then $c^a_{21} < 0$ by Corollary 2.9, and hence we are left with calculating k_0 if $-1 \leq c^a_{23} \leq 0$, $c^a_{21}+c^a_{23} = -2$, $c^{\rho_2(a)}_{13} = 0$, or $c^a_{21} = -1$, $c^a_{23} = 0$. By (1), if $c^{\rho_2(a)}_{13} = 0$ then $c^{\rho_2(a)}_{23} \neq 0$, and hence $c^a_{23} < 0$ by (C2). Thus we have to consider the elements $k\alpha_1+2\alpha_2+\alpha_3$, where $k \geq 2$, under the assumption that

$$c^a_{21} = c^a_{23} = -1,\ c^{\rho_2(a)}_{13} = 0 \quad \text{or} \quad c^a_{21} = -1,\ c^a_{23} = 0. \tag{3.2}$$

Since $c^a_{21} = -1$, Condition (3.1) gives that

$$c^{\rho_2(a)}_{12} \leq -2,$$

see [**17**, Lemma 4.8]. Further, the first set of equations in (3.2) implies that $c^{\rho_1\rho_2(a)}_{13} = 0$, and hence $c^{\rho_1\rho_2(a)}_{23} < 0$ by (1). Since $\sigma^a_2(2\alpha_1+2\alpha_2+\alpha_3) = 2\alpha_1+\alpha_3-c^a_{23}\alpha_2$, the first set of equations in (3.2) and (4) imply that $k_0 = 2$. Similarly, Corollary 2.9 tells that $k_0 = 2$ under the second set of conditions in (3.2) if and only if $c^{\rho_2(a)}_{13} \leq -2$.

It remains to consider the situation for

$$c^a_{21} = -1,\ c^a_{23} = 0,\ c^{\rho_2(a)}_{13} = -1, \tag{3.3}$$

because, indeed, equation $c^a_{23} = 0$ implies that $c^{\rho_2(a)}_{23} = 0$ by (C2), and hence $c^{\rho_2(a)}_{13} < 0$ by (1). Assuming (3.3) we obtain that $\sigma^a_2(3\alpha_1+2\alpha_2+\alpha_3) = 3\alpha_1+\alpha_2+\alpha_3$, and hence (3) implies that $k_0 = 3$ if and only if the corresponding conditions in (5) are valid.

The rest follows by looking at $\sigma_1\sigma^a_2(4\alpha_1+2\alpha_2+\alpha_3)$, details are left to the reader: The last claim holds since $c^a_{13} = 0$ implies that $c^a_{23} \neq 0$ by (1). The assumption $\#(R^a_+ \cap (\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2)) \geq 5$ is needed to exclude the case $c^a_{21} = -1$, $c^{\rho_2(a)}_{12} = -2$, $c^{\rho_1\rho_2(a)}_{21} = -1$, where $R^a_+ \cap (\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2) = \{\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, \alpha_1\}$, by using Proposition 2.6 and Corollary 2.9, see also the proof of [**17**, Lemma 4.8]. □

THEOREM 3.13. *Let $\mathcal{C}$ be a Cartan scheme of rank three. Assume that $\mathcal{R}^{\mathrm{re}}$ is a finite irreducible root system of type $\mathcal{C}$. Then all entries of the Cartan matrices of $\mathcal{C}$ are greater or equal to -7.*

PROOF. It can be assumed that $\mathcal{C}$ is connected. We prove the theorem indirectly. To do so we may assume that $a \in A$ such that $c^a_{12} \le -8$. Then Proposition 2.6 implies that $\#(R^a_+ \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2)) \ge 5$. By Lemma 3.12 there exists $k_0 \in \{0,1,2,3,4\}$ such that $\alpha := k_0\alpha_1 + 2\alpha_2 + \alpha_3 \in R^a_+$ and $\alpha - \alpha_1 \notin R^a$. By Lemma 3.2 and the choice of k_0 the set $\{\alpha, \alpha_1\}$ is a base for $V^a(\alpha, \alpha_1)$ at a. Corollary 2.5 implies that there exists a root $\gamma \in R^a$ such that $\{\alpha, \alpha_1, \gamma\}$ is a base for $\mathbb{Z}^I$ at a. Let $d \in A$, $w \in \mathrm{Hom}(a,d)$, and $i_1, i_2, i_3 \in I$ such that $w(\alpha) = \alpha_{i_1}$, $w(\alpha_1) = \alpha_{i_2}$, $w(\gamma) = \alpha_{i_3}$.

Let $b = \rho_1(a)$. Again by Lemma 3.12 there exists $k_1 \in \{0,1,2,3,4\}$ such that $\beta := k_1\alpha_1 + 2\alpha_2 + \alpha_3 \in R^b_+$. Thus

$$R^a_+ \ni \sigma^b_1(\beta) = (-k_1 - 2c^a_{12} - c^a_{13})\alpha_1 + 2\alpha_2 + \alpha_3.$$

Further,

$$-k_1 - 2c^a_{12} - c^a_{13} - k_0 > -c^a_{12}$$

since $k_0 \le 2$ if $c^a_{13} = 0$. Hence $\alpha_{i_1} + (1 - c^a_{12})\alpha_{i_2} \in R^d$, that is, $c^d_{i_2 i_1} < c^a_{12} \le -8$. We conclude that there exists no lower bound for the entries of the Cartan matrices of $\mathcal{C}$, which is a contradiction to the finiteness of $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. This proves the theorem. □

REMARK 3.14. The bound in the theorem is not sharp. After completing the classification one can check that the entries of the Cartan matrices of $\mathcal{C}$ are always at least -6. The entry -6 appears for example in the Cartan scheme corresponding to the root system with number 53, see Corollary 2.9.

PROPOSITION 3.15. *Let $\mathcal{C}$ be an irreducible connected simply connected Cartan scheme of rank three. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$. Let e be the number of vertices, k the number of edges, and f the number of (2-dimensional) faces of the object change diagram of $\mathcal{C}$. Then $e - k + f = 2$.*

PROOF. Vertices of the object change diagram correspond to elements of A. Since $\mathcal{C}$ is connected and $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is finite, the set A is finite. Consider the equivalence relation on $I \times A$, where (i,a) is equivalent to (j,b) for some $i,j \in I$ and $a,b \in A$, if and only if $i = j$ and $b \in \{a, \rho_i(a)\}$. (This is also known as the pushout of $I \times A$ along the bijections $\mathrm{id} : I \times A \to I \times A$ and $\rho : I \times A \to I \times A$, $(i,a) \mapsto (i, \rho_i(a))$.) Since $\mathcal{C}$ is simply connected, $\rho_i(a) \ne a$ for all $i \in I$ and $a \in A$. Then edges of the object change diagram correspond to equivalence classes in $I \times A$. Faces of the object change diagram can be defined as equivalence classes of triples $(i,j,a) \in I \times I \times A \setminus \{(i,i,a) \mid i \in I, a \in A\}$, where (i,j,a) and (i',j',b) are equivalent for some $i,j,i',j' \in I$ and $a,b \in A$ if and only if $\{i,j\} = \{i',j'\}$ and $b \in \{(\rho_j\rho_i)^m(a), \rho_i(\rho_j\rho_i)^m(a) \mid m \in \mathbb{N}_0\}$. Since $\mathcal{C}$ is simply connected, (R4) implies that the face corresponding to a triple (i,j,a) is a polygon with $2m^a_{i,j}$ vertices.

For each face choose a triangulation by non-intersecting diagonals. Let d be the total number of diagonals arising this way. Now consider the following two-dimensional simplicial complex C: The 0-simplices are the objects. The 1-simplices are the edges and the chosen diagonals of the faces of the object change diagram. The 2-simplices are the $f+d$ triangles. Clearly, each edge is contained in precisely two triangles. By [**45**, Ch. III, (3.3), 2,3] the geometric realization X of C is a closed 2-dimensional surface without boundary. The space X is connected and compact.

Any two morphisms of $\mathcal{W}(\mathcal{C})$ with same source and target are equal because $\mathcal{C}$ is simply connected. By [**17**, Thm. 2.6] this equality follows from the Coxeter relations. A Coxeter relation means for the object change diagram that for the corresponding face and vertex the two paths along the sides of the face towards the opposite vertex yield the same morphism. But since this relation involves only one face which is simply connected, the corresponding paths are homotopic. Hence X is simply connected and therefore homeomorphic to a two-dimensional sphere by [**45**, Ch. III, Satz 6.9]. Its Euler characteristic is $2=e-(k+d)+(f+d)=e-k+f$. □

REMARK 3.16. Assume that $\mathcal{C}$ is connected and simply connected, and let $a\in A$. Then any pair of opposite 2-dimensional faces of the object change diagram can be interpreed as a plane in $\mathbb{Z}^I$ containing at least two positive roots $\alpha,\beta\in R^a_+$. Indeed, let $b\in A$ and $i_1,i_2\in I$ with $i_1\neq i_2$. Since $\mathcal{C}$ is connected and simply connected, there exists a unique $w\in \mathrm{Hom}(a,b)$. Then $V^a(w^{-1}(\alpha_{i_1}),w^{-1}(\alpha_{i_2}))$ is a plane in $\mathbb{Z}^I$ containing at least two positive roots. One can easily check that this plane is independent of the choice of the representative of the face determined by $(i_1,i_2,b)\in I\times I\times A$. Further, let $w_0\in \mathrm{Hom}(b,d)$, where $d\in A$, be the longest element in $\mathrm{Hom}(b,\mathcal{W}(\mathcal{C}))$. Let $j_1,j_2\in I$ such that $w_0(\alpha_{i_n})=-\alpha_{j_n}$ for $n=1,2$. Then (j_1,j_2,d) determines the plane

$$V^a((w_0w)^{-1}(\alpha_{j_1}),(w_0w)^{-1}(\alpha_{j_2}))=V^a(w^{-1}(\alpha_{i_1}),w^{-1}(\alpha_{i_2})).$$

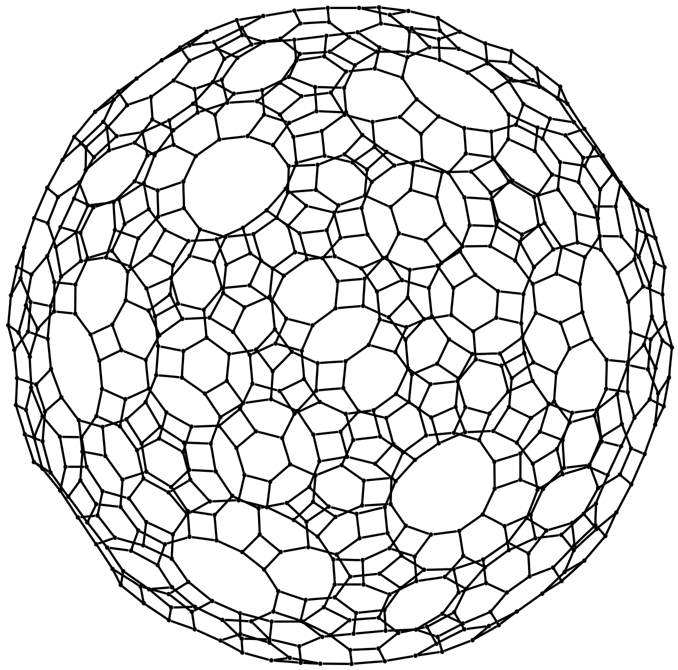
This way we attached to any pair of (2-dimensional) opposite faces of the object change diagram a plane containing at least two positive roots. Let $<$ be a semigroup ordering on $\mathbb{Z}^I$ such that $0<\gamma$ for all $\gamma\in R^a_+$. Let $\alpha,\beta\in R^a_+$ with $\alpha\neq\beta$, and assume that α and β are the smallest elements in $R^a_+\cap V^a(\alpha,\beta)$ with respect to $<$. Then $\{\alpha,\beta\}$ is a base for $V^a(\alpha,\beta)$ at a by Lemma 3.3. By Corollary 2.5 there exists $b\in A$ and $w\in\mathrm{Hom}(a,b)$ such that $w(\alpha),w(\beta)\in R^b_+$ are simple roots. Hence any plane in $\mathbb{Z}^I$ containing at least two elements of R^a_+ can be obtained by the construction in the previous paragraph.

It remains to show that different pairs of opposite faces give rise to different planes. This follows from the fact that for any $b\in A$ and $i_1,i_2\in I$ with $i_1\neq i_2$ the conditions

$$d\in A,\ u\in\mathrm{Hom}(b,d),\ j_1,j_2\in I,\ u(\alpha_{i_1})=\alpha_{j_1},\ u(\alpha_{i_2})=\alpha_{j_2}$$

have precisely two solutions: $u=\mathrm{id}_b$ on the one side, and $u=w_0w_{i_1i_2}$ on the other side, where $w_{i_1i_2}=\cdots\sigma_{i_1}\sigma_{i_2}\sigma_{i_1}\mathrm{id}_b\in\mathrm{Hom}(b,\mathcal{W}(\mathcal{C}))$ is the longest product

FIGURE 2. The object change diagram of the last root system of rank three



of reflections σ_{i_1}, σ_{i_2}, and w_0 is an appropriate longest element of $\mathcal{W}(\mathcal{C})$. The latter follows from the fact that u has to map the base $\{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\}$ for $\mathbb{Z}^I$ at b, where $I = \{i_1, i_2, i_3\}$, to another base, and any base consisting of two simple roots can be extended precisely in two ways to a base of $\mathbb{Z}^I$: by adding the third simple root or by adding a uniquely determined negative root.

It follows from the construction and by [**16**, Lemma 6.4] that the faces corresponding to a plane $V^a(\alpha, \beta)$, where $\alpha, \beta \in R^a_+$ with $\alpha \neq \beta$, have as many edges as the cardinality of $V^a(\alpha, \beta) \cap R^a$ (or twice the cardinality of $V^a(\alpha, \beta) \cap R^a_+$).

THEOREM 3.17. *Let $\mathcal{C}$ be a connected simply connected Cartan scheme of rank three. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. Let $a \in A$ and let M be the set of planes containing at least two elements of R^a_+. Then*

$$\sum_{V \in M} \#(V \cap R^a_+) = 3(\#M - 1).$$

PROOF. Let e, k, f be as in Proposition 3.15. Then $\#M = f/2$ by Remark 3.16. For any vertex $b \in A$ there are three edges starting at b, and any edge is bounded by two vertices. Hence counting vertices in two different ways one obtains that $3e = 2k$. Proposition 3.15 gives that $e - k + f = 2$. Hence $2k = 3e = 3(2 - f + k)$, that is, $k = 3f - 6$.

Any plane V corresponds to a face which is a polygon consisting of $2\#(V\cap R^a_+)$ edges, see Remark 3.16. Summing up the edges twice over all planes (that is summing up over all faces of the object change diagram), each edge is counted twice. Hence

$$2\sum_{V\in M} 2\#(V\cap R^a_+) = 2k = 2(3f-6),$$

which is the formula claimed in the theorem. □

COROLLARY 3.18. *Let $\mathcal{C}$ be a connected simply connected Cartan scheme of rank three. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. Then there exists an object $a\in A$ and $\alpha,\beta,\gamma\in R^a_+$ such that $\{\alpha,\beta,\gamma\}=\{\alpha_1,\alpha_2,\alpha_3\}$ and*

$$\#(V^a(\alpha,\beta)\cap R^a_+)=2,\quad \#(V^a(\alpha,\gamma)\cap R^a_+)=3. \tag{3.4}$$

Further $\alpha+\gamma,\beta+\gamma,\alpha+\beta+\gamma\in R^a_+$.

PROOF. Let M be as in Thm. 3.17. Let a be any object and assume $\#(V\cap R^a_+)>2$ for all $V\in M$, then $\sum_{V\in M}\#(V\cap R^a_+)\ge 3\#M$ contradicting Thm. 3.17. Hence for all objects a there exists a plane V with $\#(V\cap R^a_+)=2$. Now consider the object change diagram and count the number of faces: Let $2q_i$ be the number of faces with $2i$ edges. Then Thm. 3.17 translates to

$$\sum_{i\ge 2} iq_i = -3+3\sum_{i\ge 2} q_i. \tag{3.5}$$

Assume that there exists no object adjacent to a square and a hexagon. Since $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is irreducible, no two squares have a common edge, see Lemma 3.12(1). Look at the edges ending in vertices of squares, and count each edge once for both polygons adjacent to it. One checks that there are at least twice as many edges adjacent to a polygon with at least 8 vertices as edges of squares. This gives that

$$\sum_{i\ge 4} 2i\cdot 2q_i \ge 2\cdot 4\cdot 2q_2.$$

By Equation (3.5) we then have $-3+3\sum_{i\ge 2} q_i \ge 4q_2+2q_2+3q_3$, that is, $q_2<\sum_{i\ge 4} q_i$. But then in average each face has more than 6 edges which contradicts Thm. 3.17. Hence there is an object a such that there exist $\alpha,\beta,\gamma\in R^a_+$ as above satisfying Equation (3.4). We have $\alpha+\gamma,\beta+\gamma,\alpha+\beta+\gamma\in R^a_+$ by Lemma 3.12(1),(2) and Corollary 2.9. □

4. The classification

In this section we explain the classification of connected simply connected Cartan schemes of rank three such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system. We formulate the main result in Theorem 4.1. The proof of Theorem 4.1 is performed using computer calculations based on results of the previous sections. Our algorithm described below is sufficiently powerful: The implementation in C terminates within a few hours on a usual computer and is available at

`http://www.mathematik.uni-kl.de/~cuntz/download/wgr3.c.`

Removing any of the theorems, the calculations would take at least several weeks.

THEOREM 4.1. *(1) Let $\mathcal{C}$ be a connected Cartan scheme of rank three with $I = \{1,2,3\}$. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. Then there exists an object $a \in A$ and a linear map $\tau \in \mathrm{Aut}(\mathbb{Z}^I)$ such that $\tau(\alpha_i) \in \{\alpha_1, \alpha_2, \alpha_3\}$ for all $i \in I$ and $\tau(R^a_+)$ is one of the sets listed in Appendix 1. Moreover, $\tau(R^a_+)$ with this property is uniquely determined.*

(2) Let R be one of the 55 subsets of $\mathbb{Z}^3$ appearing in Appendix 1. There exists up to equivalence a unique connected simply connected Cartan scheme $\mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ such that $R \cup -R$ is the set of real roots R^a in an object $a \in A$. Moreover $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$.

REMARK 4.2. Theorem 4.1 is a classification of connected simply connected Cartan schemes $\mathcal{C}$, up to equivalence in the sense of [**17**, Def. 2.1], such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$. To obtain a classification of connected Cartan schemes such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system, one additionally has to classify quotients (inverse of coverings) of our simply connected Cartan schemes, which amounts to classify all subgroups of $\mathrm{Hom}(a)$ in Table 1. We do not claim that Weyl groupoids of non-equivalent Cartan schemes are non-isomorphic as groupoids. It should also be stressed that the list of root systems in the appendix contains for each Cartan scheme the root system of precisely one object. We explain in the appendix in which sense this root system is canonical.

Let $<$ be the lexicographic ordering on $\mathbb{Z}^3$ such that $\alpha_3 < \alpha_2 < \alpha_1$. Then $\alpha > 0$ for any $\alpha \in \mathbb{N}_0^3 \setminus \{0\}$.

Let $\mathcal{C}$ be a connected Cartan scheme with $I = \{1,2,3\}$. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. Let $a \in A$. By Theorem 2.10 we may construct R^a_+ inductively by starting with $R^a_+ = \{\alpha_3, \alpha_2, \alpha_1\}$, and appending in each step a sum of a pair of positive roots which is greater than all roots in R^a_+ we already have. During this process, we keep track of all planes containing at least two positive roots, and the positive roots on them. Lemma 3.3 implies that for any known root α and new root β either $V^a(\alpha, \beta)$ contains no other known positive roots, or β is not part of the unique base for $V^a(\alpha, \beta)$ at a consisting of positive roots. In the first case the roots α, β generate a new plane. It can happen that $\mathrm{Vol}_2(\alpha, \beta) > 1$, and then $\{\alpha, \beta\}$ is not a base for $V^a(\alpha, \beta)$ at a, but we don't care about that. In the second case the known roots in $V^a(\alpha, \beta) \cap R^a_+$ together with β have to form an $\mathcal{F}$-sequence by Proposition 2.6.

Sometimes, by some theorem (see the details below) we know that it is not possible to add more positive roots to a plane. Then we can mark it as “finished”.

Remark that to obtain a finite number of root systems as output, we have to ensure that we compute only irreducible systems since there are infinitely many inequivalent reducible root systems of rank two. Hence starting with $\{\alpha_3,\alpha_2,\alpha_1\}$ will not work. However, by Corollary 3.18, starting with $\{\alpha_3,\alpha_2,\alpha_2+\alpha_3,\alpha_1,\alpha_1+\alpha_2,\alpha_1+\alpha_2+\alpha_3\}$ will still yield at least one root system for each desired Cartan scheme (notice that any roots one would want to add are lexicographically greater).

In this section, we will call *root system fragment* (or *rsf*) the following set of data associated to a set of positive roots R in construction:

- normal vectors for the planes with at least two positive roots
- labels of positive roots on these planes
- Cartan entries corresponding to the root systems of the planes
- an array of flags for finished planes
- the sum s_R of $\#(V\cap R)$ over all planes V with at least two positive roots, see Theorem 3.17
- for each root $r\in R$ the list of planes it belongs to.

These data can be obtained directly from R, but the calculation is faster if we continuously update them.

We divide the algorithm into three parts. The main part is Algorithm 4.4, see below.

The first part updates a root system fragment to a new root and uses Theorems 3.10 and 3.13 to possibly refuse doing so:

Algorithm 4.3. AppendRoot(α,B,$\tilde{B}$,$\hat{\alpha}$)
Append a root to an rsf.
Input: a root α, an rsf B, an empty rsf $\tilde{B}$, a root $\hat{\alpha}$.

Output: $\begin{cases} 0: & \text{if } \alpha \text{ may be appended, new rsf is then in } \tilde{B},\\ 1: & \text{if } \alpha \text{ may not be appended,}\\ 2: & \text{if } \alpha\in R^a_+ \text{ implies the existence of } \beta\in R^a_+\\ & \text{with } \hat{\alpha}<\beta<\alpha. \end{cases}$

1. Let r be the number of planes containing at least two elements of R. For documentation purposes let $V_1,\dots,V_r$ denote these planes. For any $i\in\{1,\dots,r\}$ let v_i be a normal vector for V_i, and let R_i be the $\mathcal{F}$-sequence of $V_i\cap R$. Set $i\leftarrow 1$, $g\leftarrow 1$, $c\leftarrow[\,]$, $p\leftarrow[\,]$, $d\leftarrow\{\,\}$. During the algorithm c will be an ordered subset of $\{1, \dots, r\}$, p a corresponding list of "positions", and d a subset of R.

2. If $i \leq r$ and $g \neq 0$, then compute the scalar product $g := (\alpha, v_i)$. (Then $g = \det(\alpha, \gamma_1, \gamma_2) = \pm\mathrm{Vol}_3(\alpha, \gamma_1, \gamma_2)$, where $\{\gamma_1, \gamma_2\}$ is the basis of V_i consisting of positive roots.) Otherwise go to Step 6.

3. If $g = 0$ then do the following: If V_i is not finished yet, then check if α extends R_i to a new $\mathcal{F}$-sequence. If yes, add the roots of R_i to d, append i to c, append the position of the insertion of α in R_i to p, let $g \leftarrow 1$, and go to Step 5.

4. If $g^2 = 1$, then use Corollary 3.11: Let γ_1 and γ_2 be the beginning and the end of the $\mathcal{F}$-sequence R_i, respectively. (Then $\{\gamma_1, \gamma_2\}$ is a base for V_i at a). Let $\delta_1 \leftarrow \alpha - \gamma_1$, $\delta_2 \leftarrow \alpha - \gamma_2$. If $\delta_1, \delta_2 \notin R$, then return 1 if $\delta_1, \delta_2 \leq \hat{\alpha}$ and return 2 otherwise.

5. Set $i \leftarrow i + 1$ and go to Step 2.

6. If there is no objection appending α so far, i.e. $g \neq 0$, then copy B to $\tilde{B}$ and include α into $\tilde{B}$: use c, p to extend existing $\mathcal{F}$-sequences, and use (the complement of) d to create new planes. Finally, apply Theorem 3.13: If there is a Cartan entry lesser than -7 then return 1, else return 0. If $g = 0$ then return 2.

The second part looks for small roots which we must include in any case. The function is based on Proposition 3.6. This is a strong restriction during the process.

Algorithm 4.4. RequiredRoot(R,B,$\hat{\alpha}$)
Find a smallest required root under the assumption that all roots $\leq \hat{\alpha}$ are known.
Input: R a set of roots, B an rsf for R, $\hat{\alpha}$ a root.

Output: $\begin{cases} 0 & \text{if we cannot determine such a root,} \\ 1, \varepsilon & \text{if we have found a small missing root } \varepsilon \text{ with } \varepsilon > \hat{\alpha}, \\ 2 & \text{if the given configuration is impossible.} \end{cases}$

1. Initialize the return value $f \leftarrow 0$.

2. We use the same notation as in Algo. 4.2, step 1. For all γ_1 in R and all $(j, k) \in \{1, \dots, r\} \times \{1, \dots, r\}$ such that $j \neq k$, $\gamma_1 \in R_j \cap R_k$, and both R_j, R_k contain two elements, let $\gamma_2, \gamma_3 \in R$ such that $R_j = \{\gamma_1, \gamma_2\}$, $R_k = \{\gamma_1, \gamma_3\}$. If $\mathrm{Vol}_3(\gamma_1, \gamma_2, \gamma_3) = 1$, then do Steps 3 to 6.

3. $\xi_2 \leftarrow \gamma_1 + \gamma_2$, $\xi_3 \leftarrow \gamma_1 + \gamma_3$.

4. If $\hat{\alpha} \geq \xi_2$: If $\hat{\alpha} \geq \xi_3$ or plane V_k is already finished, then return 2. If $f = 0$ or $\varepsilon > \xi_3$, then $\varepsilon \leftarrow \xi_3$, $f \leftarrow 1$. Go to Step 2 and continue loop.

5. If $\hat{\alpha} \geq \xi_3$: If plane V_j is already finished, then return 2. If $f = 0$ or $\varepsilon > \xi_2$, then $\varepsilon \leftarrow \xi_2$, $f \leftarrow 1$.

6. Go to Step 2 and continue loop.

7. Return f, ε.

Finally, we resursively add roots to a set, update the rsf and include required roots:

Algorithm 4.5. CompleteRootSystem(R,B,$\hat\alpha$,u,β)
Collects potential new roots, appends them and calls itself again.
Input: R a set of roots, B an rsf for R, $\hat\alpha$ a lower bound for new roots, u a flag, β a vector which is necessarily a root if $u =$ True.
Output: Root systems containing R.

1. Check Theorem 3.17: If $s_R = 3(r-1)$, where r is the number of planes containing at least two positive roots, then output R (and continue). We have found a potential root system.
2. If we have no required root yet, i.e. $u =$ False, then f, ε :=RequiredRoot$(R, B, \hat\alpha)$. If $f = 1$, then we have found a required root; we call CompleteRootSystem($R, B, \hat\alpha, True, \varepsilon$) and terminate. If $f = 2$, then terminate.
3. Potential new roots will be collected in $Y \leftarrow \{\ \}$; $\tilde B$ will be the new rsf.
4. For all planes V_i of B which are not finished, do Steps 5 to 7.
5. $\nu \leftarrow 0$.
6. For ζ in the set of roots that may be added to the plane V_i such that $\zeta > \hat\alpha$, do the following:
- set $\nu \leftarrow \nu + 1$.
- If $\zeta \notin Y$, then $Y \leftarrow Y \cup \{\zeta\}$. If moreover $u =$ False or $\beta > \zeta$, then
  - $y \leftarrow$ AppendRoot($\zeta, B, \tilde B, \hat\alpha$);
  - if $y = 0$ then CompleteRootSystem($R \cup \{\zeta\}, \tilde B, \zeta, u, \beta$).
  - if $y = 1$ then $\nu \leftarrow \nu - 1$.

7. If $\nu = 0$, then mark V_i as finished in $\tilde B$.
8. if $u =$ True and AppendRoot($\beta, B, \tilde B, \hat\alpha$) = 0, then call CompleteRootSystem($R \cup \{\beta\}, \tilde B, \beta,$ False$, \beta$).
Terminate the function call.

Note that we only used necessary conditions for root systems, so after the computation we still need to check which of the sets are indeed root systems. A short program in MAGMA confirms that Algorithm 4.4 yields only root systems, for instance using this algorithm:

Algorithm 4.6. RootSystemsForAllObjects(R)
Returns the root systems for all objects if $R = R_+^a$ determines a Cartan scheme $\mathcal{C}$ such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is an irreducible root system.

Input: R the set of positive roots at one object.
Output: the set of root systems at all objects, or $\{\}$ if R does not yield a Cartan scheme as desired.

1. $N \leftarrow [R]$, $M \leftarrow \{\}$.
2. While $|N| > 0$, do steps 3 to 5.
3. Let F be the last element of N. Remove F from N and include it to M.
4. Let C be the Cartan matrix of F. Compute the three simple reflections given by C.
5. For each simple reflection s, do:
- Compute $G := \{s(v) \mid v \in F\}$. If an element of G has positive and negative coefficients, then return $\{\}$. Otherwise mutliply the negative roots of G by -1.
- If $G \notin M$, then append G to N.

6. Return M.

REMARK 4.7. We list all 55 root systems in Appendix 1. It is also interesting to summarize some of the invariants, which is done in Table 1. Let $\mathcal{O} = \{R^a \mid a \in A\}$ denote the set of root systems in the objects of the Cartan scheme. By identifying objects with the same root system one obtains a quotient Cartan scheme of the simply connected Cartan scheme of the classification (see [**16**, Def. 3.1] for the definition of coverings). This quotient has the minimal number of objects with respect to all quotients of the Cartan scheme. In the fifth column we give the automorphism group of one (equivalently, any) object of this quotient. The last column gives the multiplicities of planes; for example 3^7 means that there are 7 different planes containing precisely 3 positive roots.

Nr.	$\lvert R^a_+\rvert$	$\lvert\mathcal{O}\rvert$	$\lvert A\rvert$	$\mathrm{Hom}(a)$	planes
1	6	1	24	A_3	$2^3, 3^4,$
2	7	4	32	$A_1 \times A_1 \times A_1$	$2^3, 3^6,$
3	8	5	40	B_2	$2^4, 3^6, 4^1,$
4	9	1	48	B_3	$2^6, 3^4, 4^3,$
5	9	1	48	B_3	$2^6, 3^4, 4^3,$
6	10	5	60	$A_1 \times A_2$	$2^6, 3^7, 4^3,$
7	10	10	60	A_2	$2^6, 3^7, 4^3,$
8	11	9	72	$A_1 \times A_1 \times A_1$	$2^7, 3^8, 4^4,$
9	12	21	84	$A_1 \times A_1$	$2^8, 3^{10}, 4^3, 5^1,$
10	12	14	84	A_2	$2^9, 3^7, 4^6,$
11	13	4	96	$G_2 \times A_1$	$2^9, 3^{12}, 4^3, 6^1,$
12	13	12	96	$A_1 \times A_1 \times A_1$	$2^{10}, 3^{10}, 4^3, 5^2,$
13	13	2	96	B_3	$2^{12}, 3^4, 4^9,$
14	13	2	96	B_3	$2^{12}, 3^4, 4^9,$

Nr.	$\lvert R^a_+\rvert$	$\lvert\mathcal{O}\rvert$	$\lvert A\rvert$	$\mathrm{Hom}(a)$	planes
15	14	56	112	A_1	$2^{11}, 3^{12}, 4^4, 5^2,$
16	15	16	128	$A_1 \times A_1 \times A_1$	$2^{13}, 3^{12}, 4^6, 5^2,$
17	16	36	144	$A_1 \times A_1$	$2^{14}, 3^{15}, 4^6, 5^1, 6^1,$
18	16	24	144	A_2	$2^{15}, 3^{13}, 4^6, 5^3,$
19	17	10	160	$B_2 \times A_1$	$2^{16}, 3^{16}, 4^7, 6^2,$
20	17	10	160	$B_2 \times A_1$	$2^{16}, 3^{16}, 4^7, 6^2,$
21	17	10	160	$B_2 \times A_1$	$2^{18}, 3^{12}, 4^7, 5^4,$
22	18	30	180	A_2	$2^{18}, 3^{18}, 4^6, 5^3, 6^1,$
23	18	90	180	A_1	$2^{19}, 3^{16}, 4^6, 5^5,$
24	19	25	200	$A_1 \times A_1 \times A_1$	$2^{20}, 3^{20}, 4^6, 5^4, 6^1,$
25	19	8	192	$G_2 \times A_1$	$2^{21}, 3^{18}, 4^6, 6^4,$
26	19	50	200	$A_1 \times A_1$	$2^{20}, 3^{20}, 4^6, 5^4, 6^1,$
27	19	25	200	$A_1 \times A_1 \times A_1$	$2^{20}, 3^{20}, 4^6, 5^4, 6^1,$
28	19	8	192	$G_2 \times A_1$	$2^{24}, 3^{12}, 4^6, 5^6, 6^1,$
29	20	27	216	B_2	$2^{20}, 3^{26}, 4^4, 5^4, 8^1,$
30	20	110	220	A_1	$2^{21}, 3^{24}, 4^6, 5^4, 7^1,$
31	20	110	220	A_1	$2^{23}, 3^{20}, 4^7, 5^5, 6^1,$
32	21	15	240	$B_2 \times A_1$	$2^{22}, 3^{28}, 4^6, 5^4, 8^1,$
33	21	30	240	$A_1 \times A_1 \times A_1$	$2^{26}, 3^{20}, 4^9, 5^4, 6^2,$
34	21	5	240	B_3	$2^{24}, 3^{24}, 4^9, 6^4,$
35	22	44	264	A_2	$2^{27}, 3^{25}, 4^9, 5^3, 6^3,$
36	25	42	336	$A_1 \times A_1 \times A_1$	$2^{33}, 3^{34}, 4^{12}, 5^2, 6^3, 8^1,$
37	25	14	336	$G_2 \times A_1$	$2^{36}, 3^{30}, 4^9, 5^6, 6^4,$
38	25	28	336	$A_1 \times A_2$	$2^{36}, 3^{30}, 4^9, 5^6, 6^4,$
39	25	7	336	B_3	$2^{36}, 3^{28}, 4^{15}, 6^6,$
40	26	182	364	A_1	$2^{35}, 3^{39}, 4^{10}, 5^4, 6^3, 8^1,$
41	26	182	364	A_1	$2^{37}, 3^{36}, 4^9, 5^6, 6^3, 7^1,$
42	27	49	392	$A_1 \times A_1 \times A_1$	$2^{38}, 3^{42}, 4^9, 5^6, 6^3, 8^1,$
43	27	98	392	$A_1 \times A_1$	$2^{39}, 3^{40}, 4^{10}, 5^6, 6^2, 7^2,$
44	27	98	392	$A_1 \times A_1$	$2^{39}, 3^{40}, 4^{10}, 5^6, 6^2, 7^2,$
45	28	420	420	1	$2^{41}, 3^{44}, 4^{11}, 5^6, 6^2, 7^1, 8^1,$
46	28	210	420	A_1	$2^{42}, 3^{42}, 4^{12}, 5^6, 6^1, 7^3,$
47	28	70	420	A_2	$2^{42}, 3^{42}, 4^{12}, 5^6, 6^1, 7^3,$
48	29	56	448	$A_1 \times A_1 \times A_1$	$2^{44}, 3^{46}, 4^{13}, 5^6, 6^2, 8^2,$
49	29	112	448	$A_1 \times A_1$	$2^{45}, 3^{44}, 4^{14}, 5^6, 6^1, 7^2, 8^1,$
50	29	112	448	$A_1 \times A_1$	$2^{45}, 3^{44}, 4^{14}, 5^6, 6^1, 7^2, 8^1,$
51	30	238	476	A_1	$2^{49}, 3^{44}, 4^{17}, 5^6, 6^1, 7^1, 8^2,$
52	31	21	504	$G_2 \times A_1$	$2^{54}, 3^{42}, 4^{21}, 5^6, 6^1, 8^3,$
53	31	21	504	$G_2 \times A_1$	$2^{54}, 3^{42}, 4^{21}, 5^6, 6^1, 8^3,$
54	34	102	612	A_2	$2^{60}, 3^{63}, 4^{18}, 5^6, 6^4, 8^3,$
55	37	15	720	B_3	$2^{72}, 3^{72}, 4^{24}, 6^{10}, 8^3,$

Table 1: Invariants of irreducible root systems of rank three, see Rem. 4.7

At first sight, one is tempted to look for a formula for the number of objects in the universal covering depending on the number of roots. There is an obvious one: consider the coefficients of $4/((1-x)^2(1-x^4))$. However, there are exceptions, for example nr. 29 with 20 positive roots and 216 objects (instead of 220).

Rank 3 Nichols algebras of diagonal type with finite irreducible arithmetic root system are classified in [**27**, Table 2]. In Table 2 we identify the Weyl groupoids of these Nichols algebras.

row in [**27**, Table 2]	1	2	3	4	5	6	7	8	9
Weyl groupoid	1	5	4	1	5	3	11	1	2
row in [**27**, Table 2]	10	11	12	13	14	15	16	17	18
Weyl groupoid	2	2	5	13	5	6	7	8	14

Table 2: Weyl groupoids of rank three Nichols algebras of diagonal type

1. Irreducible root systems of rank three

We give the roots in a multiplicative notation to save space: The word $1^x 2^y 3^z$ corresponds to $x\alpha_3 + y\alpha_2 + z\alpha_1$.

Notice that we have chosen a "canonical" object for each groupoid. Write $\pi(R^a_+)$ for the set R^a_+ where the coordinates are permuted via $\pi \in S_3$. Then the set listed below is the minimum of $\{\pi(R^a_+) \mid a \in A,\, \pi \in S_3\}$ with respect to the lexicographical ordering on the sorted sequences of roots.

Nr. 1 with 6 positive roots:
1, 2, 3, 12, 13, 123
Nr. 2 with 7 positive roots:
1, 2, 3, 12, 13, 23, 123
Nr. 3 with 8 positive roots:
1, 2, 3, 12, 13, $1^2 2$, 123, $1^2 23$
Nr. 4 with 9 positive roots:
1, 2, 3, 12, 13, $1^2 2$, 123, $1^2 23$, $1^2 23^2$
Nr. 5 with 9 positive roots:
1, 2, 3, 12, 23, $1^2 2$, 123, $1^2 23$, $1^2 2^2 3$
Nr. 6 with 10 positive roots:
1, 2, 3, 12, 13, $1^2 2$, $1^2 3$, 123, $1^2 23$, $1^3 23$
Nr. 7 with 10 positive roots:
1, 2, 3, 12, 13, 23, $1^2 2$, 123, $1^2 23$, $1^2 2^2 3$
Nr. 8 with 11 positive roots:
1, 2, 3, 12, 13, $1^2 2$, $1^2 3$, 123, $1^2 23$, $1^3 23$, $1^3 2^2 3$
Nr. 9 with 12 positive roots:

$1, 2, 3, 12, 13, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{3}23, 1^{3}2^{2}3, 1^{4}2^{2}3$
Nr. 10 with 12 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{2}23, 1^{3}23, 1^{2}2^{2}3, 1^{3}2^{2}3$
Nr. 11 with 13 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{3}2^{2}3, 1^{4}2^{2}3$
Nr. 12 with 13 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}23, 1^{4}23, 1^{4}2^{2}3$
Nr. 13 with 13 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{2}23, 1^{3}23, 1^{2}2^{2}3, 1^{3}2^{2}3, 1^{4}2^{2}3$
Nr. 14 with 13 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 123, 13^{2}, 1^{2}23, 123^{2}, 1^{2}23^{2}, 1^{3}23^{2}, 1^{3}2^{2}3^{2}$
Nr. 15 with 14 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3$
Nr. 16 with 15 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3$
Nr. 17 with 16 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3$
Nr. 18 with 16 positive roots:
$1, 2, 3, 12, 23, 1^{2}2, 123, 1^{3}2, 1^{2}23, 12^{2}3, 1^{3}23, 1^{2}2^{2}3, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{4}2^{3}3, 1^{4}2^{3}3^{2}$
Nr. 19 with 17 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{4}2, 1^{3}23, 1^{4}23, 1^{5}23, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{6}2^{2}3$
Nr. 20 with 17 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{5}2^{2}3^{2}$
Nr. 21 with 17 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{3}23, 1^{2}2^{2}3, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{5}2^{3}3, 1^{5}2^{3}3^{2}, 1^{6}2^{3}3^{2}$
Nr. 22 with 18 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{5}2^{3}3, 1^{6}2^{3}3$
Nr. 23 with 18 positive roots:
$1, 2, 3, 12, 13, 23, 1^{2}2, 123, 1^{3}2, 1^{2}23, 12^{2}3, 1^{3}23, 1^{2}2^{2}3, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{3}2^{3}3, 1^{4}2^{3}3, 1^{4}2^{3}3^{2}$
Nr. 24 with 19 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{4}2, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{6}2^{2}3, 1^{6}2^{3}3, 1^{7}2^{3}3, 1^{7}2^{3}3^{2}$
Nr. 25 with 19 positive roots:
$1, 2, 3, 12, 23, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{4}2, 1^{3}23, 1^{2}2^{2}3, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{6}2^{2}3, 1^{6}2^{3}3, 1^{6}2^{3}3^{2}$
Nr. 26 with 19 positive roots:
$1, 2, 3, 12, 13, 23, 1^{2}2, 12^{2}, 123, 1^{3}2, 1^{2}23, 12^{2}3, 1^{3}23, 1^{2}2^{2}3, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{3}2^{3}3, 1^{4}2^{3}3, 1^{4}2^{3}3^{2}$
Nr. 27 with 19 positive roots:
$1, 2, 3, 12, 13, 23, 1^{2}2, 123, 1^{3}2, 1^{2}23, 12^{2}3, 1^{3}2^{2}, 1^{3}23, 1^{2}2^{2}3, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{3}2^{3}3, 1^{4}2^{3}3, 1^{4}2^{3}3^{2}$
Nr. 28 with 19 positive roots:
$1, 2, 3, 12, 23, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{2}2^{2}3, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{3}2^{3}3, 1^{4}2^{3}3, 1^{5}2^{3}3, 1^{6}2^{3}3, 1^{6}2^{4}3$
Nr. 29 with 20 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{4}2, 1^{3}2^{2}, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{5}2^{2}, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{6}2^{2}3, 1^{6}2^{3}3, 1^{7}2^{3}3$
Nr. 30 with 20 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{4}2, 1^{3}2^{2}, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{6}2^{2}3, 1^{6}2^{3}3, 1^{7}2^{3}3, 1^{7}2^{3}3^{2}$
Nr. 31 with 20 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{2}2^{2}3, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{4}2^{3}3, 1^{5}2^{3}3, 1^{6}2^{3}3^{2}$
Nr. 32 with 21 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 123, 1^{3}2, 1^{2}23, 1^{4}2, 1^{3}2^{2}, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{5}2^{2}, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{6}2^{2}3, 1^{6}2^{3}3, 1^{7}2^{3}3, 1^{7}2^{3}3^{2}$
Nr. 33 with 21 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{2}2^{2}3, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{4}2^{3}3, 1^{5}2^{3}3, 1^{6}2^{3}3, 1^{6}2^{3}3^{2}$
Nr. 34 with 21 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{5}2^{3}3, 1^{5}2^{2}3^{2}, 1^{6}2^{3}3, 1^{6}2^{3}3^{2}, 1^{7}2^{3}3^{2}$
Nr. 35 with 22 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{3}2^{2}, 1^{3}23, 1^{2}2^{2}3, 1^{4}23, 1^{3}2^{2}3, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{4}2^{3}3, 1^{5}2^{3}3, 1^{5}2^{2}3^{2}, 1^{5}2^{3}3^{2}, 1^{6}2^{3}3^{2}$
Nr. 36 with 25 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{4}2, 1^{3}2^{2}, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{5}2^{2}, 1^{5}23, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{6}2^{2}3, 1^{7}2^{2}3, 1^{6}2^{3}3, 1^{7}2^{3}3, 1^{8}2^{3}3, 1^{8}2^{3}3^{2}$
Nr. 37 with 25 positive roots:
$1, 2, 3, 12, 13, 1^{2}2, 1^{2}3, 123, 1^{3}2, 1^{2}23, 1^{4}2, 1^{3}23, 1^{4}23, 1^{3}2^{2}3, 1^{5}23, 1^{4}2^{2}3, 1^{5}2^{2}3, 1^{6}2^{2}3, 1^{7}2^{2}3, 1^{6}2^{3}3, 1^{7}2^{3}3, 1^{8}2^{3}3,$

$1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2$

Nr. 38 with 25 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 12^2, 123, 1^3 2, 1^2 23, 12^2 3, 1^3 23, 1^2 2^2 3, 1^4 23, 1^3 2^2 3, 1^4 2^2 3, 1^3 2^3 3, 1^3 2^2 3^2, 1^4 2^3 3, 1^5 2^3 3,$
$1^4 2^3 3^2, 1^5 2^3 3^2, 1^6 2^3 3^2, 1^7 2^4 3^2$

Nr. 39 with 25 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^3 2^2, 1^3 23, 1^2 2^2 3, 1^4 23, 1^3 2^2 3, 1^4 2^2 3, 1^5 2^2 3, 1^4 2^3 3, 1^5 2^3 3, 1^5 2^2 3^2, 1^6 2^3 3,$
$1^5 2^3 3^2, 1^6 2^3 3^2, 1^7 2^3 3^2, 1^7 2^4 3^2$

Nr. 40 with 26 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^7 2^2 3, 1^6 2^3 3,$
$1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2$

Nr. 41 with 26 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^7 2^2 3, 1^6 2^3 3, 1^7 2^3 3,$
$1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2$

Nr. 42 with 27 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^7 2^2 3, 1^6 2^3 3,$
$1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2$

Nr. 43 with 27 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3, 1^6 2^3 3,$
$1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2$

Nr. 44 with 27 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^7 2^2 3, 1^6 2^3 3, 1^7 2^3 3,$
$1^7 2^2 3^2, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2$

Nr. 45 with 28 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3,$
$1^6 2^3 3, 1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2$

Nr. 46 with 28 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3, 1^6 2^3 3,$
$1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2, 1^9 2^4 3^2$

Nr. 47 with 28 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3, 1^6 2^3 3,$
$1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2, 1^{11} 2^4 3^2$

Nr. 48 with 29 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3,$
$1^6 2^3 3, 1^7 2^3 3, 1^7 2^2 3^2, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2$

Nr. 49 with 29 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3,$
$1^6 2^3 3, 1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2, 1^9 2^4 3^2$

Nr. 50 with 29 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3,$
$1^6 2^3 3, 1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2, 1^{11} 2^4 3^2$

Nr. 51 with 30 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3,$
$1^6 2^3 3, 1^7 2^3 3, 1^7 2^2 3^2, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2, 1^9 2^4 3^2$

Nr. 52 with 31 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 23, 1^5 2, 1^4 23, 1^6 2, 1^5 23, 1^4 2^2 3, 1^6 23, 1^5 2^2 3, 1^7 23, 1^6 2^2 3, 1^7 2^2 3, 1^8 2^2 3,$
$1^9 2^2 3, 1^{10} 2^2 3, 1^9 2^3 3, 1^{10} 2^3 3, 1^{11} 2^3 3, 1^{10} 2^3 3^2, 1^{11} 2^3 3^2, 1^{12} 2^3 3^2$

Nr. 53 with 31 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^2 3^2, 1^7 2^2 3,$
$1^6 2^3 3, 1^7 2^3 3, 1^7 2^2 3^2, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^3 3^2, 1^9 2^3 3^2, 1^9 2^4 3^2, 1^{11} 2^4 3^2$

Nr. 54 with 34 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^3 3, 1^7 2^2 3,$
$1^6 2^3 3, 1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^8 2^4 3, 1^8 2^3 3^2, 1^9 2^4 3, 1^9 2^3 3^2, 1^9 2^4 3^2, 1^{11} 2^4 3^2, 1^{11} 2^5 3^2, 1^{12} 2^5 3^2$

Nr. 55 with 37 positive roots:

$1, 2, 3, 12, 13, 1^2 2, 1^2 3, 123, 1^3 2, 1^2 23, 1^4 2, 1^3 2^2, 1^3 23, 1^4 23, 1^3 2^2 3, 1^5 2^2, 1^5 23, 1^4 2^2 3, 1^5 2^2 3, 1^6 2^2 3, 1^5 2^3 3, 1^7 2^2 3,$
$1^6 2^3 3, 1^7 2^3 3, 1^8 2^3 3, 1^7 2^3 3^2, 1^9 2^3 3, 1^8 2^4 3, 1^8 2^3 3^2, 1^9 2^4 3, 1^9 2^3 3^2, 1^{10} 2^4 3, 1^9 2^4 3^2, 1^{11} 2^4 3^2, 1^{11} 2^5 3^2, 1^{12} 2^5 3^2,$
$1^{13} 2^5 3^2$

CHAPTER 6

Minimal fields of definition for simplicial arrangements in the real projective plane

M. CUNTZ

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For each simplicial arrangement in the real projective plane of the catalogue of Grünbaum [**23**], we determine the minimal extension of the rationals over which there exists a realization of its incidence structure. For the infinite families we use the symmetries of the incidence. For the sporadic arrangements we give an algorithm that uses Gröbner bases.

1. Introduction

Recently I. Heckenberger and the author have classified the so-called finite Weyl groupoids of rank three [**19**]. A *Weyl groupoid* is a generalization of the Weyl group, see [**17**] for an introduction. It was B. Mühlherr who noticed that the root systems of rank three Weyl groupoids yield simplicial arrangements in the real projective plane. Thanks to the catalogue of B. Grünbaum [**23**], it was easy to identify them: It turned out that 53 of the 67 sporadic arrangements in the large component of his Hasse diagram come from Weyl groupoids. This is motivation enough to investigate simplicial arrangements from this new viewpoint, especially since it is still an open question whether the catalogue of Grünbaum is complete.

To the Weyl groupoids are associated certain *root systems*. With respect to the simple roots, the coefficients of the roots are rational integers. The Weyl groups are obtained as a special case, but for example the Coxeter group of type H_3 is not included in this setting. One reason is that there is no arrangement over the rationals with the same incidence structure as the arrangement of type H_3. Thus as a first step, it is important to understand which number fields are required to "realize" the incidence structure of an arrangement.

In this note, we develop a technique to compute these fields of definition. For the infinite series, we use the symmetry of the incidence structure to deduce that the solutions are in fact unique up to projectivity. The known sporadic arrangements are dealt by an algorithm that uses Gröbner bases to obtain enough restrictions to determine the field extension.

The infinite families $\mathcal{R}(1)$, $\mathcal{R}(2)$ require the following fields of definition (see Theorem 3.6 and Corollary 3.7 for details):

$$\mathbb{Q}(\zeta) \cap \mathbb{R} \quad \text{for } \zeta \text{ a root of unity.}$$

The known sporadic arrangements all have a realization over one of (see Theorem 4.1 for details):

$$\mathbb{Q}, \quad \mathbb{Q}(\sqrt{2}), \quad \mathbb{Q}(\sqrt{3}), \quad \mathbb{Q}(\sqrt{5}) \quad \text{or} \quad \mathbb{Q}[X]/(X^3 - 3X + 25).$$

This note is organized as follows. We start with a section in which we prove that any simplicial arrangement has a realization over an algebraic number field. In the following section we compute the fields of definition for the infinite series. In the last section we treat the sporadic arrangements.

Acknowledgement. I wish to thank J. Maslowski and G. Malle for helpful discussions.

2. Algebraic realizations

We first recall some definitions (compare [**40**, 1.2, 5.1]).

DEFINITION 2.1. Let K be a field and V a finite dimensional vector space over K. A *projective arrangement* $(\mathcal{A}, V)$ is a finite set of projective hyperplanes in $\mathbb{P}(V)$. Let $L(\mathcal{A})$ be the set of all nonempty intersections of elements of $\mathcal{A}$. If $K \subseteq \mathbb{R}$ and every component of the complement of $\bigcup_{H\in\mathcal{A}} H$ in V is an open simplicial cone, then we call $\mathcal{A}$ a *simplicial arrangement*.

Throughout this note, all simplicial arrangements will be in the projective plane over a subfield of $\mathbb{R}$, i.e. $K \subseteq \mathbb{R}$, $V = K^3$. We write "(,)" for the usual inner product on $\mathbb{R}^3$.

DEFINITION 2.2. An *incidence structure* is a triple (P, L, I) where P is a set of "points", L is a set of "lines" and $I \subseteq P \times L$ is the *incidence relation*.

DEFINITION 2.3. Given an incidence structure I, we call a *realization of I over K* an arrangement $(\mathcal{A}, K^3)$ such that the poset $L(\mathcal{A})$ (with respect to inclusion) is given exactly by I. Conversely, from an arrangement $\mathcal{A}$ we obtain an incidence structure from the poset $L(\mathcal{A})$.

PROPOSITION 2.4. *If I is the incidence of an arrangement in $\mathbb{P}(\mathbb{R}^3)$, then there exists a finite field extension K of $\mathbb{Q}$ such that $K \subset \mathbb{R}$ and such that I admits a realization over K.*
The same holds for simplicial arrangements: There exists a finite field extension $K \subset \mathbb{R}$ of $\mathbb{Q}$ such that I admits a simplicial realization over K for which the same triples of lines give open simplicial cones.

PROOF. Let $v_1, \dots, v_n \in \mathbb{R}^3$ be normal vectors of the n planes of an arrangement. If $g_{i,j} \neq 0$ is an element of the intersection of $v_i^\perp$ and $v_j^\perp$, then

$$(g_{i,j}, v_i) = 0 = (g_{i,j}, v_j), \tag{2.1}$$

and without loss of generality, we may assume

$$(g_{i,j}, g_{i,j}) = 1. \tag{2.2}$$

For each k such that $g_{i,j}$ is not on plane k we have

$$(g_{i,j}, v_k)x_{i,j,k} = 1 \tag{2.3}$$

for some $x_{i,j,k} \in \mathbb{R}$, and if $g_{i,j}$ is in the intersection of $v_k^\perp$ and $v_l^\perp$, then we may assume

$$g_{i,j} = g_{k,l}. \tag{2.4}$$

Equations (2.1), (2.2), (2.3), (2.4) define an ideal $\mathfrak{I}$ in the polynomial ring over $\mathbb{Q}$ with the coordinates of all $v_i, g_{i,j}$ and $x_{i,j,k}$ as indeterminates. If the corresponding variety (over $\mathbb{R}$) is non-empty, then we have at least one algebraic solution:

Assume first that $\mathfrak{I}$ has dimension 0. Then clearly all points on the variety $\mathcal{V}(\mathfrak{I})$ have algebraic coordinates, so if $\mathcal{V}(\mathfrak{I}) \neq \emptyset$ then we also have an algebraic solution. Now assume that the dimension of $\mathfrak{I}$ is greater than 0. Since $\mathcal{V}(\mathfrak{I}) \neq \emptyset$, there exists a hyperplane which has non trivial intersection with $\mathcal{V}(\mathfrak{I})$. But $\mathbb{Q}$ is dense in $\mathbb{R}$, so in the space of hyperplanes that meet $\mathcal{V}(\mathfrak{I})$ there also exists one defined by a rational form. Adding this to the ideal we get an ideal of dimension $\dim(\mathfrak{I}) - 1$; by induction we obtain an algebraic solution.

For the case of simplicial arrangements it remains to translate the fact that we need a triangulation. Each triple (i, j, l) of planes that yields an open simplicial cone gives a set of inequalities:

Write a v_k with respect to the basis (v_i, v_j, v_l). Notice that for this we need to include the generator $\det(v_i, v_j, v_l)x = 1$ for some new variable x to our ideal, otherwise (v_i, v_j, v_l) is not a basis. The base change takes place over our polynomial ring, since x is the inverse of the determinant.

Now view all vectors with respect to the basis (v_i, v_j, v_l). The coordinates of v_k are greater or equal to 0 if and only if the planes orthogonal to v_i, v_j, v_l form an open simplicial cone which intersects trivially with any hyperplane of $\mathcal{A}$: Assume first that

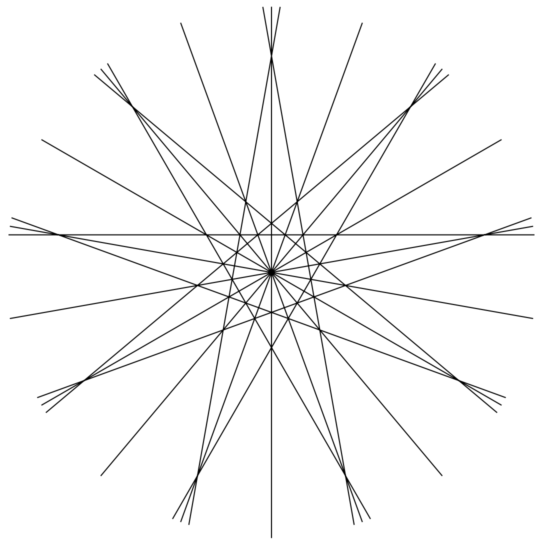

FIGURE 1. The arrangement $\mathcal{A}(18,1)$

$(w, v_k) = 0$ and that the coordinates of w are all greater or equal to 0. Then from $(w, v_k) = 0$ we see that either $w = 0$ or there exist coordinates c, d of v_k with $c > 0$ and $d < 0$. Conversely, if the coordinates of v_k are all greater or equal to 0 and w lies on plane k, i.e. $(w, v_k) = 0$, then either $w = 0$ or w has some negative coordinate.

Thus for each open simplicial cone we obtain a set of inequalities. The same argument as for arbitrary arrangements gives: Either there is no solution, or there exists an algebraic solution. □

3. The infinite families

There are three known infinite families of simplicial arrangements in $\mathbb{P}(\mathbb{R}^3)$. They are denoted by $\mathcal{R}(0)$, $\mathcal{R}(1)$, $\mathcal{R}(2)$ in [**23**]. Family $\mathcal{R}(0)$ consists of *near pencils*. It is clear from definition that all near pencils may be realized over $\mathbb{Q}$, hence we will ignore them in this note.

Family $\mathcal{R}(1)$ consists of the following arrangements: Starting with a regular convex n-gon in the Euclidean plane, the arrangement $\mathcal{A}(2n, 1)$ is obtained by taking the n lines determined by the sides of the n-gon together with the n lines of mirror symmetry of that n-gon (see Figure 1 for an example).

Finally for $n = 4m+1$, from $\mathcal{A}(4m, 1)$ one obtains a new arrangement by adjoining the "line at infinity". These are the arrangements in the family $\mathcal{R}(2)$.

DEFINITION 3.1. We will need the arrangement $\mathcal{A}(2n, 1)$ more explicitly, so here is a more precise definition. We write I_n for its incidence structure. Let $\zeta = \exp \frac{2\pi \mathrm{i}}{n}$.

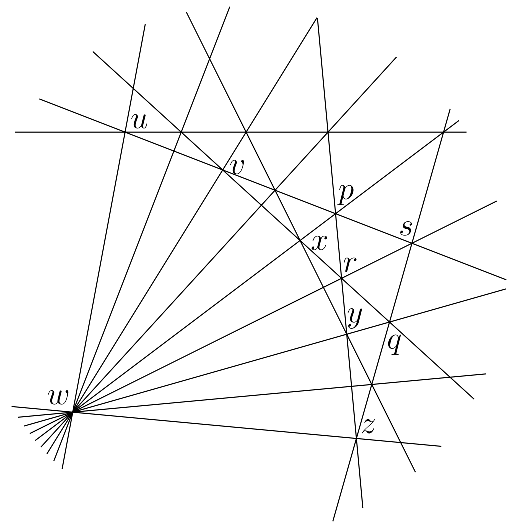

FIGURE 2. Proof of Lemma 3.2

We choose our labels for the points in such a way that 1 is the point in the center and $2, \ldots, n+1$ are the vertices of the n-gon in counterclockwise ordering: $p_1 := (0 : 0 : 1)$ and the n-gon has vertices

$$p_{i+2} := (\mathrm{Re}(\zeta^i) : \mathrm{Im}(\zeta^i) : 1)$$

for $i = 0, \ldots, n-1$. For the next lemma, we need some information about the incidence: Consider Figure 2 as a part of the n-gon and identify $p_1, \ldots, p_5$ with w, u, v, x, y. Using symmetries of the n-gon one can check that the incidences between the points p, q, r, s and the lines of the figure do not depend on n.

LEMMA 3.2. *A realization of the incidence structure I_n is uniquely determined by the choice of points $p_1, \ldots, p_5$ in $\mathbb{P}(\mathbb{R}^3)$, where p_i corresponds to the intersection point labeled by i.*

PROOF. First observe that the complete image is given by the points labeled $1, \ldots,$ $n+1$. We show that if the points labeled $1, r, \ldots, r+3$ for $1 < r \leq n-4$ are given, then one can construct the point $r+4$. Identify the points $p_1, p_r, \ldots, p_{r+3}$ with w, u, v, x, y in Figure 2. If we write $\iota(a, b, c, d)$ for the intersection point of the lines (a, b) and (c, d), then

$$p = \iota(u, v, x, w), \quad q = \iota(v, x, y, w),$$
$$r = \iota(v, x, p, y), \quad s = \iota(u, v, w, r).$$

Thus the next point p_{r+4} is $z = \iota(p, y, s, q)$. By induction we obtain the claim. □

DEFINITION 3.3. Call a tuple $(p_1, \ldots, p_5)$ of points in $\mathbb{P}(\mathbb{R}^3)$ a *solution* for I_n, if the construction of Lemma 3.2 leads to the incidence structure I_n. We write

$$N : \mathbb{P}(\mathbb{R}^3)^5 \to \mathbb{P}(\mathbb{R}^3), \quad (w, u, v, x, y) \mapsto z,$$

i.e. $N(w,u,v,x,y)=z$ is the next point according to Lemma 3.2 or Figure 2.

REMARK 3.4. If $\zeta=\exp(2\pi \mathrm{i} k/n)$ for some k with $\gcd(k,n)=1$, then the points $p_1=(0:0:1)$ and

$$p_{i+1}=(\mathrm{Re}(\zeta^i):\mathrm{Im}(\zeta^i):1)$$

for $i=1,\dots,4$ are a solution for I_n.

LEMMA 3.5. *Given a realization of I_n, the points labeled $1,r,r+1,r+2$ are in general position for $1<r<n-1$.*

PROOF. If three different points with labels in $\{1,r,\dots,r+2\}$ were not in general position, then they would lie on a common line. □

THEOREM 3.6. *Let $n\in\mathbb{N}$, $n>2$ and $K\subseteq\mathbb{R}$ a field. If $p_1,\dots,p_m\in\mathbb{P}(K^3)$ are a realization of I_n, then $\mathbb{Q}(\mathrm{Re}(\zeta))=\mathbb{Q}(\zeta)\cap\mathbb{R}\subseteq K$, where $\zeta=\exp(2\pi \mathrm{i}/n)$. Moreover, up to projectivity there exists only one realization of I_n.*

PROOF. Assume that $p_1,\dots,p_5$ are a solution to the indicence structure I_n. Applying a projectivity, we may assume without loss of generality that $p_1=(0:0:1)$, $p_2=(1:0:1)$, $p_3=(0:1:1)$, $p_4=(1:1:1)$ and that $p_5=(x:y:z)$ is an indeterminate point. Since $p_1,\dots,p_4$ have rational coordinates, this really is a projectivity on $\mathbb{P}(K^3)$.

We first construct a map $D:\mathbb{P}(K^3)\to\mathbb{P}(K^3)$ in the following way: Let $p_6=N(p_1,\dots,p_5)$ be the unique next point given by Lemma 3.2. Consider the projectivity π given by (notice that the points are in general position by Lemma 3.5)

$$p_1\mapsto p_1,\quad p_3\mapsto p_2,\quad p_4\mapsto p_3,\quad p_5\mapsto p_4.$$

Since p_1,p_3,p_4,p_5,p_6 are a solution to the incidence structure, the points p_1,p_2,p_3,p_4, $\pi(p_6)$ will be a solution as well. Hence we have now two different possible choices for a fifth point: p_5 and $\pi(p_6)$ which we will denote $D(p_5):=\pi(p_6)$.

We now compute $D(p_5)$ for $p_5=(x:y:z)$. We certainly have $x\neq 0$ because otherwise $p_5=(0:y:z)\in\langle(0:0:1),(0:1:1)\rangle$ which contradicts I_n. One computes

$$p_6=((x-y+z)(x+y-z):xy-y^2+2yz-z^2:xz-2y^2+4yz-2z^2)$$

and $\pi(p_6)=D(p_5)$ is

$$\left(x:\frac{(x-y+z)(x+y-z)}{x}:\frac{x^2-xy+xz-y^2+2yz-z^2}{x}\right).$$

Evaluating these functions it turns out that $D(D(p_5))=D(p_5)$ i.e. $D^2=D$. This means that $D(p_5)$ is a solution fixed by D and we thus obtain a projectivity ψ such that $p_1,p_2,\psi(p_2)=p_3,\psi^2(p_2)=p_4,\psi^3(p_2)=D(p_5)$ is a solution and $\psi(p_1)=p_1$, $\psi^n(p_2)=p_2$.

Let φ be a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ with $\psi = \mathbb{P}(\varphi)$. Since p_1, p_2, $\psi(p_2)$, $\psi^2(p_2)$ are in general position and are fixed by ψ^n, we have $\varphi^n = \lambda\mathrm{id}$ for some $\lambda \in \mathbb{R}^\times$. Choose an $\varepsilon \in \mathbb{C}$ with $\varepsilon^n = 1/\lambda$; we may assume $\varepsilon \in \mathbb{R}$ if n is odd or $\lambda > 0$, and $\varepsilon \in \mathrm{i}\mathbb{R}$ otherwise.

But then $\varepsilon\varphi = \mathrm{diag}(\xi, \eta, 1)$ with respect to some basis (b_1, b_2, b_3) and for suitable $\xi, \eta \in \mathbb{C}$ with $\xi^k = \eta^m = 1$, $n = \mathrm{lcm}(k, m)$. The constant term in the minimal polynomial of $\varphi|_{\langle b_1, b_2\rangle}$ is $\xi\eta\varepsilon^{-2}$ and has to be real, thus $\xi = \pm\eta^{-1}$. In both cases, there exists a basis $(\tilde{b}_1, \tilde{b}_2, b_3)$ such that $(\varepsilon\varphi)|_{\langle\tilde{b}_1, \tilde{b}_2\rangle}$ is a rotation of order n. Thus for a solution fixed by D, there exists a projectivity π'

$$q_1 \mapsto p_1, \quad q_2 \mapsto p_2, \quad q_3 \mapsto p_3, \quad q_4 \mapsto p_4,$$

where q_1, q_2, q_3, q_4, q_5 is our prefered solution from Remark 3.4 for $\zeta = \exp(2\pi\mathrm{i}/n)$, and such that $\pi(p_6) = \pi'(q_5)$, explicitly:

$$\pi'(q_5) = \left(\zeta + \zeta^{-1} : \frac{\zeta^4 + \zeta^2 + 1}{\zeta(\zeta^2 + 1)} : \frac{\zeta^4 - \zeta^3 + \zeta^2 - \zeta + 1}{\zeta(\zeta^2 + 1)}\right).$$

(Notice that we only need to know that $\zeta(\zeta^2 + 1) \neq 0$ to compute this expression, the relation $\zeta^n = 1$ is not used.) Using $D(p_5) = \pi'(q_5)$ we obtain

$$x = \zeta + \zeta^{-1}, \quad y = z + 1,$$

thus at least $\mathbb{Q}(\mathrm{Re}(\zeta)) \subseteq K$. For the uniqueness consider the projectivity (reflection) σ given by

$$p_1 \mapsto p_1, \quad p_5 \mapsto p_2, \quad p_4 \mapsto p_3, \quad p_3 \mapsto p_4.$$

This maps p_2 to

$$\sigma(p_2) = (x(z - y) : yz - y^2 : x^2 - xy - y^2 + yz),$$

again a new solution different from p_5. One computes $D(\sigma(p_2)) = ((y - z)^2 : (z - x)(x - 2y + z) : -x^2 + xy + xz + y^2 - 3yz + z^2)$ which is a solution fixed by D and thus equal to $\pi'(q_5)$. Collecting these relations yields a 0-dimensional ideal in $\mathbb{R}[x, y, z]$ with exactly one solution, $(x : y : z) = \pi'(q_5)$, so in fact $D = \mathrm{id}$. $\square$

COROLLARY 3.7. *Let I be the incidence structure of $\mathcal{A}(4m+1, 1)$, an arrangement of family $\mathcal{R}(2)$. If $\mathcal{A}$ is a realization of I over a field $K \subseteq \mathbb{R}$, then $\mathbb{Q}(\mathrm{Re}(\zeta)) = \mathbb{Q}(\zeta) \cap \mathbb{R} \subseteq K$, where $\zeta = \exp(2\pi\mathrm{i}/(4m))$. Moreover, there exists a realization of I over $\mathbb{Q}(\mathrm{Re}(\zeta))$, thus this is the field of definition.*

PROOF. Since $\mathcal{A}(4m+1, 1)$ is a descendant of $\mathcal{A}(4m, 1)$, by Theorem 3.6 we have $\mathbb{Q}(\mathrm{Re}(\zeta)) = \mathbb{Q}(\zeta) \cap \mathbb{R} \subseteq K$, where $\zeta = \exp(2\pi\mathrm{i}/(4m))$. The second assertion holds by Remark 3.4 since adding the line at infinity does not enlarge the required field. $\square$

4. The known sporadic arrangements

THEOREM 4.1. *For each connected component of the Hasse diagram of sporadic arrangements* [**23**, Figure 4] *exists a well defined unique field of definition* K*:*

(1) *The component of* $\mathcal{A}(6,1)$ *has* $K = \mathbb{Q}$.
(2) *The component of* $\mathcal{A}(16,1)$ *has* $K = \mathbb{Q}(\sqrt{2})$.
(3) *The component of* $\mathcal{A}(24,1)$ *has* $K = \mathbb{Q}(\sqrt{3})$.
(4) *The component of* $\mathcal{A}(10,1)$ *has* $K = \mathbb{Q}(\sqrt{5})$.
(5) *The component of* $\mathcal{A}(15,5)$ *has* $K = \mathbb{Q}(x)/(x^3 - 3x + 25)$. *This field is not Galois over* $\mathbb{Q}$*; its splitting field is* $\mathbb{Q}(x)/(x^6+3x^5+5x^4+5x^3+5x^2+3x+1)$ *and has Galois group* S_3.

PROOF. For each minimal and maximal arrangement in [**23**, Figure 4], we compute the fields of definition by the algorithm below. These are:

(1) $\mathcal{A}(10,1)$, $\mathcal{A}(13,4)$, $\mathcal{A}(14,4)$, $\mathcal{A}(16,5)$, $\mathcal{A}(31,1)$ for the component on the left.
(2) $\mathcal{A}(21,4)$, $\mathcal{A}(21,6)$, $\mathcal{A}(25,2)$, $\mathcal{A}(37,3)$ for the component in the middle.
(3) $\mathcal{A}(16,1)$, $\mathcal{A}(17,8)$, $\mathcal{A}(25,5)$, $\mathcal{A}(15,5)$, $\mathcal{A}(21,7)$, $\mathcal{A}(24,1)$, $\mathcal{A}(37,2)$ for the remaining components.

Notice that for the component in the middle the field of definition is always $\mathbb{Q}$, so we do not need to consider the minimal arrangements. □

REMARK 4.2. The only maximal arrangement in the large component in the middle which does not come from a Weyl groupoid is $\mathcal{A}(21,6)$. The fields of definition for the arrangements $\mathcal{A}(10,1)$, $\mathcal{A}(16,1)$, $\mathcal{A}(24,1)$ are given by Theorem 3.6.

4.1. A procedure to determine fields of definition.

DEFINITION 4.3. We will say that an incidence structure I is *generated* by points $\lambda_1,\dots,\lambda_m$ if the set of all points is the smallest set P with

(1) $\lambda_1,\dots,\lambda_m \in P$,
(2) for all $\lambda,\mu,\nu,\rho \in P$ such that $\langle\lambda,\mu\rangle$, $\langle\nu,\rho\rangle$ are lines in I we have $\langle\lambda,\mu\rangle \cap \langle\nu,\rho\rangle \in P$.

The main part of the procedure determines a set of polynomials:

Algorithm 4.4. FindRelations(I)
Computes equations which have solutions over the field of definition of an arrangement.

Input: an incidence structure I.
Output: the Gröbner basis of an ideal $\mathfrak{I}$ of relations satisfied by the coordinates of the points of I.

1. Find a small set of labels $\lambda_1, \dots, \lambda_m$ which generate the incidence structure and such that $\lambda_1, \dots, \lambda_4$ are in general position (for this use the incidence structure: if two points are on the same line and a third is not, then we know that their coordinates are linearly independent).
2. The points $p_1, \dots, p_4$ corresponding to the labels $\lambda_1, \dots, \lambda_4$ may be chosen in general position with coordinates in $\mathbb{Q}$; $p_5, \dots, p_m$ have indeterminate coordinates: $p_i = (x_i : y_i : z_i)$.
3. Over $F = \mathbb{Q}(x_5, y_5, z_5, \dots, x_m, y_m, z_m)$, compute the intersection spaces and new hyperplanes until we have them all. An intersection space is stored as $\langle v \rangle$ for some $v \in F^3$. Multiplying v by the least common multiple of the denominators of the coordinates of v if necessary, we obtain an element of $\mathbb{Q}[x_5, \dots, z_m]^3$.
4. For each pair of planes, compute the "difference" of the intersection space $\langle v \rangle$ and the space $\langle w \rangle$ computed in the last step:
   - Either they have a common non-zero entry, say $v_1 \neq 0 \neq w_1$. Then the "differences" are $v_2 w_1 - w_2 v_1$ and $v_3 w_1 - w_3 v_1$.
   - Or else $v_i w_i = 0$ for all $i = 1, 2, 3$. Then the "differences" are $v_1 - w_1$, $v_2 - w_2$, $v_3 - w_3$.

   Collect these differences in a set $R \subset \mathbb{Q}[x_5, \dots, z_m]$.
5. Compute a Gröbner basis of the ideal $\mathfrak{I}$ generated by the elements of R.
6. For each triple of points q_1, q_2, q_3 in general position (use the incidence structure), compute the determinant of the matrix with rows q_1, q_2, q_3. Collect these in a set D.
7. Let B be the basis of $\mathfrak{I}$. As long as an element f of B is divisible by an element g of D, replace f by f/g. We obtain a new set B'.
8. Compute the ideal $\mathfrak{I}'$ generated by B' and its basis B''. If $\mathfrak{I}' \neq \mathfrak{I}$ then go back to step 7 with $\mathfrak{I} \leftarrow \mathfrak{I}'$, $B \leftarrow B''$.
9. Return B''.

REMARK 4.5. It appears that in practice, if I is the incidence structure of a simplicial arrangement then we always have $m = 5$ in step 1.

REMARK 4.6. The elements of D are polynomials which must be different from 0. The usual technique is to add a new variable v for each $f \in D$ and the equation $fv = 1$. But here D is a very large set, so this would increase the number of variables considerably. Of course, one could also add only one variable v and consider the equation $v \prod_{f \in D} f = 1$, but this is a Polynomial of very high degree.

After using this algorithm, it remains to perform the steps:

(1) Examine the resulting Gröbner basis B''. We obtain a subfield K of the field of definition.
(2) Compute a realization of the incidence structure over K. Then we are sure that K is minimal.

Ad (1). If the ideal $\langle B''\rangle$ has dimension 0 then it easy to determine the field. If the ideal has dimension > 0 then we can hope to extract some information about the field extension from B'': For instance, we can consider the cases that a given coordinate of (x_5, y_5, z_5) is 0 or not, and hence without loss of generality 0 or 1. In both cases we eliminate a variable and obtain an ideal in a smaller ring. This is sufficient to treat all simplicial arrangements of the catalogue.

Ad (2). This is the easiest part. Since we have a generating set of points given by step 1 of the algorithm, it suffices to choose a solution over K for $\lambda_1, \dots, \lambda_m$ and to check that it realizes the incidence structure.

REMARK 4.7. The above algorithm is also applicable to arrangements which are not simplicial.

4.2. Some examples.

4.2.1. $\mathcal{A}(10,1)$. This is the incidence structure I for $\mathcal{A}(10,1)$ (the sets of lines going through the points $1,\dots,16$):
$\{1,2,3,4,5\}$, $\{4,9\}$, $\{1,10\}$, $\{3,6\}$, $\{5,7\}$, $\{2,8\}$, $\{5,9,10\}$, $\{2,6,10\}$, $\{4,6,7\}$, $\{1,7,8\}$, $\{3,8,9\}$, $\{4,8,10\}$, $\{1,6,9\}$, $\{3,7,10\}$, $\{5,6,8\}$, $\{2,7,9\}$.
Our algorithm suggests to start with the points labeled by

$$1, 8, 9, 10, 11.$$

For the first four points, we choose

$$(1:0:0), (1:1:0), (1:0:1), (1:1:1);$$

the fifth point will be $(x : y : z)$. The ideal is generated by 9 polynomials in x, y, z. It has a Gröbner basis with 2 generators: $x - 2z$, $y^2 - yz - z^2$. Now either $x \neq 0$ or $x = 0$. For $x = 0$ the only solution is $(x, y, z) = (0, 0, 0)$, hence $x \neq 0$, say $x = 2$. But then $z = 1$ and

$$f(y) = y^2 - y - 1 = 0.$$

The polynomial f is the minimal polynomial of $-\zeta_5 - \zeta_5^{-1}$, thus a field of definition should include $\sqrt{5}$. This agrees with Theorem 3.6.

4.2.2. $\mathcal{A}(15,5)$. This is the incidence structure I for $\mathcal{A}(15,5)$ (the sets of lines going through the points $1,\ldots,34$):

$\{1,2,3,15,8\}$, $\{1,14,4\}$, $\{1,5,6\}$, $\{1,7\}$, $\{1,9\}$, $\{1,13,10\}$, $\{11,1,12\}$, $\{2,4\}$, $\{2,13,6\}$, $\{11,2,14,7,10\}$, $\{12,2,9\}$, $\{12,13,3,4,7\}$, $\{3,5,10\}$, $\{11,3,9\}$, $\{3,14\}$, $\{4,5\}$, $\{4,6,9\}$, $\{4,8,10\}$, $\{11,4,15\}$, $\{5,7,8\}$, $\{14,5,9\}$, $\{11,13,5\}$, $\{12,15,5\}$, $\{15,6,7\}$, $\{11,6,8\}$, $\{12,6,10\}$, $\{14,6\}$, $\{13,8,9\}$, $\{12,14,8\}$, $\{15,9,10\}$, $\{13,14,15\}$, $\{2,5\}$, $\{3,6\}$, $\{7,9\}$.

Our algorithm suggests to start with the points labeled by

$$1, 2, 19, 22, 28.$$

Again, for the first four points we choose

$$(1:0:0), (1:1:0), (1:0:1), (1:1:1);$$

the fifth point will be $(x:y:z)$. The ideal is generated by 40 polynomials in x,y,z. It has a Gröbner basis (computed for instance with MAGMA) with 8 generators. There are 3105 determinants which may be used to reduce this basis, and 8 more come from the fact that certain points are different. After three cycles of steps 7, 8 of the above algorithm, we obtain an ideal with basis

$$x-2y, \quad y^3-5y^2z+4yz^2-z^3.$$

Now either $x=1$ or $x=0$. For $x=0$ the only solution is $(x,y,z)=(0,0,0)$, hence $x=1$. But then $y=1/2$ and

$$f(z)=z^3-2z^2+5/4z-1/8=0.$$

The polynomial f is irreducible, thus a solution would at least contain a root α of f. After computing a solution over $\mathbb{Q}(\alpha)$, one checks that it is indeed simplicial.

The usual simplification algorithm on f yields the nicer polynomial $z^3-3z+25$ defining the same number field (evaluate f at $x/6+2/3$ and multiply by 216).

4.2.3. $\mathcal{A}(37,2)$. We omit the incidence structure I for $\mathcal{A}(37,2)$ because it takes too much space. The ideal is generated by 489 polynomials in x,y,z. MAGMA computes a Gröbner basis with 5 generators. Since there are 971970 potential determinants, we concentrate on those given by the five starting points. Together with the polynomials coming from the differences of points we get 287 polynomials which must be different from 0. After three cycles of steps 7, 8 of the above algorithm, we obtain an ideal with basis

$$x-5y+5z, \quad y^2-6yz+6z^2.$$

Again, if $x=0$ then $(x,y,z)=(0,0,0)$, hence $x=1$. But then

$$y-z-1/5=0, \quad z^2-4/5z+1/25=0,$$

thus the field of definition is $\mathbb{Q}(\sqrt{3})$.

CHAPTER 7

Crystallographic arrangements: Weyl groupoids and simplicial arrangements

M. CUNTZ
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We introduce the simple notion of a "crystallographic arrangement" and prove a one-to-one correspondence between these arrangements and the connected simply connected Cartan schemes for which the real roots are a finite root system (up to equivalence on both sides). Thus the classification of "finite Weyl groupoids" leads to a complete classification of this large subclass of the class of simplicial arrangements.

1. Introduction

Originally, Weyl groupoids were introduced as the underlying symmetry structure of certain Nichols algebras with the purpose to understand pointed Hopf algebras via the Lifting method of Andruskiewitsch and Schneider [**5**]. For example the upper triangular part of a small quantum group, also called Frobenius-Lusztig kernel, is such a Nichols algebra. Heckenberger classified these structures in the case of Nichols algebras of diagonal type [**26**]. Later, this invariant was defined in a very general setting [**32**]. It plays a similar role as the Weyl group plays for semisimple Lie algebras and algebraic groups.

After an axiomatic approach to Weyl groupoids had been initiated in [**34**] and axioms had been formulated in terms of categories in [**17**], Heckenberger and the author started the classification of the finite case [**16**], [**19**]. The case of rank three [**19**] revealed a connection to geometric combinatorics: Heckenberger and Welker [**33**] proved that the "root systems" obtained from Weyl groupoids define a simplicial arrangement in the same way as reflection groups do. For example in rank three, 53 of the 67 sporadic simplicial arrangements in the large component of the Hasse diagram in [**23**] may be obtained from Weyl groupoids (notice that the Weyl *groups* only yield 2 arrangements, type A_3 and B_3). The classification of simplicial arrangements in the real projective plane is still an open problem. These new insights considerably increase the importance of the theory of Weyl groupoids which henceforth may be considered as a subject in its own right.

The axioms for a Weyl groupoid in [**34**] or [**17**] are rather complicated and seem to be somewhat artificial. In this article we give an equivalent, more intuitive definition of "finite Weyl groupoids", or more precisely "connected simply connected Cartan schemes for which the real roots form a finite root system" in terms of simplicial arrangements. It turns out that almost two pages of definitions and axioms [**17**] (see also Section 3) reduce to one single axiom (see (I) below). This shows that Weyl groupoids are a very natural structure after all. More importantly, our result shows that the classification of Weyl groupoids of rank three [**19**] applies to the very large class of arrangements satisfying axiom (I). Further, the classification of finite Weyl groupoids [**18**] will yield a complete classification of these arrangements in arbitrary dimension. It is conceivable that this new definition could be a step towards a classification of the complete class of simplicial arrangements or at least of the class of free arrangements.

We briefly sketch our result.

Let $\mathcal{A}$ be a simplicial arrangement in $\mathbb{R}^r$, i.e. the complement of the union of all hyperplanes in $\mathcal{A}$ decomposes into open simplicial cones called the chambers. For each hyperplane of $\mathcal{A}$, choose a normal vector. Let R be the set of all these normal vectors including their negatives. For each chamber[1] K of $\mathcal{A}$ we have a basis B^K of $\mathbb{R}^r$ consisting of the normal vectors (in R) of the walls of K pointing to the inside (see Section 2 for more details). Assume that

(I) $\quad R \subseteq \sum_{\alpha\in B^K} \mathbb{Z}\alpha \quad$ for all chambers K.

We call a pair $(\mathcal{A}, R)$ with property (I) a *crystallographic arrangement*[2]. The main result of this article is (see Theorem 5.4):

THEOREM 1.1. *There is a one-to-one correspondence between crystallographic arrangements and connected simply connected Cartan schemes for which the real roots are a finite root system (up to equivalence on both sides):*

(1) *Let $(\mathcal{A}, R)$ be a crystallographic arrangement. Then R is the set of real roots at some object of a connected Cartan scheme $\mathcal{C}$.*
(2) *Every connected simply connected Cartan scheme for which the real roots are a finite root system comes from a unique crystallographic arrangement up to equivalence.*

This paper is organized as follows. We start with a section in which we define the crystallographic arrangements and prove the key lemma (Lemma 2.9). In the following section we recall the definition of Cartan schemes, Weyl groupoids and root systems

[1]We use the letter K because we will need the C for "Cartan matrices" later.

[2]Since the word "integral" is already in use for arrangements, "crystallographic" seems to be the most appropriate.

following [**17**]. In Section 4 we construct a Cartan scheme and a root system for a given crystallographic arrangement. In the last section we summarize the results.

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2. Crystallographic and additive arrangements

Let $r \in \mathbb{N}$, $V := \mathbb{R}^r$. For $\alpha \in V^*$ we write $\alpha^\perp = \ker(\alpha)$. We first recall the definition of a simplicial arrangement (compare [**40**, 1.2, 5.1]).

DEFINITION 2.1. Let $\mathcal{A}$ be a simplicial arrangement in V, i.e. $\mathcal{A} = \{H_1, \dots, H_n\}$ where $H_1, \dots, H_n$ are distinct linear hyperplanes in V and every component of $V \setminus \bigcup_{H \in \mathcal{A}} H$ is an open simplicial cone (a cone as in equation (2.1) for a basis $\alpha_1^\vee, \dots, \alpha_r^\vee$ of V). Let $\mathcal{K}(\mathcal{A})$ be the set of connected components of $V \setminus \bigcup_{H \in \mathcal{A}} H$; they are called the *chambers* of $\mathcal{A}$.

For each H_i, $i = 1, \dots, n$ we choose an element $x_i \in V^*$ such that $H_i = x_i^\perp$. Let

$$R = \{\pm x_1, \dots, \pm x_n\} \subseteq V^*.$$

Further, for $\alpha \in R$ we write

$$N(\alpha) = \{v \in V \mid \alpha(v) \geq 0\}.$$

For each chamber $K \in \mathcal{K}(\mathcal{A})$ set

$$\begin{aligned} W^K &= \{H \in \mathcal{A} \mid \dim(H \cap \overline{K}) = r - 1\}, \\ B^K &= \{\alpha \in R \mid \alpha^\perp \in W^K, \quad N(\alpha) \cap K = K\} \subseteq R. \end{aligned}$$

Here, $\overline{K}$ denotes the closure of K. The elements of W^K are the *walls* of K and B^K "is" the set of normal vectors of the walls of K pointing to the inside. Note that

$$\overline{K} = \bigcap_{\alpha \in B^K} N(\alpha).$$

Moreover, if $\alpha_1^\vee, \dots, \alpha_r^\vee$ is the dual basis to $B^K = \{\alpha_1, \dots, \alpha_r\}$, then

$$K = \Big\{ \sum_{i=1}^r a_i \alpha_i^\vee \mid a_i > 0 \quad \text{for all} \quad i = 1, \dots, r \Big\}. \tag{2.1}$$

The following simple lemma is fundamental for the theory.

LEMMA 2.2. *Let $K \in \mathcal{K}(\mathcal{A})$. If $B^K = \{\alpha_1, \dots, \alpha_r\}$, then for all α in R we have $\alpha \in \pm \sum_i \mathbb{R}_{\geq 0} \alpha_i$.*

PROOF. Now let $\alpha = \sum_i a_i\alpha_i \in R$ for some $a_1, \dots, a_r \in \mathbb{R}$. Write $\{1, \dots, r\} = I_1 \cup I_2 \cup I_3$ such that $a_i > 0$ for $i \in I_1$, $a_i < 0$ for $i \in I_2$ and $a_i = 0$ for $i \in I_3$. Assume that $I_1 \neq \emptyset \neq I_2$. Let $\alpha_1^\vee, \dots, \alpha_r^\vee \in V$ be the dual basis to $\alpha_1, \dots, \alpha_r$ and define

$$v := \sum_{i\in I_1} \frac{1}{|I_1|a_i}\alpha_i^\vee - \sum_{i\in I_2} \frac{1}{|I_2|a_i}\alpha_i^\vee + \sum_{i\in I_3} \alpha_i^\vee.$$

Then $\alpha(v) = 0$, thus $v \in \alpha^\perp$. But for all $\beta \in B^K$, $\beta(v) > 0$, hence $v \in K \cap \alpha^\perp$ which is a contradiction to the fact that K is a chamber and $\alpha^\perp$ a hyperplane of $\mathcal{A}$. □

DEFINITION 2.3. Let $\mathcal{A}$ be a simplicial arrangement and $R \subseteq V^*$ a finite set such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. For $K \in \mathcal{K}(\mathcal{A})$ set

$$R_+^K = R \cap \sum_{\alpha\in B^K} \mathbb{R}_{\geq 0}\alpha.$$

We call $(\mathcal{A}, R)$ a *crystallographic arrangement* if for all $K \in \mathcal{K}(\mathcal{A})$:

(I) $R \subseteq \sum_{\alpha\in B^K} \mathbb{Z}\alpha$.

We call $(\mathcal{A}, R)$ an *additive arrangement* if for all $K \in \mathcal{K}(\mathcal{A})$:

(A) For all $\alpha \in R_+^K \backslash B^K$ there exist $\beta, \gamma \in R_+^K$ with $\alpha = \beta + \gamma$.

Two crystallographic arrangements $(\mathcal{A}, R)$, $(\mathcal{A}', R')$ in V are called *equivalent* if there exists $\psi \in \mathrm{Aut}(V^*)$ with $\psi(R) = R'$. We write $(\mathcal{A}, R) \cong (\mathcal{A}', R')$.

REMARK 2.4. By Lemma 2.2, axiom (I) implies $R \subseteq \pm \sum_{\alpha\in B^K} \mathbb{N}_0\alpha$ for all chambers K.

EXAMPLE 2.5. Let $\mathcal{C}$ be a connected Cartan scheme and assume that $\mathcal{R}^{re}(\mathcal{C})$ is a finite root system (see Section 3). Let a be an object of $\mathcal{C}$. Then $(\mathcal{A} = \{\alpha^\perp \mid \alpha \in R_+^a\}, R^a)$ is an additive, crystallographic arrangement, where we embed R_+^a into the dual space $V^* \cong \mathbb{R}^r$. We will restate this in Section 5, it is a result of [**33**] and [**19**].

EXAMPLE 2.6. Crystallographic reflection arrangements (arrangements from Weyl groups) are a special case of example 2.5. Note that as crystallographic arrangements, arrangements of type B_r and C_r are not equivalent in rank $r \geq 3$. The reader may want to compare the subsequent development with the theory of cristallographic reflection arrangements.

LEMMA 2.7. *Let $(\mathcal{A}, R)$ be as above. Then for all $K \in \mathcal{K}(\mathcal{A})$:*

(1) $R = R_+^K \dot\cup - R_+^K$,
(2) $R_+^K \cap \mathbb{R}\alpha = \{\alpha\}$ *for all* $\alpha \in R_+^K$.

Further, if $(\mathcal{A}, R)$ is additive then it is crystallographic.

PROOF. This follows immediately from the definition. □

Our goal is now to prove that any crystallographic arrangement comes from a "finite Weyl groupoid", and that any "finite Weyl groupoid" is given by some crystallographic arrangement.

In the sequel, let $(\mathcal{A}, R)$ be a crystallographic arrangement and K_0, K be adjacent chambers, i.e. $W^K \cap W^{K_0} = \{\alpha_1^\perp\}$ for some $\alpha_1 \in R_+^{K_0}$. Let $B^{K_0} = \{\alpha_1, \dots, \alpha_r\}$.

LEMMA 2.8. *Let $\beta \in B^K$. Then*

$$\beta = -\alpha_1 \quad \textit{or} \quad \beta \in \sum_{i=1}^r \mathbb{N}_0 \alpha_i.$$

PROOF. Let $\beta \in B^K$ and assume $\beta \in -\sum_{i=1}^r \mathbb{N}_0 \alpha_i$ (using Rem. 2.4). Since $N(\gamma) \cap N(\delta) \subseteq N(\gamma + \delta)$ for all $\gamma, \delta \in V^*$, we have $\overline{K}_0 = \bigcap_{i=1}^r N(\alpha_i) \subseteq N(-\beta)$. But $\overline{K} = \bigcap_{\gamma \in B^K} N(\gamma)$ and thus we obtain

$$\overline{K}_0 \cap \overline{K} \subseteq N(\beta) \cap N(-\beta) = \beta^\perp.$$

Since $\alpha_1^\perp$ is a wall of K_0 and K, we have $\dim(\overline{K}_0 \cap \overline{K}) = r - 1$ and $\overline{K}_0 \cap \overline{K} \subseteq \alpha_1^\perp$. We conclude $\alpha_1^\perp = \beta^\perp$. □

The following lemma will be the key for all arguments concerning crystallographic arrangements in this paper.

LEMMA 2.9. *Write $B^K = \{-\alpha_1, \beta_2, \dots, \beta_r\}$ for some $\beta_2, \dots, \beta_r \in R$. Then there exists a permutation $\tau \in S_r$ with $\tau(1) = 1$ and such that*

$$\beta_i = c_{\tau(i)} \alpha_1 + \alpha_{\tau(i)}, \qquad i = 2, \dots, r$$

for certain $c_2, \dots, c_r \in \mathbb{N}_0$.

PROOF. Let σ be the linear map

$$\sigma : V \to V, \quad \alpha_1 \mapsto -\alpha_1, \quad \alpha_i \mapsto \beta_i \quad \text{for} \quad i = 2, \dots, r.$$

With respect to the basis B^{K_0}, σ is a matrix of the form

$$\begin{pmatrix} -1 & c_2 & \cdots & c_r \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

for some $c_2, \dots, c_r \in \mathbb{N}_0$ and $A \in \mathbb{N}_0^{(r-1)\times(r-1)}$ by Lemma 2.8. But by the same argument with the role of K_0 and K interchanged, with respect to $\{-\alpha_1, \beta_2, \dots, \beta_r\}$, the $\alpha_2, \dots, \alpha_r$ have coefficients in $\mathbb{N}_0$, thus $A^{-1} \in \mathbb{N}_0^{(r-1)\times(r-1)}$. This is clearly only possible if A is a permutation matrix. □

REMARK 2.10. Lemma 2.9 may also be proved for an arrangement $(\mathcal{A}, R)$ for which a slightly stronger version of axiom (A) holds only at the chamber K_0. But it is not clear whether this stronger axiom transports from K_0 to adjacent chambers, so we do not elaborate on this.

DEFINITION 2.11. Let K, K' be adjacent chambers and $\{\alpha^\perp\} = W^K \cap W^{K'}$ for $\alpha \in B^K$. Then by Lemma 2.9 there exist unique $c_\beta \in \mathbb{N}_0$, $\beta \in B^K \backslash \{\alpha\}$ such that

$$\varphi_{K,K'} : B^K \to B^{K'}, \quad \beta \mapsto c_\beta \alpha + \beta, \quad \alpha \mapsto -\alpha$$

is a bijection.

DEFINITION 2.12. Now fix a chamber K_0 and an ordering $B^{K_0} = \{\alpha_1, \dots, \alpha_r\}$. Then for any sequence $\mu_1, \dots, \mu_m \in \{1, \dots, r\}$ we get a unique chain of chambers $K_0, \dots, K_m$ such that K_{i-1} and K_i are adjacent and

$$W^{K_{i-1}} \cap W^{K_i} = \{\varphi_{K_{i-1},K_i} \cdots \varphi_{K_0,K_1}(\alpha_{\mu_i})^\perp\}$$

for $i = 1, \dots, m$. We will write

$$\alpha_j^{K_i} := \varphi_{K_{i-1},K_i} \cdots \varphi_{K_0,K_1}(\alpha_j)$$

for $i = 1, \dots, m$ and $j = 1, \dots, r$. For $i = 0$ we set $\alpha_j^{K_0} := \alpha_j$. But notice that this depends on the chosen chain from K_0 to K_i, that is, the $\mu_1, \dots, \mu_i$.

For $i = 0, \dots, m-1$, denote by $\sigma_{\mu_{i+1}}^{K_i}$ the linear map given by

$$\sigma_{\mu_{i+1}}^{K_i}(\alpha) = \varphi_{K_i,K_{i+1}}(\alpha)$$

for all $\alpha \in B^{K_i}$. By Lemma 2.9 (we have $A = \mathrm{id}$ now),

$$\sigma_{\mu_{i+1}}^{K_i} \in \mathrm{Aut}(V^*), \quad (\sigma_{\mu_{i+1}}^{K_i})^2 = \mathrm{id}.$$

Note that $\sigma_{\mu_{i+1}}^{K_i}$ is a reflection. For a sequence $\mu_1, \dots, \mu_m$ we may abbreviate

$$\sigma_{\mu_m} \cdots \sigma_{\mu_2} \sigma_{\mu_1}^{K_0} := \sigma_{\mu_m}^{K_{m-1}} \cdots \sigma_{\mu_2}^{K_1} \sigma_{\mu_1}^{K_0}$$

since $K_1, \dots, K_m$ are uniquely determined by the sequence of μ_i.

3. Cartan schemes and Weyl groupoids

We briefly recall the notions of Cartan schemes, Weyl groupoids and root systems, following [**17, 16**]. The foundations of the general theory have been developed in [**34**] using a somewhat different terminology.

DEFINITION 3.1. Let I be a non-empty finite set and $\{\alpha_i \mid i \in I\}$ the standard basis of $\mathbb{Z}^I$. By [**37**, §1.1] a *generalized Cartan matrix* $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \le 0$ for all $i, j, k \in I$ with $j \ne k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

DEFINITION 3.2. Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

DEFINITION 3.3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{3.1}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are compositions of maps σ^a_i with $i \in I$ and $a \in A$, where σ^a_i is considered as an element in $\mathrm{Hom}(a, \rho_i(a))$. The category $\mathcal{W}(\mathcal{C})$ is a groupoid in the sense that all morphisms are isomorphisms.

As above, for notational convenience we will often neglect upper indices referring to elements of A if they are uniquely determined by the context. For example, the morphism $\sigma^{\rho_{i_2}\cdots\rho_{i_k}(a)}_{i_1} \cdots \sigma^{\rho_{i_k}(a)}_{i_{k-1}} \sigma^a_{i_k} \in \mathrm{Hom}(a, b)$, where $k \in \mathbb{N}$, $i_1, \dots, i_k \in I$, and $b = \rho_{i_1} \cdots \rho_{i_k}(a)$, will be denoted by $\sigma_{i_1} \cdots \sigma^a_{i_k}$ or by $\mathrm{id}_b\sigma_{i_1} \cdots \sigma_{i_k}$. The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$.

DEFINITION 3.4. A Cartan scheme is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$. The Cartan scheme is called *simply connected*, if $\mathrm{Hom}(a, a) = \{\mathrm{id}_a\}$ for all $a \in A$.

Two Cartan schemes $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I', A', (\rho'_i)_{i\in I'}, (C'^a)_{a\in A'})$ are termed *equivalent*, if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1(\rho_i(a)) = \rho'_{\varphi_0(i)}(\varphi_1(a)), \qquad c^{\varphi_1(a)}_{\varphi_0(i)\varphi_0(j)} = c^a_{ij} \tag{3.2}$$

for all $i, j \in I$ and $a \in A$. We write then $\mathcal{C} \cong \mathcal{C}'$.

Let $\mathcal{C}$ be a Cartan scheme. For all $a \in A$ let

$$(R^{\mathrm{re}})^a = \{\mathrm{id}_a\sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \,|\, k \in \mathbb{N}_0,\, i_1, \dots, i_k, j \in I\} \subseteq \mathbb{Z}^I.$$

The elements of the set $(R^{\mathrm{re}})^a$ are called *real roots* (at a). The pair $(\mathcal{C}, ((R^{\mathrm{re}})^a)_{a\in A})$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. A real root $\alpha \in (R^{\mathrm{re}})^a$, where $a \in A$, is called positive (resp. negative) if $\alpha \in \mathbb{N}^I_0$ (resp. $\alpha \in -\mathbb{N}^I_0$). In contrast to real roots associated to a single generalized Cartan matrix, $(R^{\mathrm{re}})^a$ may contain elements which are neither positive nor negative. A good general theory, which is relevant for example for the study of Nichols algebras, can be obtained if $(R^{\mathrm{re}})^a$ satisfies additional properties.

DEFINITION 3.5. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subseteq \mathbb{Z}^I$, and define $m^a_{i,j} = |R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma^a_i(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m^a_{i,j}$ is finite, then $(\rho_i\rho_j)^{m^a_{i,j}}(a) = a$.

The axioms (R2) and (R3) are always fulfilled for $\mathcal{R}^{\mathrm{re}}$. The root system $\mathcal{R}$ is called *finite* if for all $a \in A$ the set R^a is finite. By [**17**, Prop. 2.12], if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R} = \mathcal{R}^{\mathrm{re}}$, and hence $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$ in that case.

4. Constructing a Weyl groupoid and a root system

We now construct a Weyl groupoid for a given crystallographic arrangement $(\mathcal{A}, R)$ of rank r. Let $I := \{1, \dots, r\}$. Fix a chamber $K_0 \in \mathcal{K}(\mathcal{A})$ and an ordering $B^{K_0} = \{\alpha_1, \dots, \alpha_r\}$. Consider the set

$$\hat{A} := \{(\mu_1, \dots, \mu_m) \mid m \in \mathbb{N},\ \ \mu_1, \dots, \mu_m \in I\}$$

and write $a.\nu$ for the sequence $(\mu_1, \dots, \mu_m, \nu)$ if $a = (\mu_1, \dots, \mu_m)$, $\nu \in I$. We have a map

$$\pi : \hat{A} \to \mathrm{End}(V^*), \quad (\mu_1, \dots, \mu_m) \mapsto \sigma_{\mu_m} \dots \sigma_{\mu_2}\sigma^{K_0}_{\mu_1}$$

which yields an equivalence relation $\sim$ on $\hat{A}$ via

$$v \sim w \quad :\Longleftrightarrow \quad \pi(v) = \pi(w)$$

for $v, w \in \hat{A}$. Let

$$A := \hat{A}_{/\sim}.$$

Each sequence $a = (\mu_1, \dots, \mu_m) \in \hat{A}$ determines a unique map φ_a by

$$\varphi_a := \varphi_{K_{m-1},K_m} \cdots \varphi_{K_0,K_1}$$

where $K_0, \dots, K_m$ is the sequence of chambers corresponding to a.

LEMMA 4.1. *The set A is finite.*

PROOF. Since each sequence $a \in \hat{A}$ leads to a unique chamber K and φ_a is a bijection $B^{K_0} \to B^K$, the set A has at most $|\mathcal{K}(\mathcal{A})|r!$ elements and thus is finite. □

REMARK 4.2. Two equivalent sequences certainly lead to the same chamber. For $a \in A$, we will write K_a for this unique chamber.

DEFINITION 4.3. For each $a \in \hat{A}$ we construct a Matrix $C^a \in \mathbb{Z}^{r\times r}$ in the following way:

Let $i, j \in I$ and let K' be the chamber adjacent to K_a with $W^{K_a} \cap W^{K'} = \{\varphi_a(\alpha_i)^\perp\}$. By Lemma 2.9 there exist integers $c_{i,1}, \dots, c_{i,r}$ such that

$$\varphi_{a.i}(\alpha_j) = -c_{i,j}\varphi_a(\alpha_i) + \varphi_a(\alpha_j).$$

We set

$$C^a := (c_{i,j})_{1\le i,j\le r}.$$

LEMMA 4.4. *Let $a \in \hat{A}$. Then C^a satisfies* (M1) *and* (M2).

PROOF. By Lemma 2.9, $c_{i,i} = 2$ and $c_{i,j} \le 0$ for $i \ne j$. Thus the matrix C^a satisfies (M1).

Without loss of generality take $a = ()$, $m = 0$ and $K_0 = K$. Assume that $c_{i,j} = 0$ and let K', K'' be the chambers with $W^K \cap W^{K'} = \{\alpha_i^\perp\}$, $W^K \cap W^{K''} = \{\alpha_j^\perp\}$. Thus $\sigma_i^K(\alpha_j) = \alpha_j$. But then

$$R \cap \langle \alpha_i, \alpha_j\rangle = \pm\{\alpha_i, \alpha_j\}$$

(otherwise we would have a hyperplane intersecting a chamber) and therefore $\sigma_j^K(\alpha_i) \in \pm\{\alpha_i, \alpha_j\}$ which is only possible if $\sigma_j^K(\alpha_i) = \alpha_i$. Hence $c_{j,i} = 0$ and axiom (M2) holds for C^a. □

Notice that $C^a = C^b$ for $a, b \in \hat{A}$ with $\pi(a) = \pi(b)$, so we obtain a unique Cartan matrix for each element of A, which we will also denote C^a, $a \in A$.

We now define the maps $\rho_i : A \to A$, $i \in I$. First, set

$$\hat{\rho}_i : \hat{A} \to \hat{A}, \quad a \mapsto a.i.$$

Since $\pi(a) = \pi(b)$ implies $\varphi_a = \varphi_b$ for $a, b \in A$, this induces well-defined maps

$$\rho_i : A \to A, \quad \overline{a} \mapsto \overline{a.i},$$

where $\overline{a}$ is the equivalence class of a in A. By Lemma 2.9,

$$\rho_i^2 = \mathrm{id}, \quad c_{i,j}^a = c_{i,j}^{\rho_i(a)}$$

for all $i, j \in I$, $a \in A$, hence (C1) and (C2) are satisfied. We obtain:

PROPOSITION 4.5. *Let* $(\mathcal{A}, R)$ *be a crystallographic arrangement of rank* r *and* K_0 *a chamber. Then* I, A, $(\rho_i)_{i\in I}$, $(C^a)_{a\in A}$ *defined as above define a Cartan scheme which we denote* $\mathcal{C} = \mathcal{C}(\mathcal{A}, R, K_0)$. *Further, we obtain a Weyl groupoid* $\mathcal{W}(\mathcal{C}) = \mathcal{W}(\mathcal{A}, R, K_0)$.

REMARK 4.6. The Cartan scheme $\mathcal{C} = \mathcal{C}(\mathcal{A}, R, K_0)$ depends on the ordering chosen on B^{K_0}, but a different ordering clearly yields an equivalent Cartan scheme.

We may now construct a root system of type $\mathcal{C}(\mathcal{A}, R, K_0)$. Let $a = (\mu_1, \ldots, \mu_m) \in \hat{A}$ and let

$$\phi_a : V^* \to \mathbb{R}^r$$

be the coordinate map for elements of V^* with respect to the basis $\varphi_a(\alpha_1), \ldots, \varphi_a(\alpha_r)$ (in this ordering). Set

$$R^a := \phi_a(R).$$

Again, notice that $R^a = R^b$ for $a, b \in \hat{A}$ with $\pi(a) = \pi(b)$, so we get a unique set which we also denote R^a for each element of $a \in A$.

LEMMA 4.7. *Let* $i, j \in I$, $i \neq j$. *Starting with* K_0, *let* $K_0, K_1, \ldots$ *be the unique chain given by* $i, j, i, j, \ldots$ *Then* $K_m = K_0$ *for some smallest* $m \in \mathbb{N}$ *and*

$$\varphi_{K_{m-1},K_m} \cdots \varphi_{K_1,K_2} \varphi_{K_0,K_1} = \mathrm{id}.$$

Further,

$$m = |R \cap \langle \alpha_i, \alpha_j \rangle|$$

where $B^{K_0} = \{\alpha_1, \ldots, \alpha_r\}$.

PROOF. By Lemma 2.9 and Def. 2.12 the chain $(i, j, i, j, \ldots)$ determines a sequence of endomorphisms with determinant -1 leaving the subspace $U := \langle \alpha_i, \alpha_j \rangle$ invariant.

Since we have only finitely many chambers, $K_k = K_m$ for some $k < m \in \mathbb{N}_0$, m minimal. Assume that $k > 0$. Then since ${\alpha_i^{K_k}}^\perp$ is the wall between K_{k-1} and K_k and ${\alpha_j^{K_k}}^\perp$ is the wall between K_k and K_{k+1} (or possibly with i, j exchanged), the wall between K_{m-1} and K_m is neither ${\alpha_j^{K_k}}^\perp$ nor ${\alpha_i^{K_k}}^\perp$ unless $K_{m-1} = K_{k-1}$ or $K_{m-1} = K_{k+1}$, contradicting the minimality of m. Thus $K_0 = K_m$, $m \in \mathbb{N}$ minimal.

Denote $V' := V/U^\perp$ and $\kappa : V \to V'$ the projection. The set $R \cap U$ yields a simplicial arrangement in V' of rank 2: $\{\kappa(\alpha^\perp) \mid \alpha \in R \cap U\}$. The chain of chambers $K_0, \ldots, K_m$ gives a chain in V' of consecutively adjacent chambers $\kappa(K_0), \ldots, \kappa(K_m)$. But $K_0 = K_m$, thus m has to be even and the determinant of the endomorphism is 1. We conclude that $\varphi_{K_{m-1},K_m} \cdots \varphi_{K_1,K_2} \varphi_{K_0,K_1}$ is a power of the transposition (i, j) with determinant 1, hence it is equal to id.

The last assertion also follows by considering the arrangement of rank 2 in V': Assume that $\kappa(K_0) = \kappa(K_\nu)$ for some $\nu \in \{1, \dots, m\}$. Note that as above ν has to be even. By Lemma 2.9, with respect to the basis B^{K_0}, the map $\sigma_j^{K_{\nu-1}} \dots \sigma_i^{K_0}$ is a matrix (without loss of generality $\{i, j\} = \{1, 2\}$)

$$\begin{pmatrix} 1 & 0 & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix}.$$

This implies $K_0 \subseteq K_\nu$, thus $K_0 = K_\nu$. Since m was chosen minimal, $m = \nu$ and m is the number of chambers of the arrangement in V', exactly one for each element of $|R \cap \langle \alpha_i, \alpha_j \rangle|$. □

Thus the groupoid $\mathcal{W}(\mathcal{A}, R, K_0)$ is a Coxeter groupoid in the sense of [**34**].

PROPOSITION 4.8. *The sets R^a, $a \in A$ as above are the real roots of $\mathcal{C} = \mathcal{C}(\mathcal{A}, R, K_0)$ and form a root system of type $\mathcal{C}$.*

PROOF. By axiom (I) we have $R^a \subseteq \mathbb{Z}^I$. Axioms (R1), (R2) and (R3) follow immediately from the fact that $\mathcal{A}$ is an arrangement and the definition of σ_i^K for a chamber K and $i \in I$. Remark that these sets are indeed the real roots of $\mathcal{C}(\mathcal{A}, R, K_0)$ as defined above. It remains to check axiom (R4). But this is Lemma 4.7. □

5. The correspondence

In this section we summarize the relation between crystallographic arrangements and root systems.

LEMMA 5.1. *Let $(\mathcal{A}, R)$ be a crystallographic arrangement. Then for every chamber K_0 the Cartan scheme $\mathcal{C}(\mathcal{A}, R, K_0)$ is connected. Moreover, every chamber K is attained by some sequence.*

PROOF. Let U be the union of all intersections of distinct hyperplanes of $\mathcal{A}$. Then for each pair of chambers K, K' there exists a chain of consecutively adjacent chambers $K = K_1, \dots, K_m = K'$, since $V \backslash U$ is connected. Thus every chamber K is attained from K_0 by some sequence. Now if $\overline{a}, \overline{b}$ are objects in $\mathcal{C}(\mathcal{A}, R, K_0)$, i.e. $a, b \in \hat{A}$, and a^{-1} denotes the reversed sequence, then ba^{-1} gives a morphism from $\overline{a}$ to $\overline{b}$, so $\mathcal{C}(\mathcal{A}, R, K_0)$ is connected. □

LEMMA 5.2. *Let $(\mathcal{A}, R)$ be a crystallographic arrangement. Then for every chamber K_0 the Cartan scheme $\mathcal{C}(\mathcal{A}, R, K_0)$ is simply connected.*

PROOF. This follows from the definition of the map π. □

LEMMA 5.3. *Let $(\mathcal{A}, R)$ be a crystallographic arrangement and K, K' be chambers. Then $\mathcal{C}(\mathcal{A}, R, K)$ and $\mathcal{C}(\mathcal{A}, R, K')$ are equivalent Cartan schemes.*

PROOF. Choose orderings $B^K = \{\alpha_1, \dots, \alpha_r\}$ and $B^{K'} = \{\alpha'_1, \dots, \alpha'_r\}$ which yield $\mathcal{C}(\mathcal{A}, R, K)$ and $\mathcal{C}(\mathcal{A}, R, K')$. Then to any chain $a = (\mu_1, \dots, \mu_m) \in \hat{A}$ of consecutively adjacent chambers from K to K' belongs a bijection $\varphi_a : B^K \to B^{K'}$, thus an element $\varphi_0 \in S_r$ since the set of labels is $\{1, \dots, r\}$ on both sides. Up to equivalence of Cartan schemes, we may assume without loss of generality that $\varphi_0 = \mathrm{id}$.

If A and A' denote the objects of $\mathcal{C}(\mathcal{A}, R, K)$ resp. $\mathcal{C}(\mathcal{A}, R, K')$, then

$$\varphi_1 : A' \to A, \quad b \mapsto \mu_1 \dots \mu_m.b$$

is bijective, since all paths are invertible. Clearly, φ_1 is compatible with the Cartan schemes (ρ, ρ' and the Cartan matrices) and thus it gives an equivalence of Cartan schemes $\mathcal{C}(\mathcal{A}, R, K) \cong \mathcal{C}(\mathcal{A}, R, K')$. □

We can now state the main theorem.

THEOREM 5.4. *Let $\mathfrak{A}$ be the set of all crystallographic arrangements and $\mathfrak{C}$ be the set of all connected simply connected Cartan schemes for which the real roots are a finite root system. Then the map*

$$\Lambda : \mathfrak{A}/_{\cong} \to \mathfrak{C}/_{\cong}, \quad \overline{(\mathcal{A}, R)} \mapsto \overline{\mathcal{C}(\mathcal{A}, R, K)},$$

where K is any chamber of $\mathcal{A}$, is a bijection.

PROOF. The map Λ is well-defined by Prop. 4.5, Prop. 4.8, Lemma 5.1, Lemma 5.2 and Lemma 5.3. The surjectivity of Λ is [**33**, Corollary 4.7]. If $\Lambda(\overline{(\mathcal{A}, R)}) = \Lambda(\overline{(\mathcal{A}', R')})$ then the ranks are equal, say r, and we have a bijection $\varphi_0 \in S_r$ and a bijection φ_1 between the objects compatible with the Cartan schemes. But φ_0, φ_1 define an element $\psi \in \mathrm{Aut}(\mathbb{R}^{r*})$ which is an equivalence $(\mathcal{A}, R) \cong (\mathcal{A}', R')$, thus Λ is injective. □

Recall the following theorem which is essential for the classification of finite Weyl groupoids:

THEOREM 5.5. [**19**, Thm. 2.10] *Let $\mathcal{C}$ be a Cartan scheme. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$. Let $a \in A$ and $\alpha \in R^a_+$. Then either α is simple, or it is the sum of two positive roots.*

COROLLARY 5.6. *Let $(\mathcal{A}, R)$ be as in Def. 2.1. Then $(\mathcal{A}, R)$ is crystallographic if and only if it is additive.*

PROOF. This is Thm. 5.4, Lemma 2.7 and Thm. 5.5. □

CHAPTER 8

Finite Weyl groupoids

M. CUNTZ AND I. HECKENBERGER

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Using previous results concerning the rank two and rank three cases, all connected simply connected Cartan schemes for which the real roots form a finite irreducible root system of arbitrary rank are determined. As a consequence one obtains the list of all crystallographic arrangements, a large subclass of the class of simplicial hyperplane arrangements. Supposing that the rank is at least three, the classification yields Cartan schemes of type A and B, an infinite family of series involving the types C and D, and 74 sporadic examples.

1. Introduction

In the 1970s, simplicial arrangements became a popular subject of study after Deligne [**20**] proved that the complement of a complexified finite simplicial real hyperplane arrangement is an Eilenberg-MacLane space of type $K(\pi,1)$ for some group π. It is known that the cohomology ring of such a space coincides with the group cohomology $H^*(\pi,\mathbb{Z})$. Previously, Brieskorn [**12**] had identified the fundamental groups of complements of complexified Coxeter arrangements as pure Artin braid groups. In 1980, the result of Brieskorn was extended to all real simplicial arrangements by Orlik and Solomon [**41**] based on algebraic constructions of lattices which are related to Bacławskis work on geometric lattices [**9**].

Simplicial arrangements in the real projective plane were introduced by Melchior [**38**] in 1941. Their classification remained so far an open problem. Grünbaum [**23**] provides a conjecturally complete list which contains three infinite series and a large number of exceptional examples. In higher dimensional spaces only a few examples of simplicial arrangements are known.

Let $\mathcal{A}$ be a simplicial arrangement of finitely many real hyperplanes in a Euclidean space V and let R be a set of nonzero covectors such that $\mathcal{A}=\{\alpha^\perp \mid \alpha\in R\}$. Assume that $\mathbb{R}\alpha\cap R=\{\pm\alpha\}$ for all $\alpha\in R$. The pair $(\mathcal{A},R)$ is called crystallographic, see [**14**, Def. 2.3], if for any chamber K the elements of R are integer linear combinations of the covectors defining the walls of K. For example, crystallographic Coxeter groups give rise to crystallographic arrangements in this sense, but there are many other. In this paper we solve the natural problem of classifying crystallographic arrangements by considering Cartan schemes, their root systems, and their Weyl groupoids.

Weyl groupoids have been introduced by the second author in [**26**] to obtain finiteness properties of Nichols algebras of diagonal type. The Weyl groupoid provides simplification, generalization, and unification of related finiteness results by Rosso [**44**] and Andruskiewitsch and Schneider [**8**]. A very general definition of the Weyl groupoid of a Nichols algebra and the necessary structural results have been obtained by Andruskiewitsch, Schneider, and the second author in [**7**]. An axiomatic approach to Weyl groupoids via Cartan schemes and root systems was developed in a series of papers by Yamane and the authors [**34, 17, 16, 19**]. For historical and practical reasons, the emphasis was put on connected simply connected Cartan schemes such that the real roots form a finite irreducible root system. Such Cartan schemes will be called coscorf Cartan schemes.

The connection to simplicial arrangements was established successively in [**19, 33, 14**]. If the real roots of a connected Cartan scheme form a finite irreducible root system, then they define a simplicial arrangement similarly as in the construction in [**35**, Sect. 1.15] for Coxeter groups. This was first observed in [**19**] in the case of rank three and then proved in full generality in [**33**]. The final step was achieved in [**19**] where it was shown that crystallographic arrangements can be described axiomatically as coscorf Cartan schemes. Therefore it is natural to try a classification of simplicial arrangements via Cartan schemes.

Coscorf Cartan schemes of rank at most three have been classified by the authors, see [**16**] and [**19**]. The classification of rank two is surprisingly nice: There is a natural bijection between the set of coscorf Cartan schemes of rank two with $2n$ objects and the triangulations of a convex n-gon by non-intersecting diagonals [**15**]. In contrast, up to equivalence there are only finitely many coscorf Cartan schemes of rank three. In the present paper we treat the general case. Our main result is the following.

THEOREM 1.1. *There are exactly three families of connected simply connected Cartan schemes for which the real roots form a finite irreducible root system:*

(1) *The family of Cartan schemes of rank two parametrized by triangulations of a convex n-gon by non-intersecting diagonals.*
(2) *For each rank $r > 2$, the standard Cartan schemes of type A_r, B_r, C_r and D_r, and a series of $r-1$ further Cartan schemes described explicitly in Thm. 3.21.*
(3) *A family consisting of 74 further "sporadic" Cartan schemes (including those of type F_4, E_6, E_7 and E_8).*

REMARK 1.2. We classify connected simply connected Cartan schemes $\mathcal{C}$ up to equivalence in the sense of [**17**, Def. 2.1], such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$. To obtain a classification of connected Cartan schemes such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system, one additionally has to classify quotients (inverse of coverings) of simply connected Cartan schemes, which amounts to classify all subgroups of the automorphism group of the minimal quotient of $\mathcal{C}$, see Def. 2.9.

As mentioned above, the result is known for Cartan schemes of rank at most three. We split the proof of the remaining part of the theorem in two cases depending on the rank.

We obtain the classification in rank r, $3 < r \leq 8$ by an algorithm which enumerates all root systems of coscorf Cartan schemes. We use the knowledge of rank three as a starting point and then inductively classify coscorf Cartan schemes of rank r using the classification of coscorf Cartan schemes of rank $r-1$. The structure of the algorithm is similar to the one given in [**19**], but additional algorithmic ideas and further improvements of the computational techniques are needed to make the implementation practicable.
The classification in rank greater than eight mainly relies on the analysis of the Dynkin diagrams corresponding to the Cartan matrices of the Cartan schemes. The simplicity of the arguments suggests a similar approach for the Cartan schemes of lower rank. However, this is misleading, since in lower rank there are many sporadic examples making the argumentations much more difficult. In particular, single Cartan matrices of non-standard sporadic Cartan schemes contain only little information about the roots at the corresponding object.

This paper is organized as follows. In Section 2 we repeat the definitions of Cartan schemes, Weyl groupoids, root systems and crystallographic arrangements. Section 3 is divided into two subsections: In the first one we determine the Dynkin diagrams of finite Weyl groupoids of rank greater than eight. In the second subsection we give an explicit description of the root systems of all coscorf Cartan schemes of rank greater than eight which are not standard, i.e. which have at least two different Cartan matrices. In Section 4 we describe an algorithm which enumerates all coscorf Cartan schemes of rank at most eight. In the appendix we collect invariants of coscorf Cartan schemes. Finally we give a list of root systems that contains for each sporadic example the roots of precisely one object. We explain in which sense this object is canonical.

2. Weyl groupoids and crystallographic arrangements

2.1. Cartan schemes and root systems. We briefly recall the notions of Cartan schemes, Weyl groupoids and root systems, following [**17, 16**]. The foundations of the general theory have been developed in [**34**] using a somewhat different terminology.

DEFINITION 2.1. Let I be a non-empty finite set and $\{\alpha_i \,|\, i \in I\}$ the standard basis of $\mathbb{Z}^I$. By [**37**, §1.1] a *generalized Cartan matrix* $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \leq 0$ for all $i, j, k \in I$ with $j \neq k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

DEFINITION 2.2. Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

DEFINITION 2.3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{2.1}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are compositions of maps σ^a_i with $i \in I$ and $a \in A$, where σ^a_i is considered as an element in $\mathrm{Hom}(a, \rho_i(a))$. The category $\mathcal{W}(\mathcal{C})$ is a groupoid in the sense that all morphisms are isomorphisms.

As above, for notational convenience we will often neglect upper indices referring to elements of A if they are uniquely determined by the context. For example, the morphism $\sigma^{\rho_{i_2}\cdots\rho_{i_k}(a)}_{i_1} \cdots \sigma^{\rho_{i_k}(a)}_{i_{k-1}} \sigma^a_{i_k} \in \mathrm{Hom}(a, b)$, where $k \in \mathbb{N}$, $i_1, \dots, i_k \in I$, and $b = \rho_{i_1} \cdots \rho_{i_k}(a)$, will be denoted by $\sigma_{i_1} \cdots \sigma^a_{i_k}$ or by $\mathrm{id}^b\sigma_{i_1} \cdots \sigma_{i_k}$. The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$.

DEFINITION 2.4. A Cartan scheme is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$. The Cartan scheme is called *simply connected*, if $\mathrm{Hom}(a, a) = \{\mathrm{id}^a\}$ for all $a \in A$.

Two Cartan schemes $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I', A', (\rho'_i)_{i\in I'}, (C'^a)_{a\in A'})$ are termed *equivalent*, if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1(\rho_i(a)) = \rho'_{\varphi_0(i)}(\varphi_1(a)), \qquad c^{\varphi_1(a)}_{\varphi_0(i)\varphi_0(j)} = c^a_{ij} \tag{2.2}$$

for all $i, j \in I$ and $a \in A$. We write then $\mathcal{C} \cong \mathcal{C}'$.

Let $\mathcal{C}$ be a Cartan scheme. For all $a \in A$ let

$$(R^{\mathrm{re}})^a = \{\mathrm{id}^a\sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \,|\, k \in \mathbb{N}_0,\ i_1, \dots, i_k, j \in I\} \subseteq \mathbb{Z}^I.$$

The elements of the set $(R^{\mathrm{re}})^a$ are called *real roots* (at a). The pair $(\mathcal{C},((R^{\mathrm{re}})^a)_{a\in A})$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. A real root $\alpha\in(R^{\mathrm{re}})^a$, where $a\in A$, is called positive (resp. negative) if $\alpha\in\mathbb{N}_0^I$ (resp. $\alpha\in-\mathbb{N}_0^I$). In contrast to real roots associated to a single generalized Cartan matrix, $(R^{\mathrm{re}})^a$ may contain elements which are neither positive nor negative. A good general theory, which is relevant for example for the study of Nichols algebras, can be obtained if $(R^{\mathrm{re}})^a$ satisfies additional properties.

DEFINITION 2.5. Let $\mathcal{C}=\mathcal{C}(I,A,(\rho_i)_{i\in I},(C^a)_{a\in A})$ be a Cartan scheme. For all $a\in A$ let $R^a\subseteq\mathbb{Z}^I$, and define $m_{i,j}^a=|R^a\cap(\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j)|$ for all $i,j\in I$ and $a\in A$. We say that

$$\mathcal{R}=\mathcal{R}(\mathcal{C},(R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a=R^a_+\cup-R^a_+$, where $R^a_+=R^a\cap\mathbb{N}_0^I$, for all $a\in A$.
(R2) $R^a\cap\mathbb{Z}\alpha_i=\{\alpha_i,-\alpha_i\}$ for all $i\in I$, $a\in A$.
(R3) $\sigma_i^a(R^a)=R^{\rho_i(a)}$ for all $i\in I$, $a\in A$.
(R4) If $i,j\in I$ and $a\in A$ such that $i\neq j$ and $m_{i,j}^a$ is finite, then $(\rho_i\rho_j)^{m_{i,j}^a}(a)=a$.

The axioms (R2) and (R3) are always fulfilled for $\mathcal{R}^{\mathrm{re}}$. The root system $\mathcal{R}$ is called *finite* if for all $a\in A$ the set R^a is finite. By [**17**, Prop. 2.12], if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R}=\mathcal{R}^{\mathrm{re}}$, and hence $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$ in that case.

In [**17**, Def. 4.3] the concept of an *irreducible* root system of type $\mathcal{C}$ was defined. By [**17**, Prop. 4.6], if $\mathcal{C}$ is a Cartan scheme and $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R}$ is irreducible if and only if for all $a\in A$ the generalized Cartan matrix C^a is indecomposable. If $\mathcal{C}$ is also connected, then it suffices to require that there exists $a\in A$ such that C^a is indecomposable.

2.2. Coscorf Cartan schemes and arrangements.

DEFINITION 2.6. In this article, we will abbreviate a connected simply connected Cartan scheme for which the real roots form a finite root system by a *coscorf Cartan scheme* (**co**nnected **s**imply **co**nnected, **r**eal roots, **f**inite).

Although we will not need it, we reproduce the definition of a crystallographic arrangement [**14**, Def. 2.3] because it demonstrates how large the class of arrangements which we classify actually is.

DEFINITION 2.7. Let $(\mathcal{A},V)$ be a simplicial arrangement and $R\subseteq V$ a finite set such that $\mathcal{A}=\{\alpha^\perp\mid\alpha\in R\}$ and $\mathbb{R}\alpha\cap R=\{\pm\alpha\}$ for all $\alpha\in R$. For a chamber K of $\mathcal{A}$ let B^K denote the set of normal vectors in R of the walls of K pointing to the inside. We call $(\mathcal{A},R)$ a *crystallographic arrangement* if

(I) $\quad R\subseteq\sum_{\alpha\in B^K}\mathbb{Z}\alpha\quad$ for all chambers K.

As mentioned in the introduction, by [**14**, Thm. 1.1] all results on coscorf Cartan schemes directly apply to crystallographic arrangements:

THEOREM 2.8. *There is a one-to-one correspondence between crystallographic arrangements and coscorf Cartan schemes (up to equivalence on both sides).*

DEFINITION 2.9. Let $\mathcal{C}$ be a coscorf Cartan scheme and $a \in A$. Then we call

$$\mathrm{Aut}(\mathcal{C}, a) := \{w \in \mathrm{Hom}(a, b) \mid b \in A, \ \ R^a = R^b\}$$

the *automorphism group of* $\mathcal{C}$ *at* a. This is a finite subgroup of $\mathrm{Aut}(\mathbb{Z}^r)$ because the number of all morphisms is finite. Since $\mathcal{C}$ is connected, $\mathrm{Aut}(\mathcal{C}, a) \cong \mathrm{Aut}(\mathcal{C}, b)$ for all $a, b \in A$. We will therefore write $\mathrm{Aut}(\mathcal{C})$ if we are only interested in the isomorphism class of the group.

REMARK 2.10. If $\mathcal{C}$ is a coscorf Cartan scheme then it is simply connected. The automorphism group of $\mathcal{C}$ is the automorphism group of an object of the Cartan scheme obtained from $\mathcal{C}$ by identifying all objects with equal root systems (the "smallest" quotient, see [**16**, Def. 3.1] for the definition of coverings). If we have n objects in $\mathcal{C}$ and m different root systems, then $m|\,\mathrm{Aut}(\mathcal{C}, a)| = n$.

2.3. Diagrams.

DEFINITION 2.11. Let $r, s \in \mathbb{N}$ with $r \leq s$. We will say that a finite set $\Lambda \subseteq \mathbb{Z}^s$ is a *root set of rank* r if there exists a Cartan scheme $\mathcal{C}$ of rank r and an injective linear map $w : \mathbb{Z}^r \to \mathbb{Z}^s$ such that $w((R^{\mathrm{re}})^a) = \Lambda$ for some object a. We call the set $\{w(\alpha_1), \dots, w(\alpha_r)\}$ a *base* of Λ. If $\mathcal{R}(\mathcal{C})$ is irreducible, then we call Λ *irreducible*.

If $U \leq \mathbb{R}^r$ is a subspace and $\Lambda' = (R^{\mathrm{re}})^a \cap U$ for some object a, then we call Λ' a *root subset* of $\mathcal{C}$. Remark that if $\Lambda' \neq \emptyset$, then it is a root set by [**19**, Cor. 2.5] (Cor. 3.1). Sometimes, for $\beta_1, \dots, \beta_k \in (R^{\mathrm{re}})^a$ we will write $\langle \beta_1, \dots, \beta_k \rangle$ for the root subset $(R^{\mathrm{re}})^a \cap \sum_{i=1}^k \mathbb{R}\beta_i$.

Remember the following fact ([**19**, Cor. 2.9]):

LEMMA 2.12. *Let $\mathcal{C}$ be a Cartan scheme and assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system. Let $a \in A$, $k \in \mathbb{Z}$, and $i, j \in I$ such that $i \neq j$. Then $\alpha_j + k\alpha_i \in (R^{\mathrm{re}})^a$ if and only if $0 \leq k \leq -c_{ij}^a$.*

DEFINITION 2.13. To a finite set $\Lambda \subseteq \mathbb{Z}^r$ we associate a matrix $C_\Lambda = (c_{ij})_{1 \leq i,j \leq r}$ given by

$$c_{ij} = -\max\{k \mid k\alpha_i + \alpha_j \in \Lambda\}, \quad c_{ii} = 2$$

for $1 \leq i, j \leq r$ and $i \neq j$. The matrix C_Λ is a generalized Cartan matrix and it defines linear maps $\sigma_i : \mathbb{Z}^r \to \mathbb{Z}^r$, $i = 1, \dots, r$ via

$$\sigma_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$$

for all $j = 1, \dots, r$.

REMARK 2.14. For example, if $\mathcal{C}$ is a coscorf Cartan scheme and $a \in A$, then $C^a = C_{(R^{\mathrm{re}})^a_+}$ and $\sigma_i = \sigma_i^a$ for $i = 1, \dots, r$ by Lemma 2.12.

DEFINITION 2.15. As in [**17**, Def. 3.1], we call a coscorf Cartan scheme *standard* if all its Cartan matrices are equal. Note that up to coverings, $\mathcal{W}(\mathcal{C})$ is a Weyl group in this case.

DEFINITION 2.16. Let $\mathcal{C}$ be a Cartan scheme and a an object. The *Dynkin diagram* Γ^a at a is a labeled directed graph given by the Cartan matrix $C^a = (c^a_{ij})_{i,j}$ in the following way: The vertices are the elements of I. Vertices $i, j \in I$ with $i \neq j$ are connected by an arrow pointing to i and labeled $-c^a_{ij}$ if and only if $c^a_{ij} \neq 0$.

When drawing the diagrams, if $c^a_{ij} = c^a_{ji} = -1$ then instead of drawing two labeled arrows we just connect i and j by an edge. If $c^a_{ij} \neq c^a_{ji} = -1$ then we only draw the arrow labeled c^a_{ij}.

For an object a, we will write "*a is of Dynkin type X*" if its Dynkin diagram is of type X.

3. Finite coscorf Cartan schemes of rank > 8

In this section we use that all coscorf Cartan schemes of rank < 9 are as in Section 4, see Thm. 4.1. In particular, we have a complete list of all their Dynkin diagrams (p. 166, Figure 5).

3.1. The Dynkin diagrams. Recall [**19**, Cor. 2.5] which will be the key to Prop. 3.2 and Lemma 3.5:

COROLLARY 3.1. *Let $\mathcal{C}$ be a Cartan scheme such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system. Let $a \in A$, $k \in \{1, \dots, |I|\}$, and let $\beta_1, \dots, \beta_k \in R^a_+$ be linearly independent elements. Then $R^a_+ \cap \sum_{i=1}^k \mathbb{R}\beta_i \subseteq \sum_{i=1}^k \mathbb{N}_0\beta_i$ if and only if there exist $b \in A$, $w \in \mathrm{Hom}(a, b)$, and a permutation τ of I such that $w(\beta_i) = \alpha_{\tau(i)}$ for all $i \in \{1, \dots, k\}$.*

PROPOSITION 3.2. *Let $\mathcal{C}$ be a Cartan scheme such that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system. Let $a \in A$. Then the following are equivalent.*

(1) $\sum_{i \in I} \alpha_i \in (R^{\mathrm{re}})^a$,
(2) Γ^a *is connected.*

PROOF. We proceed by induction on the rank r. The claim is true by [**15**, Prop. 3.7] for $r = 2$ and by [**19**, Lemma 3.12(2)] for $r = 3$ (alternatively one can verify this by inspecting the data in [**19**]).

Let $r > 3$. The implication (1) ⇒ (2) is [**17**, Prop. 4.6]. Hence we have to show that (2) implies (1). Without loss of generality, for each $i > 1$ there exists $j < i$ such that $\alpha_i + \alpha_j$ is a root in $(R^{\mathrm{re}})^a$. In particular $\alpha_1 + \alpha_2 \in (R^{\mathrm{re}})^a$. Let

$$\beta_1 = \alpha_1 + \alpha_2, \quad \beta_2 = \alpha_3,\ \beta_3 = \alpha_4, \dots,\ \beta_{r-1} = \alpha_r.$$

By Cor. 3.1 there exist $b \in A$, $w \in \mathrm{Hom}(a, b)$ such that $w(\beta_i) \in \{\alpha_1, \dots, \alpha_r\}$ for all $i = 1, \dots, r-1$. By the above assumption, if $\alpha_1 + \alpha_i \in (R^{\mathrm{re}})^a$ or $\alpha_2 + \alpha_i \in (R^{\mathrm{re}})^a$ for some $i > 2$ then $\{\alpha_1, \alpha_2, \alpha_i\}$ is the base of an irreducible root set and by induction $\alpha_1+\alpha_2+\alpha_i \in (R^{\mathrm{re}})^a$. Thus for each $i > 1$ there exists $j < i$ such that $\beta_i+\beta_j \in (R^{\mathrm{re}})^a$. By induction, $\sum_{i=1}^{r-1} w(\beta_i) \in (R^{\mathrm{re}})^b$ and hence $\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r-1} \beta_i \in (R^{\mathrm{re}})^a$. □

In the sequel let $\mathcal{C}$ be an irreducible coscorf Cartan scheme of rank $r > 8$, a be an object of $\mathcal{C}$ and Γ the Dynkin diagram of C^a.

LEMMA 3.3. *The diagram Γ does not contain a diagram of type E_8.*

PROOF. Assume that Γ contains a subdiagram of type E_8. Then by the list of diagrams with 8 vertices in Figure 5, Γ contains a subdiagram of affine type $\tilde{E}_8$, otherwise one gets a forbidden subdiagram of rank 8. So assume without loss of generality that $r = 9$ and that the diagram at a is $\tilde{E}_8$.

Label the vertices of the diagram E_8 by $1, \dots, 8$ and the new vertex by 9 (as in Fig. 1). Since the coscorf Cartan scheme with diagram E_8 is standard, $\sigma_1, \dots, \sigma_8$ map

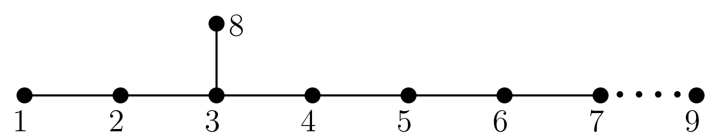

FIGURE 1. Dynkin diagram of type $\tilde{E}_8$

to objects $b_1, \dots, b_8$ where the subdiagram to the labels $1, \dots, 8$ is of type E_8 as well. But by the same argument as above, the Γ^{b_i}, $i = 1, \dots, 8$ are all of type $\tilde{E}_8$.

Now consider the map σ_9; let $b_9 = \rho_9(a)$. Since the vertices $1, 2, \dots, 6$, 8 are not connected with the vertex 9, they are not connected in Γ^{b_9} as well. Thus [**17**, Lemma 4.5] implies that the connections between the vertices $1, 2, \dots, 6, 8$ in Γ^{b_9} are the same as in Γ and that vertices 6 and 7 are connected. By the reason given at the beginning of the proof it follows that Γ^{b_9} is of Dynkin type $\tilde{E}_8$.

Altogether, the Cartan scheme is standard of Dynkin type $\tilde{E}_8$, thus it is not a coscorf Cartan scheme by the classification of finite crystallographic Coxeter groups and by [**17**, Thm. 3.3]. □

DEFINITION 3.4. Let Γ be a Dynkin diagram and assume that it has vertices i, j such that:

(1) i and j are connected by an edge,
(2) there is no vertex $k \notin \{i,j\}$ such that (i,k) and (j,k) are edges.

Let Γ' be the diagram obtained from Γ by removing the edge (i,j) and identifying the vertices i and j to a new vertex ℓ, i.e. the edges of Γ' are $\{(k,m) \mid k,m \notin \{i,j\},\ (k,m)$ edge in $\Gamma\} \cup \{(k,\ell) \mid k \notin \{i,j\},\ (k,i)$ or (k,j) edge in $\Gamma\}$.

Then we call Γ' the *contraction of* Γ *along* (i,j).

The following lemma is a useful tool for the classification:

LEMMA 3.5. *Let $\mathcal{C}$ be an irreducible coscorf Cartan scheme of rank $r > 8$ and assume that there are pairwise different $i_1,\dots,i_7 \in I$ such that in Γ^a, (i_ν,i_μ) for $\nu<\mu$ is connected if and only if $\mu-\nu=1$, and such that $(i_\nu,i_{\nu+1})$ are edges (with labels 1) for all $\nu=1,\dots,6$.*

Then there exists an irreducible coscorf Cartan scheme $\mathcal{C}' = \mathcal{C}'(I',A',(\rho'_i)_{i\in I'},(C'^a)_{a\in A'})$ and an object $a' \in A'$ such that

(1) $I' = \{\ell\} \cup I\backslash\{i_3,i_4\}$,
(2) $\Gamma'^{a'}$ *is the contraction of* Γ^a *along* (i_3,i_4),
(3) *for all* $k \notin \{i_2,i_3,i_4,i_5\}$, $\Gamma'^{\rho'_k(a')}$ *is the contraction of* $\Gamma^{\rho_k(a)}$ *along* (i_3,i_4).

PROOF. Notice first that by Lemma 3.3 there is an edge from k to i_3 if and only if $k \in \{i_2,i_4\}$ and that by Fig. 5 there is an edge from k to i_4 if and only if $k \in \{i_3,i_5\}$.

By Cor. 3.1, $\{\alpha_{i_3}+\alpha_{i_4},\alpha_j \mid j \in I, i_3 \neq j \neq i_4\}$ is a base of a finite root set Λ of rank $r-1$. Let $\mathcal{C}' = \mathcal{C}'(I',A',(\rho'_i)_{i\in I'},(C'^a)_{a\in A'})$ be a Cartan scheme, $\iota : \mathbb{Z}^{r-1} \to \mathbb{Z}^r$ a linear map and a' be an object of $\mathcal{C}'$ such that $\Lambda = \iota((R^{\mathrm{re}})^{a'})$. Remark that $\mathcal{W}(\mathcal{C}')$ is a parabolic subgroupoid of $\mathcal{W}(\mathcal{C})$ (see [**33**, Def. 2.3] for the precise definition of a parabolic subgroupoid of W(C)). For the vertices of $\Gamma'^{a'}$ we use the same labels as for Γ^a; the new vertex $\iota^{-1}(\alpha_{i_3}+\alpha_{i_4})$ is labeled ℓ.

We prove that the Dynkin diagram $\Gamma'^{a'}$ is the contraction of Γ^a along (i_3,i_4). The subdiagram to i_1,i_2,ℓ,i_5,i_6,i_7 is of type A_6: Let $k \notin \{i_2,\dots,i_5\}$ and assume that there is a connection from k to ℓ in $\Gamma'^{a'}$. Then $\alpha_k+\alpha_{i_3}+\alpha_{i_4}$ is a root in R^a. But by Prop. 3.2, either $\alpha_k+\alpha_{i_3}$ or $\alpha_k+\alpha_{i_4}$ is a root, contradicting the fact that there is no edge from k to i_3 or i_4 in Γ. Thus there is no connection from k to ℓ in $\Gamma'^{a'}$. Moreover, the edge (i_5,ℓ) is labeled by a one by Fig. 5 and the edge (i_2,ℓ) is labeled by a one because $r>8$ and the diagrams of type Γ_6^1 and Γ_6^2 are not part of an irreducible Dynkin diagram with 7 vertices. Of course, connections not involving ℓ are the same as in Γ^a by Lemma 2.12.

For $k \notin \{i_2, \ldots, i_5\}$ the diagram

$$\begin{array}{ccc} R'^{a'} & \xrightarrow{\iota} & R^a \\ \sigma_k \Big\downarrow & & \sigma_k \Big\downarrow \\ R'^{\rho'_k(a')} & \xrightarrow{\iota} & R^{\rho_k(a)} \end{array}$$

commutes because ι maps simple roots α_m with $m \notin \{i_3, i_4\}$ to simple roots and because there is no edge from k to i_3 or i_4. By the same argument as above we obtain (3). □

The following theorem classifies the possible Dynkin diagrams.

THEOREM 3.6. *Let Γ be the Dynkin diagram of an object a in a coscorf Cartan scheme $\mathcal{C}$ of rank $r > 8$. Then Γ is of type A, B, C, D or D'.*

PROOF. We proceed by induction on r. By Section 4, for $r = 8$ the diagrams are of type A_8, B_8, C_8, D_8 or E_8. Now let $r > 8$. By Lemma 3.3 and induction, each connected subdiagram of Γ of rank $r-1$ is of type A_{r-1}, B_{r-1}, C_{r-1}, D_{r-1} or D'_{r-1}.

If Γ has a subdiagram of type A_{r-1}, then using induction and Lemma 3.3, one checks that Γ is of type A_r, B_r, C_r, D_r, D'_r or $\tilde{A}_r$. If Γ is of type $\tilde{A}_r$ then by Lemma 3.5, removing an edge in the middle yields an irreducible root set of rank $r-1$ with a Dynkin diagram of type $\tilde{A}_{r-1}$ which is forbidden.

Similarly, if Γ has a subdiagram of type B_{r-1} or C_{r-1}, then Γ is of type B_r resp. C_r (notice that $r-1 > 7$).

If Γ has a subdiagram of type D_{r-1} or D'_{r-1}, then Γ is of type D_r, D'_r or we are in one of two cases:
1. The diagram Γ has the connections of a diagram of type $\tilde{D}_r$ (the affine diagram of type D) and possibly some more connections. Choose the labels as in Fig. 2. Identifying the vertices 3 and 4 does not give a Dynkin diagram of a coscorf Car-

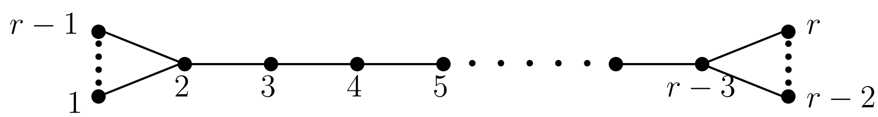

FIGURE 2. Case 1.

tan scheme, thus by Lemma 3.5 this case is impossible (again, notice that $r-1 > 7$).
2. The subdiagrams to the labels $(r-1, 2, \ldots, r-2, r)$ and $(1, \ldots, r-2, r)$ are both of type B or C. But then by Lemma 3.5, removing an edge in the middle yields an irreducible root set of rank $r-1$ with a forbidden Dynkin diagram. □

LEMMA 3.7. *Let Γ be the Dynkin diagram of an object a in a coscorf Cartan scheme $\mathcal{C}$ and $i \in I$.*

(1) $\{i, i_1, \dots, i_k\} \subseteq I$ *are connected in* Γ^a *if and only if* $\{i, i_1, \dots, i_k\}$ *are connected in* $\Gamma^{\rho_i(a)}$.

Let $j, k \in I$ *with* $|\{i, j, k\}| = 3$.

(2) *If* i *is not connected to* j *nor to* k *then the connection between* j *and* k *(including labels) is the same in* Γ^a *and* $\Gamma^{\rho_i(a)}$.
(3) *If* i *is connected to* j *and* i *is not connected to* k *then* j *is connected to* k *in* Γ^a *if and only if they are connected in* $\Gamma^{\rho_i(a)}$.

PROOF. Use [**17**, Lemma 4.5], axiom (C2), [**17**, Prop. 4.6]. □

Now Section 4 allows us to give more details about the Cartan schemes.

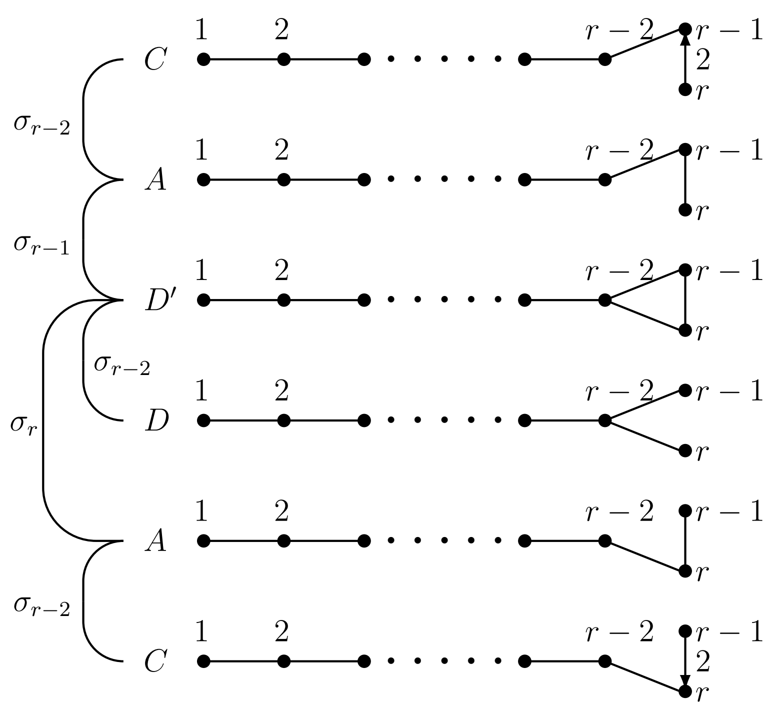

FIGURE 3. Dynkin diagrams for the series

PROPOSITION 3.8. *Let* Γ *be the Dynkin diagram of an object* a *in a coscorf Cartan scheme* $\mathcal{C}$ *of rank* $r > 8$.

(1) *If* Γ *is of type* B *then* $\mathcal{C}$ *is a standard Cartan scheme.*
(2) *If* Γ *is of type* A, C *or* D, *then either* $\mathcal{C}$ *is a standard Cartan scheme, or there is an object of Dynkin type* D'.

(3) *Assume that there is an object of Dynkin type D' in $\mathcal{C}$.*
If Γ is of type D' with labels as in Fig. 3, then the diagrams that appear in $\mathcal{C}$ are the diagrams of Fig. 3 with the same labels, possibly without the diagrams of type C or D.
If $\mathcal{C}$ has an object a with diagram of type D resp. C and if there is a $j \in I$ such that $\rho_j(a)$ is not of Dynkin type D resp. C, then $j = r - 2$ and $\rho_j(a)$ is of Dynkin type D' resp. A.
The simple reflections σ_{r-1} and σ_r always map an object of Dynkin type D' to an object of Dynkin type A and vice versa (as in Fig. 3).

PROOF. We proceed by induction on r and prove (1)-(3) simultaneously. For $r = 8$ and Γ not of type E_8 all the above claims hold by inspecting the resulting data of Section 4. Now let $r > 8$.

If Γ is of type B then by induction hypothesis, Lemma 3.5 and Thm. 3.6, the maps $\sigma_1, \ldots, \sigma_r$ map to objects of Dynkin type B_r, thus $\mathcal{C}$ is standard.

Assume that Γ is of type A, C or D and that $\mathcal{C}$ is not standard. Then there is an object in $\mathcal{C}$ with diagram Γ and $j \in I$ such that applying σ_j leads to an object of different Dynkin type. Choose the labels as in Fig. 3. Then by Lemma 3.5, removing the edge $(4, 5)$ yields a diagram Γ' of the same type belonging to a Cartan scheme $\mathcal{C}'$ of rank $r - 1$. If $\mathcal{C}'$ was standard, then the maps $\sigma_1, \sigma_2, \sigma_7, \ldots, \sigma_r$ would preserve the diagram Γ; but since this is also the case for $\sigma_3, \ldots, \sigma_6$ by Lemma 3.7, this would contradict the assumption that σ_j maps to a different diagram. Hence $\mathcal{C}'$ is not standard. Now if Γ' is of type A, then by induction either σ_{r-1} or σ_r maps (in $\mathcal{C}'$) to a diagram of type D'. But these maps are not affected by the deletion of $(4, 5)$, so σ_{r-1} or σ_r map Γ to a diagram of type D' in $\mathcal{C}$. If Γ is of type D then an easy calculation shows that $j \in \{r - 2, r - 1, r\}$. But then using $\mathcal{C}'$ we get that $j = r - 2$ and that σ_j maps to a diagram of type D'.

If Γ is of type C, then by the same argument as for type D we get to an object of Dynkin type A. We just proved that in this case an object of Dynkin type D' also occurs in $\mathcal{C}$. Thus we have proved (2): If Γ is of type A, C or D and $\mathcal{C}$ is not standard, then there exists an object b of Dynkin type D'. The morphisms needed to get from a to b are as explained in (3) by Lemma 3.5. □

3.2. The root systems. Let $r \in \mathbb{N}$. Recall that we denote $\{\alpha_1, \ldots, \alpha_r\}$ the standard basis of $\mathbb{Z}^r$. We use the following notation: For $1 \le i, j \le r$, let

$$\eta_{i,j} := \begin{cases} \sum_{k=i}^{j} \alpha_k & i \le j \\ 0 & i > j \end{cases}.$$

DEFINITION 3.9. Let $Z \subseteq \{1, \ldots, r-1\}$. Let $\Phi_{r,Z}$ denote the set of roots

$$\begin{aligned} &\eta_{i,j-1}, && 1 \leq i < j \leq r, \\ &\eta_{i,r-2} + \alpha_r, && 1 \leq i < r, \\ &\eta_{i,r} + \eta_{j,r-2}, && 1 \leq i < j < r, \\ &\eta_{j,r} + \eta_{j,r-2}, && j \in Z. \end{aligned}$$

Let $Y \subseteq \{1, \ldots, r-1\}$. Let $\Psi_{r,Y}$ denote the set of roots

$$\begin{aligned} &\eta_{i,j}, && 1 \leq i \leq j \leq r, \\ &\eta_{i,r} + \eta_{j,r-1}, && 1 \leq i < j < r, \\ &\eta_{j,r} + \eta_{j,r-1}, && j \in Y. \end{aligned}$$

Further, denote $\Psi'_{r,Y}$ the set obtained from $\Psi_{r,Y}$ by exchanging α_{r-1} and α_r.

REMARK 3.10. The sets $\Phi_{r,\emptyset}$ resp. $\Psi_{r,\{1,\ldots,r-1\}}$ are the sets of positive roots of the Weyl groups of type D_r resp. C_r, compare [**11**, VI. 4.6, 4.8].

Recall that by Def. 2.13 we write C_Λ for the generalized Cartan matrix given by a set Λ.

PROPOSITION 3.11. *Let* $Y, Z \subseteq \{1, \ldots, r-1\}$.

(1) *The Dynkin diagram of* $C_{\Phi_{r,Z}}$ *is of type* D'_r *if* $r-1 \in Z$ *and of type* D_r *if* $r-1 \notin Z$.
(2) *The Dynkin diagram of* $C_{\Psi_{r,Y}}$ *is of type* C_r *if* $r-1 \in Y$ *and of type* A_r *if* $r-1 \notin Y$.

PROOF. This is clear by definition. □

PROPOSITION 3.12. *Let* $Y, Z \subseteq \{1, \ldots, r-1\}$ *with* $0 \leq |Y| < |Z| < r$. *Then*

$$\begin{aligned}
\sigma_i(\pm\Phi_{r,Z}) &= \pm\Phi_{r,(i\ i+1)(Z)} && \text{for } i = 1, \ldots, r-2, \\
\sigma_i(\pm\Psi_{r,Y}) &= \pm\Psi_{r,(i\ i+1)(Y)} && \text{for } i = 1, \ldots, r-2, \\
\sigma_i(\pm\Psi'_{r,Y}) &= \pm\Psi'_{r,(i\ i+1)(Y)} && \text{for } i = 1, \ldots, r-2, \\
\sigma_{r-1}(\pm\Phi_{r,Z}) &= \pm\Phi_{r,Z} && \text{if } r-1 \notin Z, \\
\sigma_r(\pm\Phi_{r,Z}) &= \pm\Phi_{r,Z} && \text{if } r-1 \notin Z, \\
\sigma_{r-1}(\pm\Phi_{r,Z}) &= \pm\Psi_{r,Z\setminus\{r-1\}} && \text{if } r-1 \in Z, \\
\sigma_r(\pm\Phi_{r,Z}) &= \pm\Psi'_{r,Z\setminus\{r-1\}} && \text{if } r-1 \in Z, \\
\sigma_{r-1}(\pm\Psi_{r,Y}) &= \pm\Phi_{r,Y\cup\{r-1\}} && \text{if } r-1 \notin Y, \\
\sigma_r(\pm\Psi_{r,Y}) &= \pm\Psi_{r,Y} && \text{if } r-1 \notin Y, \\
\sigma_{r-1}(\pm\Psi_{r,Y}) &= \pm\Psi_{r,Y} && \text{if } r-1 \in Y, \\
\sigma_r(\pm\Psi_{r,Y}) &= \pm\Psi_{r,Y} && \text{if } r-1 \in Y,
\end{aligned}$$

$$\begin{array}{rclr} \sigma_r(\pm\Psi'_{r,Y}) & = & \pm\Phi_{r,Y\cup\{r-1\}} & \textit{if } r-1\notin Y, \\ \sigma_{r-1}(\pm\Psi'_{r,Y}) & = & \pm\Psi'_{r,Y} & \textit{if } r-1\notin Y, \\ \sigma_r(\pm\Psi'_{r,Y}) & = & \pm\Psi'_{r,Y} & \textit{if } r-1\in Y, \\ \sigma_{r-1}(\pm\Psi'_{r,Y}) & = & \pm\Psi'_{r,Y} & \textit{if } r-1\in Y, \end{array}$$

where in "$\sigma_i(\Lambda)$" the map σ_i is the map given by Λ as in Def. 2.13, $(i\ \ i+1)$ is the transposition and $\pm\Lambda = \Lambda \cup -\Lambda$.

PROOF. Let $\beta_j := \eta_{j,r} + \eta_{j,r-2}$. Then one computes (at $\Phi_{r,Z}$)

$$\sigma_i(\beta_j) = \begin{cases} \beta_j & i \notin \{j-1, j\} \\ \beta_{j-1} & i = j-1 \\ \beta_{j+1} & i = j \end{cases}$$

for all $i = 1, \ldots, r-2$ and $j = 1, \ldots, r-1$. So $\sigma_1, \ldots, \sigma_{r-2}$ act as transpositions on $\beta_1, \ldots, \beta_{r-1}$. The situation is similar for Ψ and Ψ'. The other claims are an easy (although tiring) calculation. □

Prop. 3.11, Prop. 3.12, and Def. 2.13 immediately give:

COROLLARY 3.13. *Let $Z \subseteq \{1, \ldots, r-1\}$. Then there exists a coscorf Cartan scheme $\mathcal{C}$ such that $(R^{\mathrm{re}})^a_+ = \Phi_{r,Z}$ for an object a.*

REMARK 3.14. The Dynkin diagrams of the coscorf Cartan scheme of Cor. 3.13 and their connections are given by Fig. 3. The connections σ_{r-2} in the figure depend on Z resp. Y. For example σ_{r-2} maps an object $\Phi_{r,Z}$ of Dynkin type D to an object of Dynkin type D' if and only if $r-2 \in Z$; if $\Psi_{r,Y}$ is of Dynkin type C (as in the first diagram of Fig. 3) then σ_{r-2} maps to an object of Dynkin type A if and only if $r-2 \notin Z$.

PROPOSITION 3.15. *Let $Z_1, Z_2, Y_1, Y_2 \subseteq \{1, \ldots, r-1\}$ with $|Z_1| = |Z_2| = |Y_1| + 1 = |Y_2| + 1$. Then there exists a coscorf Cartan scheme $\mathcal{C}$ with objects a, b, c, d such that $(R^{\mathrm{re}})^a_+ = \Phi_{r,Z_1}$, $(R^{\mathrm{re}})^b_+ = \Phi_{r,Z_2}$, $(R^{\mathrm{re}})^c_+ = \Psi_{r,Y_1}$ and $(R^{\mathrm{re}})^d_+ = \Psi'_{r,Y_2}$.*

PROOF. By Prop. 3.12, $\sigma_1, \ldots, \sigma_{r-2}$ act as transpositions on $\{1, \ldots, r-1\}$ and generate the group $\mathrm{Sym}(\{1, \ldots, r-1\})$. Thus for the given Z_1, Z_2 there exists a product of σ_i's, $i = 1, \ldots, r-2$ mapping Φ_{r,Z_1} to Φ_{r,Z_2}. The proof for the other assertions is similar. □

REMARK 3.16. The coscorf Cartan scheme which has the root systems $\Phi_{r,\{1\}}$ and $\Psi_{r,\emptyset}$ has no object with Cartan matrix of type C_r. The coscorf Cartan scheme which has the root systems $\Phi_{r,\{1,\ldots,r-1\}}$ and $\Psi_{r,\{1,\ldots,r-2\}}$ has no object with Cartan matrix of type D_r.

DEFINITION 3.17. Let $\mathcal{C}$ be a coscorf Cartan scheme of rank r. If there exists a $Z \subseteq \{1,\dots,r-1\}$ such that $\Phi_{r,Z} = (R^{\mathrm{re}})^a_+$ for some object a, then we say that $\mathcal{C}$ *is of type* $D'(r,|Z|)$. If there exists a $Y \subseteq \{1,\dots,r-1\}$ such that $\Psi_{r,Y} = (R^{\mathrm{re}})^a_+$ for some object a, then we say that $\mathcal{C}$ *is of type* $D'(r,|Y|+1)$.

Notice that this is well-defined by Prop. 3.15. Thus if $\mathcal{C}$ is of type $D'(r,0)$ then it is standard of type D and if $\mathcal{C}$ is of type $D'(r,r)$ then it is standard of type C.

THEOREM 3.18. *Let* $Z \subseteq \{1,\dots,r-1\}$, $\mathcal{C}$ *be the coscorf Cartan scheme with* $(R^{\mathrm{re}})^a_+ = \Phi_{r,Z}$ *for an object* a, *and* $m := |Z|$. *Then* $\mathrm{Aut}(\mathcal{C})$ *is isomorphic to a reflection group of type* $C_m \times D_{r-m}$, *where* $D_1 := C_0$ *:=trivial group,* $D_2 := A_1 \times A_1$, $D_3 := A_3$, $C_1 := A_1$.

PROOF. By Prop. 3.15 we may assume $Z = \{1,\dots,m\}$. Let $\beta_j := \eta_{j,r} + \eta_{j,r-2}$ where $j = 1,\dots,r-1$ as in Prop. 3.12. Assume first that $1 < m < r-1$. Then σ^a_m is the only simple reflection which maps to an object with a different root system. The maps $\sigma^a_1,\dots,\sigma^a_{m-1}$ permute $\beta_1,\dots,\beta_m$ and $\sigma^a_{m+1},\dots,\sigma^a_r$ generate a reflection group of type D_{r-m}. Write $W(X_m)$ for the reflection group of type X_m. Then we have at least $S_m \times W(D_{r-m}) \le \mathrm{Aut}(\mathcal{C})$.

By Prop. 3.15 there is a morphism w leading to an object b with $(R^{\mathrm{re}})^b_+ = \Psi_{r,\{r-m+1,\dots,r-1\}}$. At b, $\sigma^d_{r-m+1},\dots,\sigma^d_r$ generate the group $W(C_m)$ and all these morphisms map to objects with the same root system. Conjugating this group back to a, we get $W(C_m) \times W(D_{r-m}) \le \mathrm{Aut}(\mathcal{C})$. We obtain the same result for $m=1$ and $m=r-1$ similarly.

It remains to check that there are no more morphisms. We achieve this by counting all morphisms to a fixed object a, i.e. by determining $n := |\mathrm{Hom}(\mathcal{W}(\mathcal{C}),a)|$. Since the Cartan scheme is connected and simply connected, n is the number of objects. Now $|W(C_m)\times W(D_{r-m})| = 2^m m! 2^{r-m-1}(r-m-1)!$ and we have $\binom{r-1}{m-1}+\binom{r}{m}$ different root sets. Thus we must prove $n = 2^{r-1}(m+r)(r-1)!$.

We proceed by induction on r and m and choose the object a with $(R^{\mathrm{re}})^a_+ = \Phi_{r,\{1,\dots,m\}}$. For $r \le 2$ the formula is an easy verification. For $m=0$ the set $\pm(R^{\mathrm{re}})^a_+$ is a root system of type D, thus $n = |W(D_r)| = 2^{r-1}r!$. Now let $m>0$. Write $J = \{2,\dots,r\}$ and $\mathcal{W}_J(\mathcal{C})$ for the parabolic subgroupoid of rank $r-1$ to J. One can check that $\mathrm{Hom}(\mathcal{W}(\mathcal{C}),a)$ is the union of the following "cosets":

(1)	$\mathrm{id}^a\sigma_i\sigma_{i-1}\dots\sigma_1\mathcal{W}_J(\mathcal{C})$	$i=0,\dots,m-1,$
(2)	$\mathrm{id}^a\sigma_i\sigma_{i-1}\dots\sigma_1\mathcal{W}_J(\mathcal{C})$	$i=m,\dots,r-1,$
(2)	$\mathrm{id}^a\sigma_i\sigma_{i+1}\dots\sigma_r\sigma_{r-2}\sigma_{r-3}\dots\sigma_1\mathcal{W}_J(\mathcal{C})$	$i=r,\dots,m+1,$
(1)	$\mathrm{id}^a\sigma_i\sigma_{i+1}\dots\sigma_{r-1}\sigma_r\sigma_{r-1}\dots\sigma_1\mathcal{W}_J(\mathcal{C})$	$i=m,\dots,1.$

Hereby, the parabolic subgroupoids $\mathcal{W}_J(\mathcal{C})$ are of type $D'(r-1,m-1)$ resp. $D'(r-1,m)$ in the rows labeled (1) resp. (2). Remark that for $m=r-1$, $\mathcal{W}_J(\mathcal{C})$ is of type $D'(r-1,r-1)$ in the rows labeled (2); this is standard of type C and has $2^{r-1}(r-1)!$ morphisms. Hence by induction $n = 2m2^{r-2}(m+r-2)(r-2)! + 2(r-m)2^{r-2}(m+r-1)(r-2)! = 2^{r-1}(m+r)(r-1)!$. □

Our goal is now to prove that the above coscorf Cartan schemes are the only ones with an object of Dynkin type D' in rank ≥ 9.

PROPOSITION 3.19. *Let $r \geq 8$ and let $\mathcal{C}$ be a coscorf Cartan scheme of rank r. Let a an object of $\mathcal{C}$ and assume that Γ^a is of type D'_r. Then $\Phi_{r,\{r-1\}} \subseteq (R^{\mathrm{re}})^a_+$.*

PROOF. Choose the labels for the vertices of Γ^a as in Fig. 3. We proceed by induction on r. For $r=8$ the claim is true by Section 4, so let $r>8$. Since the subdiagram to the labels $2,\dots,r$ is of type D', by induction we have $M := \Phi_{r,\{r-1\}} \cap \langle \alpha_2,\dots,\alpha_r\rangle \subseteq (R^{\mathrm{re}})^a_+$. Let $\beta := \eta_{1,r} + \eta_{2,r-2}$. One computes

$$M \cup \left\{ \sigma_1(\gamma) \mid \gamma = \sum_{i=2}^r a_i\alpha_i \in M,\ \ a_2 \neq 0 \right\} \cup \{\alpha_1,\beta\} = \Phi_{r,\{r-1\}}. \tag{3.1}$$

But σ_1 maps to an object with the same Dynkin diagram, so $M \subseteq (R^{\mathrm{re}})^{\rho_1(a)}_+$. Further $\beta = \sigma_2(\sigma_1(\alpha_2 + 2\eta_{3,r-2} + \alpha_{r-1} + \alpha_r))$. Since σ_2 also maps to an object with the same Dynkin diagram, with Equation (3.1) we obtain $\Phi_{r,\{r-1\}} \subseteq (R^{\mathrm{re}})^a_+$. □

THEOREM 3.20. *Let $r \geq 8$, let $\mathcal{C}$ be a coscorf Cartan scheme and a an object of $\mathcal{C}$. Assume that Γ^a is of type D'_r or D_r with labels for the vertices as in Fig. 3. Then there exists a subset $Z \subseteq \{1,\dots,r-1\}$ such that $(R^{\mathrm{re}})^a_+ = \Phi_{r,Z}$.*

PROOF. Notice first that by Prop. 3.19, $\Phi_{r,\{r-1\}} \subseteq (R^{\mathrm{re}})^a_+$ if Γ^a is of type D'_r. Further, we know by Cor. 3.13 that $\Phi_{r,Z}$ is a root set of rank r for all $Z \subseteq \{1,\dots,r-1\}$.

Now assume that Γ^a is of type D_r or D'_r and let $\alpha \in (R^{\mathrm{re}})^a_+$. Denote $Z_0 = \{1,\dots,r-1\}$. We prove by induction on the height $\mathrm{ht}(\alpha)$ of α that $\alpha \in \Phi_{r,Z_0}$. If $\mathrm{ht}(\alpha)=1$ then α is simple and we are done. So assume $\mathrm{ht}(\alpha)>1$. Applying σ_i^a for $i=1,\dots,r-2$ leads to an object of Dynkin type D_r or D'_r. If the height of $\sigma_i^a(\alpha)$ is smaller than $\mathrm{ht}(\alpha)$ for such an i, then by induction $\sigma_i^a(\alpha)$ is in Φ_{r,Z_0}. Let σ_i be the reflection corresponding to Φ_{r,Z_0} as in Def. 2.13. Then $\sigma_i = \sigma_i^a$ since the Dynkin diagram of Φ_{r,Z_0} is of type D'. But then $\alpha \in \sigma_i(\Phi_{r,Z_0}) = \{-\alpha_i\} \cup \Phi_{r,Z_0} \backslash \{\alpha_i\}$.

Assume that $\mathrm{ht}(\sigma_i^a(\alpha)) \geq \mathrm{ht}(\alpha)$ for $i = 1, \ldots, r-2$. Then writing $\alpha = \sum_{i=1}^r a_i\alpha_i$ we obtain

$$\begin{aligned} a_2 - a_1 &\geq a_1, \\ a_3 - a_2 + a_1 &\geq a_2, \\ \ldots & \\ a_{r-2} - a_{r-3} + a_{r-4} &\geq a_{r-3}, \\ (a_r + a_{r-1}) - a_{r-2} + a_{r-3} &\geq a_{r-2}. \end{aligned}$$

This means that

$$a_r + a_{r-1} - a_{r-2} \geq a_{r-2} - a_{r-3} \geq \ldots \geq a_2 - a_1 \geq a_1 \geq 0. \tag{3.2}$$

Now if Γ is of type D' resp. D then we compute $\beta = \sigma_r\sigma_{r-1}\sigma_r^a(\alpha)$ resp. $\beta = \sigma_{r-1}\sigma_r^a(\alpha)$. Notice that in both cases $\beta \in (R^{\mathrm{re}})_+^b$ for some object b of type D or D'. Again, if $\mathrm{ht}(\beta) < \mathrm{ht}(\alpha)$ then we are done by induction. Assuming the converse, in both cases we obtain

$$0 \leq \mathrm{ht}(\beta) - \mathrm{ht}(\alpha) = 2a_{r-2} - 2a_{r-1} - 2a_r.$$

With (3.2) this gives $a_r + a_{r-1} - a_{r-2} = 0$, thus $a_k - a_{k-1} = 0$ for $k = 2, \ldots, r-2$ and $a_1 = 0$. But then $a_{r-2} = \ldots = a_1 = 0$, which implies $a_r + a_{r-1} = 0$ and hence $\alpha = 0$ contradicting $\alpha \in (R^{\mathrm{re}})_+^a$. □

Collecting the last results we obtain the main theorem of this section:

THEOREM 3.21. *Let $\mathcal{C}$ be a coscorf Cartan scheme of rank $r > 8$ and let*

$$\mathcal{R}_+ := \{(R^{\mathrm{re}})_+^a \mid a \in \mathcal{C}\}.$$

Then there are two possibilities:
(1) The Cartan scheme $\mathcal{C}$ is standard ($|\mathcal{R}_+| = 1$) of type A, B, C, D.
(2) Up to equivalence the root sets of $\mathcal{C}$ are given by

$$\mathcal{R}_+ = \{\Phi_{r,Z}, \Psi_{r,Y}, \Psi'_{r,Y} \mid Z, Y \subseteq \{1, \ldots, r-1\},\ |Z| = s, |Y| = s-1\}$$

for some $s \in \{1, \ldots, r-1\}$.

In particular, if $\mathcal{C}$ is not standard then it has

$$\binom{r-1}{s-1} + \binom{r}{s}$$

different root sets and $2^{r-1}(m+r)(r-1)!$ objects.

4. Finite coscorf Cartan schemes of rank < 9

In this section we explain the classification of coscorf Cartan schemes of rank less or equal to 8. The proof is performed using computer calculations based on the knowledge of the case of rank two and three ([**17**], [**19**]). Our algorithm described below is sufficiently powerful: The implementation in C++ terminates within a few hours on a usual computer.

THEOREM 4.1.

(1) *Let $\mathcal{C}$ be a connected Cartan scheme of rank r, with $3 < r < 9$ and $I = \{1, \dots, r\}$. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. If $\mathcal{C}$ is not equivalent to a Cartan scheme as in Cor. 3.13, then there exists an object $a \in A$ and a linear map $\tau \in \mathrm{Aut}(\mathbb{Z}^I)$ such that $\tau(\alpha_i) \in \{\alpha_1, \dots, \alpha_r\}$ for all $i \in I$ and $\tau(R^a_+)$ is one of the sets listed in Appendix 2. Moreover, $\tau(R^a_+)$ with this property is uniquely determined.*
(2) *Let R be one of the 24 subsets of $\mathbb{Z}^r$, $3 < r < 9$ appearing in Appendix 2. There exists up to equivalence a unique coscorf Cartan scheme $\mathcal{C}$ such that $R \cup -R$ is the set of real roots R^a in an object $a \in A$. Moreover $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$.*
(3) *Let $\mathcal{C}$ be a coscorf Cartan scheme of rank r and $a \in A$. Then the Dynkin diagram Γ^a is one of the diagrams listed in Figure 5, p. 166.*

4.1. The idea. The classification of rank three has been achieved in [**19**]. Thus here we assume that the rank r is greater or equal to 4. Let $<$ be the lexicographic ordering on $\mathbb{Z}^r$ such that $\alpha_r < \alpha_{r-1} < \ldots < \alpha_1$. Then $\alpha > 0$ for any $\alpha \in \mathbb{N}_0^r \setminus \{0\}$.

The following theorem ([**19**, Thm. 2.10]) is crucial for the algorithm.

THEOREM 4.2. *Let $\mathcal{C}$ be a Cartan scheme. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$. Let $a \in A$ and $\alpha \in R^a_+$. Then either α is simple, or it is the sum of two positive roots.*

By Theorem 4.2 we may construct R^a_+ inductively by starting with $R^a_+ = \{\alpha_r, \alpha_{r-1}, \dots, \alpha_1\}$, and appending in each step a sum of a pair of positive roots which is greater than all roots in R^a_+ we already have. During this process, we keep track of all root subsets containing at least two positive roots, and their positive roots.

This is an overview of the algorithm without any details:

Algorithm 4.3. EnumerateRootSetsOverview(R)
Enumerate all root systems containing the roots R.
Input: a set of positive roots R.

Output: all root sets containing R.

1. If $R \cup -R$ is a root set, output $R \cup -R$ and continue.
2. For all subspaces U generated by elements of R, check that $R \cap U$ could be extended to a root set.
3. Set $Y := \{\alpha + \beta \mid \alpha, \beta \in R,\ \alpha \neq \beta\} \backslash R$.
4. For all $\alpha \in Y$ with $\alpha > \max R$, call **EnumerateRootSetsOverview**($R \cup \alpha$).

But this first approach is completely impracticable. We will need many improvements to reach our goal.

4.2. Some technical remarks. In fact, depending on the rank it is not always useful to compute all root subsets of all ranks because for instance a coscorf Cartan scheme of rank 7 can have up to 139251 root subsets of rank 4, in which case we spend more time organizing the root subsets than we spare using the restrictions they give. Thus in the following, $\rho < r$ will be the rank up to which we compute all root subsets.

Remark that to obtain a finite number of root systems as output, we have to ensure that we compute only irreducible systems since there are infinitely many inequivalent reducible root systems of rank two. Hence starting with $\{\alpha_r, \alpha_{r-1}, \dots, \alpha_1\}$ will not work. Instead, for each irreducible coscorf Cartan scheme we take a root system R' of rank $r-1$ and start with the sets

$$R_j := \Big\{ \sum_{i=1}^{r-1} \beta_i \alpha_{i+1} \mid \sum_{i=1}^{r-1} \beta_i \alpha_i \in R' \Big\} \cup \{\alpha_1, \alpha_1 + \alpha_j\}, \quad j = 2, \dots, r.$$

Before starting the algorithm, we collect all irreducible root subsets of rank up to ρ of all irreducible coscorf Cartan schemes of rank $r-1$ in a list Ξ (including all root subsets with permuted coordinates). During the algorithm, if a fragment of a root subset is found to be irreducible, then it is part of an irreducible root subset of rank $r-1$, and hence it lies in Ξ. We also store the list Υ of all roots for all ranks $2, \dots, \rho$ that appear in Ξ. This way we never need to fill the memory with coordinates but only with labels pointing to the root in Υ.

DEFINITION 4.4. Let R_+ be the set of positive roots of a fragment of a root subset (see 4.3) of rank three and M the set of planes containing at least two elements of R_+. We call the number

$$\varepsilon_{R_+} := 3 + \sum_{V \in M} (|V \cap R_+| - 3)$$

the *Euler invariant* of R_+.

If R_+ is a root subset of rank three, then $\varepsilon_{R_+} = 0$ by [**19**, Thm. 3.17].

4.3. The rsf. In this section, we will call *root system fragment* (or *rsf*) the following set of data associated to a set of positive roots R in construction:

- An ordered set of positive vectors R.
- For each rank $2, \dots, \rho$, the sequence of fragments of root subsets. Each such fragment consists of:
  1. A subspace U of $\mathbb{Q}^r$, a matrix used for a membership test for this subspace, and a matrix needed to compute the coordinates of a given element with respect to the basis.
  2. A hash value allowing us to perform a fast equality test for the subspaces.
  3. Labels from Υ for the roots of R in U with respect to the basis of U.
  4. Positions of the roots of R in U in the lexicographically ordered set R.
  5. The adjacency matrix of the Dynkin diagram (so far) of U and a flag whether it is connected.
- For the fragments of root subsets of rank two: all entries of the Cartan matrices and flags indicating if the root subset is “finished”.
- The Euler invariants of all fragments of rank three root subsets.
- Adjacency matrices of all fragments of parabolic subgroupoids.
- A flag “isvalid” telling if all the above data are consistent.

These data are continuously updated during the algorithm.

4.4. More remarks. Although the algorithm looks similar to the algorithm enumerating the root systems of rank three in [**19**], this version is much more work to implement for several reasons: The main reason is that we need linear algebra for the subspaces generated by the roots of a root subset. This includes an implementation of small rational numbers, Gauß algorithm, a fast membership test and hash-values for the subspaces. Further we need a good memory management for these subspaces to avoid duplicate versions of them. But there are even more functions needed, for example a test to decide if the Dynkin diagrams are connected.

Of course all these functions exist in computer algebra systems, but unfortunately they do not reach the desired performance mainly for two reasons: The first reason is that all these systems use arbitrary-precision arithmetic and in our situation the coefficients of the roots never get bigger than 14. The second reason is that these systems spend much time interpreting the code and dynamically determining the types of variables. For instance the computation of all root systems of rank 7 takes several weeks using a computer algebra prototype and takes only 12 minutes with the C++ version.

Since the C++ version needs approximatively 3 hours for rank 8, we guess that a version on any computer algebra system would take little less than a year. Besides, the computer algebra prototype uses a huge amount of memory.

4.5. The algorithm. This is the main recursion of the algorithm:

Algorithm 4.5. EnumerateRootSets(D[,γ])
Enumerate all root sets containing the roots of D .
Input: an rsf D, possibly a required root γ.
Output: all root sets containing the roots of D.

1. If the Euler invariants of all fragments of rank three root subsets in D are 0, then check if D yields a root system. If yes, output D and continue.
2. If no required root γ is known, then: For all fragments U of irreducible root subsets in D, search for possible completions in Ξ. If U may not be completed, then return $\emptyset$. Otherwise try to determine a smallest root γ which is missing and which will be included in any case, call **EnumerateRootSets**(D,γ) if successful and return.
3. Denote R the positive roots of D. Set $Y \leftarrow \{\alpha+\beta \mid \alpha,\beta \in R,\ \alpha \neq \beta\} \backslash R$.
4. For all $\alpha \in Y$ with $[\gamma \geq]\alpha > \max R$, call $\tilde{D}$:=**AppendRoot**(D,α); if $\tilde{D}$ is valid, then call **EnumerateRootSets**(D[,γ]).

In practice we use a global list Ω in which we note which R of an rsf has already been treated. The first step in "EnumerateRootSets" is to check if R is in Ω. The following proposition gives more details:

PROPOSITION 4.6. *If R contains a root $\beta = \sum_i a_i\alpha_i$ with $a_1 > 1$, then we can include the images of R under $\sigma_2,\dots,\sigma_r$ into Ω, and by the way we check if there is a contradiction (for example if these images contain roots with positive and negative coefficients).*

PROOF. Assume that R contains a root $\beta = \sum_i a_i\alpha_i$ with $a_1 > 1$ and that β is the greatest root in R. Then all roots of the form $m\alpha_i + \alpha_j$ with $m \in \mathbb{N}$ and $i, j > 1$ are smaller than β. Hence if R is to become a root set R^a_+ some day (after including roots which are greater than β), then its Cartan entries c_{ij} with $i, j > 1$ are already known:

$$c_{ij} = -\max\{m \in \mathbb{N} \mid m\alpha_i + \alpha_j \in R^a_+\}.$$

The same holds for the entries c_{i1}, $i > 1$. Therefore, the reflections $\sigma_2,\dots,\sigma_r$ are known.

Now let $i > 1$ and consider $R' := \sigma_i(R)$. If we include R' into Ω, then we have to ensure that all root sets constructed upon R' have or will be handled at some point. Thus assume $R'' = R' \cup E$ where E consists of roots greater than all roots of R'. The roots of $\sigma_i(E)$ which are greater than β do not pose a problem, because they will be considered in future. So let $\delta := k\alpha_1 + \gamma \in E$ with $\gamma \in \langle \alpha_2, \dots, \alpha_r \rangle$ and $\sigma_i(\delta) < \beta$. But

$$\sigma_i(\delta) = k\alpha_1 - kc_{i1}\alpha_i + \sigma_i(\gamma) > k\alpha_1,$$

so $\sigma_i(\delta)$ is not a root from the starting set of roots. If $\sigma_i(\delta) \notin R$, then $R \cup \{\sigma_i(\delta)\}$ is a set of roots which has already been considered in an earlier stage of the algorithm. □

REMARK 4.7. If we use Prop. 4.6, then it is essential to append new roots from Y in lexicographical order in step 4 of Algo. 4.2.

The time consuming part of the algorithm is the function “AppendRoot”:

Algorithm 4.8. AppendRoot(D, α)
Append a root to an rsf.
Input: an rsf D, a root α.
Output: an rsf $\tilde{D}$ consisting of D with α included.

1. Copy the data of D to a new rsf $\tilde{D}$.
2. The non-zero coordinates of α define a parabolic subgroupoid P of rank s which will be irreducible in $\tilde{D}$. If $s = 2$, then update the adjacency matrix for all parabolic subgroupoids containing α. Otherwise, if the Dynkin diagram of P is not connected, then set isvalid:=**false** and return.
3. For each positive root β in R compute the root subset $U := \langle \alpha, \beta \rangle$. If U is new, then include it to the rsf $\tilde{D}$.
4. For each root subset U of rank $2, \dots, \rho$ in D, test if α is in U. If it is, then include it into U in $\tilde{D}$, update the adjacency matrix and test if the Dynkin diagram is connected; compute its coordinates with respect to the basis: if they are not all non-negative integers, then return $\tilde{D}$ with isvalid:=**false**.

   If U has rank 2, update the Cartan entries: here we can test if the sequence of Cartan entries is valid and return $\tilde{D}$ with isvalid:=**false** if it is not.

   If U has rank 3, then update its Euler invariant.

   If $\alpha \notin U$, then remember U.
5. For all root subsets U of rank e we have remembered, create a root subset U' of rank $e+1$ by including α. Test if it is new by using its hash value. If $e = 2$, then initialize the Euler invariant of U'.
6. Return $\tilde{D}$ with isvalid:=**true**.

Finally, we still need a function to check which of the rsf is indeed a root set (see [**19**, Algo. 4.5]):

Algorithm 4.9. RootSetsForAllObjects(R)
Returns the roots for all objects if $R = R_+^a$ *determines a Cartan scheme* $\mathcal{C}$ *such that* $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ *is an irreducible root system.*
Input: R the set of positive roots at one object.
Output: the set of roots at all objects, or $\emptyset$ if R does not yield a Cartan scheme as desired.

1. $N \leftarrow [R]$, $M \leftarrow \emptyset$.
2. While $|N| > 0$, do steps 3 to 5.
3. Let F be the last element of N. Remove F from N and include it to M.
4. Compute the r simple reflections given by C_F.
5. For each simple reflection s, do:
- Compute $G := \{s(v) \mid v \in F\}$. If an element of G has positive and negative coefficients, then return $\emptyset$. Otherwise multiply the negative roots of G by -1.
- If $G \notin M$, then append G to N.

6. Return M.

1. Sporadic coscorf Cartan schemes

1.1. Summary. We will call *sporadic* the irreducible coscorf Cartan schemes of rank ≥ 3 not included in the series described in Section 3 because they do not seem to fit into a pattern. Among them are those of type F_4, E_6, E_7, E_8. In this section we summarize invariants of the sporadic coscorf Cartan schemes.

Rank	2	3	4	5	6	7	8	$r > 8$
Number	∞	55	18	14	13	12	12	$r + 3$

FIGURE 4. Number of irreducible coscorf Cartan schemes

Fig. 4 shows an overview of the output of the above algorithms and Section 3. We thus have $50 + 11 + 6 + 4 + 2 + 1 = 74$ sporadic coscorf Cartan schemes (in rank three only 5 coscorf Cartan schemes are not sporadic because $A_3 = D_3$).

On an *i7* with $2,8$ GHz, our C++ implementation of the algorithm (including the final check whether the sets are root sets and the computation of "canonical" objects) needs 6 min., 2 min., 16 min., 12 min., 170 min. for the ranks 4, 5, 6, 7, 8 respectively.

Remark that using Prop. 4.6 and the set Ω reduces the runtime in all cases except for the case of rank 8 where the runtime is increased by 12 minutes.

1.2. Dynkin diagrams. Figure 5 displays all Dynkin diagrams of irreducible cos-corf Cartan schemes of arbitrary rank. They are obtained from the data in Section 2 and [**19**] by Lemma 2.12.

1.3. Automorphism groups and planes.

REMARK 1.1. We collect the following invariants in Table 1.

Let $\mathcal{R}_+ = \{R^a_+ \mid a \in A\}$ denote the set of root sets in the objects of the Cartan scheme. By identifying objects with the same roots one obtains a quotient Cartan scheme of the simply connected Cartan scheme of the classification (see [**16**, Def. 3.1] for the definition of coverings). This quotient has the minimal number of objects with respect to all quotients of the Cartan scheme.

In the fifth column we give the automorphism group of one (equivalently, any) object of this quotient (this is $\mathrm{Aut}(\mathcal{C})$).

The last column contains a list of all Dynkin diagrams appearing in the Cartan scheme: the number i stands for the diagram Γ^i_r of Figure 5 if the root system is of rank r.

Remark that (except for the last column) the data for the Cartan schemes of rank three is also in [**19**, Table 1]. But notice that in [**19**] the standard Cartan schemes and the ones from the infinite series were not omitted, thus the scheme number n here corresponds to the one labeled $n+5$ in [**19**].

r	Nr.	$\lvert R^a_+\rvert$	$\lvert\mathcal{R}_+\rvert$	$\lvert A\rvert$	$\mathrm{Aut}(\mathcal{C})$	Dynkin diagrams
3	1	10	5	60	$A_1 \times A_2$	A, C, D', 6
3	2	10	10	60	A_2	A, B, C, 15
3	3	11	9	72	$A_1 \times A_1 \times A_1$	A, B, C, 6, 15
3	4	12	21	84	$A_1 \times A_1$	A, B, C, 2, 6, 15, 20
3	5	12	14	84	A_2	A, C, 6, 15
3	6	13	4	96	$G_2 \times A_1$	A, B, 2, 15
3	7	13	12	96	$A_1 \times A_1 \times A_1$	A, B, C, D', 1, 7, 11, 15
3	8	13	2	96	B_3	C, 6
3	9	13	2	96	B_3	B, 5
3	10	14	56	112	A_1	A, B, C, D', 2, 6, 7, 11, 15, 16, 20
3	11	15	16	128	$A_1 \times A_1 \times A_1$	A, C, 6, 7, 15, 20

r	Nr.	$\lvert R^a_+\rvert$	$\lvert\mathcal{R}_+\rvert$	$\lvert A\rvert$	$\mathrm{Aut}(\mathcal{C})$	Dynkin diagrams
3	12	16	36	144	$A_1 \times A_1$	A, B, C, 2, 6, 7, 15, 18, 20
3	13	16	24	144	A_2	A, B, C, 1, 5, 11, 15, 19
3	14	17	10	160	$A_1 \times B_2$	A, B, C, D', 6, 8, 11, 15
3	15	17	10	160	$B_2 \times A_1$	A, B, C, 2, 5, 7, 15
3	16	17	10	160	$C_2 \times A_1$	A, B, C, 1, 2, 5, 6, 13, 15
3	17	18	30	180	A_2	A, C, 2, 7, 15, 20
3	18	18	90	180	A_1	A, B, C, 1, 2, 5, 6, 11, 13, 15, 16, 19, 20
3	19	19	25	200	$A_1 \times A_1 \times A_1$	A, B, C, 1, 2, 3, 5, 6, 11, 13, 15, 16, 19, 20
3	20	19	8	192	$G_2 \times A_1$	B, C, 2, 4, 11, 15
3	21	19	50	200	$A_1 \times A_1$	A, B, C, 1, 2, 5, 6, 11, 13, 15, 16, 19, 21
3	22	19	25	200	$A_1 \times A_1 \times A_1$	A, B, C, 1, 2, 5, 6, 11, 13, 15, 16, 19
3	23	19	8	192	$G_2 \times A_1$	B, C, 1, 6, 7, 11
3	24	20	27	216	C_2	A, B, C, D', 2, 3, 10, 11, 15, 16
3	25	20	110	220	A_1	A, B, C, 1, 2, 3, 5, 6, 11, 13, 15, 16, 19, 21
3	26	20	110	220	A_1	A, B, C, 1, 2, 5, 6, 7, 11, 13, 15, 16, 20, 22
3	27	21	15	240	$A_1 \times C_2$	A, B, C, 1, 2, 3, 11, 15, 16, 19
3	28	21	30	240	$A_1 \times A_1 \times A_1$	A, B, C, 1, 2, 5, 6, 7, 13, 15, 20
3	29	21	5	240	C_3	A, C, 2, 7, 15
3	30	22	44	264	A_2	A, B, C, 2, 5, 6, 7, 13, 15, 16
3	31	25	42	336	$A_1 \times A_1 \times A_1$	A, B, C, 2, 3, 4, 5, 6, 8, 10, 11, 15, 16, 18, 19, 20
3	32	25	14	336	$A_1 \times G_2$	A, B, C, 1, 2, 6, 8, 13, 15, 20
3	33	25	28	336	$A_1 \times A_2$	A, B, C, D', 1, 2, 5, 6, 7, 11, 12, 13, 14, 15, 16
3	34	25	7	336	B_3	A, C, 2, 7, 15

r	Nr.	$\lvert R^a_+\rvert$	$\lvert\mathcal{R}_+\rvert$	$\lvert A\rvert$	$\mathrm{Aut}(\mathcal{C})$	Dynkin diagrams
3	35	26	182	364	A_1	A, B, C, 1, 2, 3, 5, 6, 8, 10, 11, 13, 15, 16, 17, 18, 19, 20, 22
3	36	26	182	364	A_1	A, B, C, 1, 2, 6, 8, 11, 13, 15, 16, 20, 21, 22
3	37	27	49	392	$A_1 \times A_1 \times A_1$	A, B, C, 1, 2, 3, 6, 8, 11, 13, 15, 16, 20, 22
3	38	27	98	392	$A_1 \times A_1$	A, B, C, 1, 2, 5, 6, 8, 11, 13, 15, 16, 20, 21, 22
3	39	27	98	392	$A_1 \times A_1$	A, B, C, 1, 2, 6, 7, 8, 11, 13, 15, 16, 20, 21, 22
3	40	28	420	420	1	A, B, C, 1, 2, 3, 5, 6, 7, 8, 11, 13, 15, 16, 20, 21, 22
3	41	28	210	420	A_1	A, B, C, 1, 2, 5, 6, 7, 8, 11, 13, 15, 16, 21, 22
3	42	28	70	420	A_2	A, B, C, 2, 5, 6, 8, 13, 15, 16, 21
3	43	29	56	448	$A_1 \times A_1 \times A_1$	A, B, C, 2, 3, 5, 6, 7, 8, 11, 13, 15, 16, 20, 22
3	44	29	112	448	$A_1 \times A_1$	A, B, C, 1, 2, 3, 5, 6, 7, 8, 11, 13, 15, 16, 21, 22
3	45	29	112	448	$A_1 \times A_1$	A, B, C, 2, 3, 5, 6, 7, 8, 11, 13, 15, 16, 21, 22
3	46	30	238	476	A_1	A, B, C, 2, 3, 5, 6, 7, 8, 11, 13, 15, 16, 21, 22
3	47	31	21	504	$A_1 \times G_2$	A, B, C, D', 1, 2, 6, 9, 11, 13, 15
3	48	31	21	504	$A_1 \times G_2$	A, B, C, 2, 3, 5, 6, 7, 8, 13, 15, 16
3	49	34	102	612	A_2	A, B, C, 2, 3, 6, 7, 8, 11, 15, 16, 20, 22
3	50	37	15	720	B_3	A, C, 2, 7, 8, 15, 20
4	1	15	10	360	$A_2 \times A_2$	A, D'
4	2	17	10	480	B_3	A, D, D', 1, 10
4	3	18	6	576	$B_3 \times A_1$	A, C, D', 1
4	4	21	36	864	A_3	A, B, C, D, D', 1, 8, 9, 10
4	5	22	10	960	$C_3 \times A_1$	A, C, D, D', 1, 3, 10

r	Nr.	$\lvert R^a_+\rvert$	$\lvert\mathcal{R}_+\rvert$	$\lvert A\rvert$	$\mathrm{Aut}(\mathcal{C})$	Dynkin diagrams
4	6	24	1	1152	F_4	1
4	7	25	12	1440	A_4	A, B, C, D', 9
4	8	28	20	1920	$B_3 \times A_1$	A, B, C, D, D', 1, 2, 3, 8, 9, 10
4	9	30	16	2304	$G_2 \times G_2$	A, D', 1, 4, 8
4	10	32	28	2688	$B_3 \times A_1$	A, B, C, D, D', 1, 5, 6, 7, 8, 9, 10
4	11	32	7	2688	B_4	A, C, D', 1, 3
5	1	25	6	4320	A_5	A, D, 6
5	2	30	12	8640	A_5	A, D, D', 6
5	3	33	15	11520	$B_4 \times A_1$	A, D, D', 1, 2, 6, 7
5	4	41	7	26880	B_5	A, C, D, D', 1, 6
5	5	46	56	40320	A_5	A, B, C, D, D', 2, 4, 5, 6, 7
5	6	49	21	48384	$F_4 \times A_1$	A, C, D, D', 1, 2, 3, 6, 7
6	1	36	1	51840	E_6	3
6	2	46	7	161280	D_6	A, D, 3, 4
6	3	63	14	725760	E_6	A, D', 3, 4
6	4	68	21	967680	B_6	A, D, D', 1, 2, 3, 4, 5
7	1	63	1	2903040	E_7	1
7	2	91	8	23224320	E_7	A, D, 1, 2
8	1	120	1	696729600	E_8	1

Table 1: Invariants of sporadic coscorf Cartan schemes, see Rem. 1.1

2. Irreducible root systems

We give the roots in a multiplicative notation[1] to save space: For instance the word $\prod_{i=1}^r i^{x_i}$ corresponds to $\sum_{i=1}^r x_i\alpha_{r+1-i}$.

Notice that we have chosen a "canonical" object for each groupoid. Write $\pi(R^a_+)$ for the set R^a_+ where the coordinates are permuted via $\pi \in S_r$. Then the set listed below is the minimum of $\{\pi(R^a_+) \mid a \in A,\ \pi \in S_r\}$ with respect to the lexicographical ordering on the sorted sequences of roots.

[1]We use the lexicographical ordering induced by $\alpha_1 > \alpha_2 > \ldots > \alpha_r$. This is convenient because it is the usual ordering in computer algebra systems. The index "$r+1-i$" ensures that the lists of roots start with $1, 2, 3, \ldots$

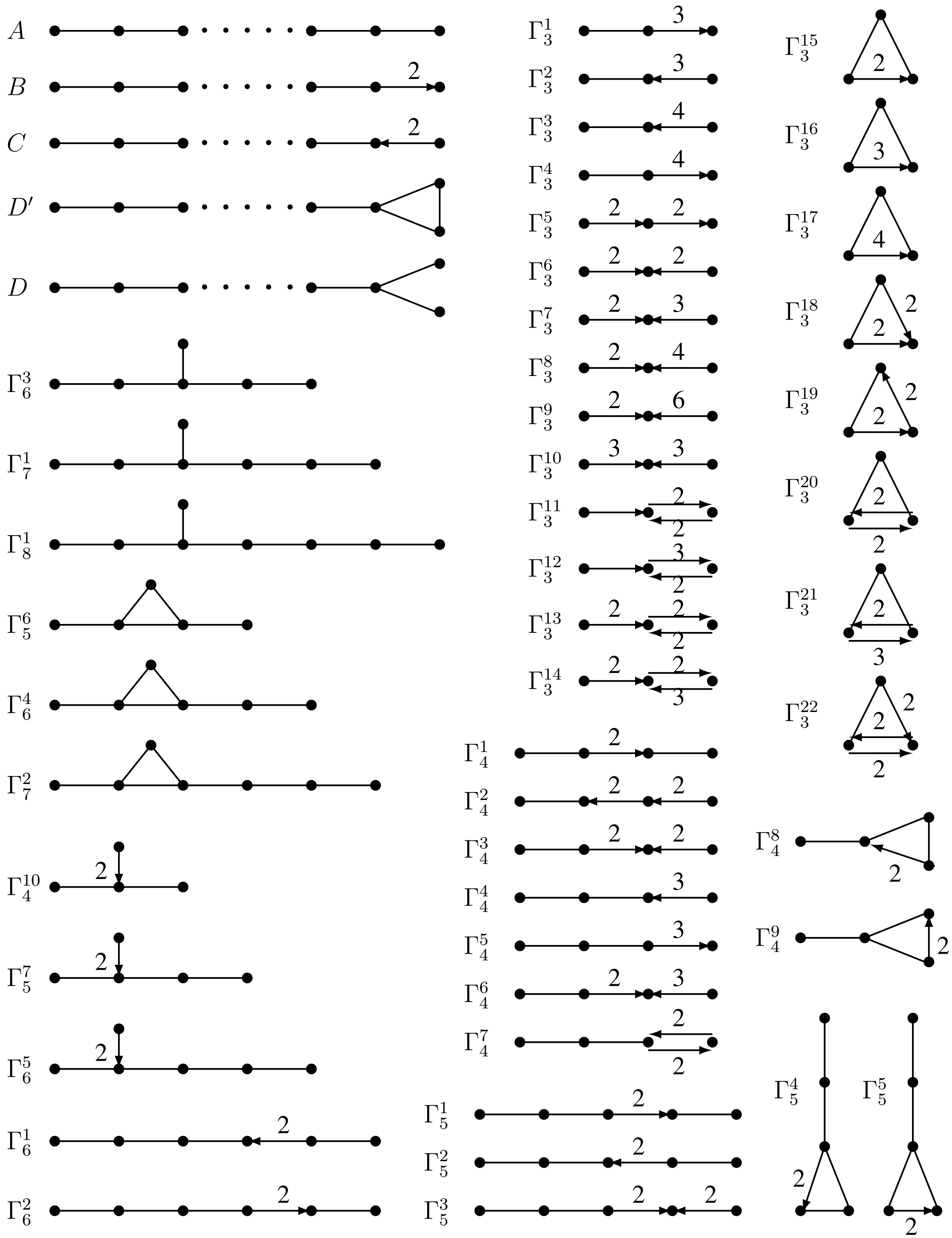

FIGURE 5. The Dynkin diagrams in irreducible coscorf Cartan schemes of rank ≥ 3

2.1. Rank 3.

Nr. 1 with 10 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{2}23$, $1^{3}23$

Nr. 2 with 10 positive roots: 1, 2, 3, 12, 13, 23, $1^{2}2$, 123, $1^{2}23$, $1^{2}2^{2}3$

Nr. 3 with 11 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{2}23$, $1^{3}23$, $1^{3}2^{2}3$

Nr. 4 with 12 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{3}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$

Nr. 5 with 12 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{2}23$, $1^{3}23$, $1^{2}2^{2}3$, $1^{3}2^{2}3$

Nr. 6 with 13 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$

Nr. 7 with 13 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}23$, $1^{4}23$, $1^{4}2^{2}3$

Nr. 8 with 13 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{2}23$, $1^{3}23$, $1^{2}2^{2}3$, $1^{3}2^{2}3$, $1^{4}2^{2}3$

Nr. 9 with 13 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, 123, 13^{2}, $1^{2}23$, 123^{2}, $1^{2}23^{2}$, $1^{3}23^{2}$, $1^{3}2^{2}3^{2}$

Nr. 10 with 14 positive roots: 1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$

Nr. 11 with 15 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$

Nr. 12 with 16 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$

Nr. 13 with 16 positive roots:
1, 2, 3, 12, 23, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $12^{2}3$, $1^{3}23$, $1^{2}2^{2}3$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{4}2^{3}3$, $1^{4}2^{3}3^{2}$

Nr. 14 with 17 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{4}2$, $1^{3}23$, $1^{4}23$, $1^{5}23$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{6}2^{2}3$

Nr. 15 with 17 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{5}2^{2}3^{2}$

Nr. 16 with 17 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{3}23$, $1^{2}2^{2}3$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{5}2^{3}3$, $1^{5}2^{3}3^{2}$, $1^{6}2^{3}3^{2}$

Nr. 17 with 18 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{5}2^{3}3$, $1^{6}2^{3}3$

Nr. 18 with 18 positive roots:
1, 2, 3, 12, 13, 23, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $12^{2}3$, $1^{3}23$, $1^{2}2^{2}3$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{3}2^{3}3$, $1^{4}2^{3}3$, $1^{4}2^{3}3^{2}$

Nr. 19 with 19 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{4}2$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{6}2^{2}3$, $1^{6}2^{3}3$, $1^{7}2^{3}3$, $1^{7}2^{3}3^{2}$

Nr. 20 with 19 positive roots:
1, 2, 3, 12, 23, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{4}2$, $1^{3}23$, $1^{2}2^{2}3$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{6}2^{2}3$, $1^{6}2^{3}3$, $1^{6}2^{3}3^{2}$

Nr. 21 with 19 positive roots:
1, 2, 3, 12, 13, 23, $1^{2}2$, 12^{2}, 123, $1^{3}2$, $1^{2}23$, $12^{2}3$, $1^{3}23$, $1^{2}2^{2}3$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{3}2^{3}3$, $1^{4}2^{3}3$, $1^{4}2^{3}3^{2}$

Nr. 22 with 19 positive roots:
1, 2, 3, 12, 13, 23, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $12^{2}3$, $1^{3}2^{2}$, $1^{3}23$, $1^{2}2^{2}3$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{3}2^{3}3$, $1^{4}2^{3}3$, $1^{4}2^{3}3^{2}$

Nr. 23 with 19 positive roots:
1, 2, 3, 12, 23, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{2}2^{2}3$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{3}2^{3}3$, $1^{4}2^{3}3$, $1^{5}2^{3}3$, $1^{6}2^{3}3$, $1^{6}2^{4}3$

Nr. 24 with 20 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{4}2$, $1^{3}2^{2}$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{5}2^{2}$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{6}2^{2}3$, $1^{6}2^{3}3$, $1^{7}2^{3}3$

Nr. 25 with 20 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{4}2$, $1^{3}2^{2}$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{6}2^{2}3$, $1^{6}2^{3}3$, $1^{7}2^{3}3$, $1^{7}2^{3}3^{2}$

Nr. 26 with 20 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{2}2^{2}3$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{4}2^{3}3$, $1^{5}2^{3}3$, $1^{6}2^{3}3^{2}$

Nr. 27 with 21 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, 123, $1^{3}2$, $1^{2}23$, $1^{4}2$, $1^{3}2^{2}$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{5}2^{2}$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{6}2^{2}3$, $1^{6}2^{3}3$, $1^{7}2^{3}3$, $1^{7}2^{3}3^{2}$

Nr. 28 with 21 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{2}2^{2}3$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{4}2^{3}3$, $1^{5}2^{3}3$, $1^{6}2^{3}3$, $1^{6}2^{3}3^{2}$

Nr. 29 with 21 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{5}2^{3}3$, $1^{5}2^{2}3^{2}$, $1^{6}2^{3}3$, $1^{6}2^{3}3^{2}$, $1^{7}2^{3}3^{2}$

Nr. 30 with 22 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{3}2^{2}$, $1^{3}23$, $1^{2}2^{2}3$, $1^{4}23$, $1^{3}2^{2}3$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{4}2^{3}3$, $1^{5}2^{3}3$, $1^{5}2^{2}3^{2}$, $1^{5}2^{3}3^{2}$, $1^{6}2^{3}3^{2}$

Nr. 31 with 25 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{4}2$, $1^{3}2^{2}$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{5}2^{2}$, $1^{5}23$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{6}2^{2}3$, $1^{7}2^{2}3$, $1^{6}2^{3}3$, $1^{7}2^{3}3$, $1^{8}2^{3}3$, $1^{8}2^{3}3^{2}$

Nr. 32 with 25 positive roots:
1, 2, 3, 12, 13, $1^{2}2$, $1^{2}3$, 123, $1^{3}2$, $1^{2}23$, $1^{4}2$, $1^{3}23$, $1^{4}23$, $1^{3}2^{2}3$, $1^{5}23$, $1^{4}2^{2}3$, $1^{5}2^{2}3$, $1^{6}2^{2}3$, $1^{7}2^{2}3$, $1^{6}2^{3}3$, $1^{7}2^{3}3$, $1^{8}2^{3}3$, $1^{7}2^{3}3^{2}$, $1^{8}2^{3}3^{2}$, $1^{9}2^{3}3^{2}$

Nr. 33 with 25 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 12^2, 123, 1^32, 1^223, 12^23, 1^323, 1^22^23, 1^423, 1^32^23, 1^42^23, 1^32^33, 1^32^23^2, 1^42^33, 1^52^33, 1^42^33^2, 1^52^33^2, 1^62^33^2, 1^72^43^2$

Nr. 34 with 25 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^32^2, 1^323, 1^22^23, 1^423, 1^32^23, 1^42^23, 1^52^23, 1^42^33, 1^52^33, 1^52^23^2, 1^62^33, 1^52^33^2, 1^62^33^2, 1^72^33^2, 1^72^43^2$

Nr. 35 with 26 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2$

Nr. 36 with 26 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2$

Nr. 37 with 27 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2$

Nr. 38 with 27 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2$

Nr. 39 with 27 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^72^23, 1^62^33, 1^72^33, 1^72^23^2, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2$

Nr. 40 with 28 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2$

Nr. 41 with 28 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2, 1^92^43^2$

Nr. 42 with 28 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2, 1^{11}2^43^2$

Nr. 43 with 29 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^72^23^2, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2$

Nr. 44 with 29 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2, 1^92^43^2$

Nr. 45 with 29 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2, 1^{11}2^43^2$

Nr. 46 with 30 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^72^23^2, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2, 1^92^43^2$

Nr. 47 with 31 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^323, 1^52, 1^423, 1^62, 1^523, 1^42^23, 1^623, 1^52^23, 1^723, 1^62^23, 1^72^23, 1^82^23, 1^92^23, 1^{10}2^23, 1^92^33, 1^{10}2^33, 1^{11}2^33, 1^{10}2^33^2, 1^{11}2^33^2, 1^{12}2^33^2$

Nr. 48 with 31 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^23^2, 1^72^23, 1^62^33, 1^72^33, 1^72^23^2, 1^82^33, 1^72^33^2, 1^82^33^2, 1^92^33^2, 1^92^43^2, 1^{11}2^43^2$

Nr. 49 with 34 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^33, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^82^43, 1^82^33^2, 1^92^43, 1^92^33^2, 1^92^43^2, 1^{11}2^43^2, 1^{11}2^53^2, 1^{12}2^53^2$

Nr. 50 with 37 positive roots:

$1, 2, 3, 12, 13, 1^22, 1^23, 123, 1^32, 1^223, 1^42, 1^32^2, 1^323, 1^423, 1^32^23, 1^52^2, 1^523, 1^42^23, 1^52^23, 1^62^23, 1^52^33, 1^72^23, 1^62^33, 1^72^33, 1^82^33, 1^72^33^2, 1^92^33, 1^82^43, 1^82^33^2, 1^92^43, 1^92^33^2, 1^{10}2^43, 1^92^43^2, 1^{11}2^43^2, 1^{11}2^53^2, 1^{12}2^53^2, 1^{13}2^53^2$

2.2. Rank 4.

Nr. 1 with 15 positive roots:

$1, 2, 3, 4, 12, 13, 14, 23, 123, 124, 134, 1^224, 1234, 1^2234, 1^22^234$

Nr. 2 with 17 positive roots:

$1, 2, 3, 4, 12, 13, 14, 1^{2}2, 123, 124, 134, 1^{2}23, 1^{2}24, 1234, 1^{2}234, 1^{3}234, 1^{3}2^{2}34$

Nr. 3 with 18 positive roots:

$1, 2, 3, 4, 12, 13, 24, 1^{2}2, 123, 124, 1^{2}23, 1^{2}24, 1234, 1^{2}2^{2}4, 1^{2}234, 1^{2}2^{2}34, 1^{3}2^{2}34, 1^{3}2^{2}3^{2}4$

Nr. 4 with 21 positive roots:

$1, 2, 3, 4, 12, 13, 14, 23, 1^{2}2, 123, 124, 134, 1^{2}23, 1^{2}24, 1234, 1^{2}2^{2}3, 1^{2}234, 1^{3}234, 1^{2}2^{2}34, 1^{3}2^{2}34, 1^{3}2^{2}3^{2}4$

Nr. 5 with 22 positive roots:

$1, 2, 3, 4, 12, 13, 24, 1^{2}2, 1^{2}3, 123, 124, 1^{2}23, 1^{2}24, 1234, 1^{3}23, 1^{2}2^{2}4, 1^{2}234, 1^{3}234, 1^{2}2^{2}34, 1^{3}2^{2}34, 1^{4}2^{2}34, 1^{4}2^{2}3^{2}4$

Nr. 6 with 24 positive roots (type F_4):

$1, 2, 3, 4, 12, 13, 24, 1^{2}2, 123, 124, 1^{2}23, 1^{2}24, 1234, 1^{2}2^{2}4, 1^{2}23^{2}, 1^{2}234, 1^{2}2^{2}34, 1^{2}23^{2}4, 1^{3}2^{2}34, 1^{2}2^{2}3^{2}4, 1^{3}2^{2}3^{2}4, 1^{4}2^{2}3^{2}4, 1^{4}2^{3}3^{2}4, 1^{4}2^{3}3^{2}4^{2}$

Nr. 7 with 25 positive roots:

$1, 2, 3, 4, 12, 13, 23, 34, 1^{2}2, 123, 134, 234, 1^{2}23, 1234, 13^{2}4, 1^{2}2^{2}3, 1^{2}234, 123^{2}4, 1^{2}2^{2}34, 1^{2}23^{2}4, 1^{3}23^{2}4, 1^{2}2^{2}3^{2}4, 1^{3}2^{2}3^{2}4, 1^{3}2^{2}3^{3}4, 1^{3}2^{2}3^{3}4^{2}$

Nr. 8 with 28 positive roots:

$1, 2, 3, 4, 12, 13, 34, 1^{2}2, 1^{2}3, 123, 134, 1^{2}23, 1^{2}34, 1234, 1^{3}23, 1^{2}234, 1^{2}3^{2}4, 1^{3}2^{2}3, 1^{3}234, 1^{2}23^{2}4, 1^{3}2^{2}34, 1^{3}23^{2}4, 1^{4}23^{2}4, 1^{3}2^{2}3^{2}4, 1^{4}2^{2}3^{2}4, 1^{5}2^{2}3^{2}4, 1^{5}2^{2}3^{3}4, 1^{5}2^{2}3^{3}4^{2}$

Nr. 9 with 30 positive roots:

$1, 2, 3, 4, 12, 13, 34, 1^{2}2, 123, 134, 1^{3}2, 1^{2}23, 1234, 1^{3}2^{2}, 1^{3}23, 1^{2}234, 1^{3}2^{2}3, 1^{3}234, 1^{2}23^{2}4, 1^{4}2^{2}3, 1^{3}2^{2}34, 1^{3}23^{2}4, 1^{4}2^{2}34, 1^{3}2^{2}3^{2}4, 1^{4}2^{2}3^{2}4, 1^{5}2^{2}3^{2}4, 1^{5}2^{3}3^{2}4, 1^{6}2^{3}3^{2}4, 1^{6}2^{3}3^{3}4, 1^{6}2^{3}3^{3}4^{2}$

Nr. 10 with 32 positive roots:

$1, 2, 3, 4, 12, 13, 34, 1^{2}2, 1^{2}3, 123, 134, 1^{3}2, 1^{2}23, 1^{2}34, 1234, 1^{3}23, 1^{2}234, 1^{2}3^{2}4, 1^{4}23, 1^{3}234, 1^{2}23^{2}4, 1^{4}2^{2}3, 1^{4}234, 1^{3}23^{2}4, 1^{4}2^{2}34, 1^{4}23^{2}4, 1^{5}23^{2}4, 1^{4}2^{2}3^{2}4, 1^{5}2^{2}3^{2}4, 1^{6}2^{2}3^{2}4, 1^{6}2^{2}3^{3}4, 1^{6}2^{2}3^{3}4^{2}$

Nr. 11 with 32 positive roots:

$1, 2, 3, 4, 12, 13, 24, 1^{2}2, 1^{2}3, 123, 124, 1^{2}23, 1^{2}24, 1234, 1^{3}23, 1^{2}2^{2}3, 1^{2}2^{2}4, 1^{2}234, 1^{3}2^{2}3, 1^{3}234, 1^{2}2^{2}34, 1^{4}2^{2}3, 1^{3}2^{2}34, 1^{4}2^{2}34, 1^{3}2^{3}34, 1^{4}2^{3}34, 1^{4}2^{2}3^{2}4, 1^{5}2^{3}34, 1^{4}2^{3}3^{2}4, 1^{5}2^{3}3^{2}4, 1^{6}2^{3}3^{2}4, 1^{6}2^{4}3^{2}4$

2.3. Rank 5.

Nr. 1 with 25 positive roots:

$1, 2, 3, 4, 5, 12, 13, 14, 23, 25, 123, 124, 125, 134, 235, 1234, 1235, 1245, 1^{2}234, 12^{2}35, 12345, 1^{2}2345, 12^{2}345, 1^{2}2^{2}345, 1^{2}2^{2}3^{2}45$

Nr. 2 with 30 positive roots:

$1, 2, 3, 4, 5, 12, 13, 14, 23, 35, 123, 124, 134, 135, 235, 1^{2}24, 1234, 1235, 1345, 1^{2}234, 123^{2}5, 12345, 1^{2}2^{2}34, 1^{2}2345, 123^{2}45, 1^{2}2^{2}345, 1^{2}23^{2}45, 1^{2}2^{2}3^{2}45, 1^{3}2^{2}3^{2}45, 1^{3}2^{2}3^{2}4^{2}5$

Nr. 3 with 33 positive roots:

$1, 2, 3, 4, 5, 12, 13, 14, 35, 1^{2}2, 123, 124, 134, 135, 1^{2}23, 1^{2}24, 1234, 1235, 1345, 1^{2}234, 1^{2}235, 12345, 1^{3}234, 1^{2}23^{2}5, 1^{2}2345, 1^{3}2^{2}34, 1^{3}2345, 1^{2}23^{2}45, 1^{3}2^{2}345, 1^{3}23^{2}45, 1^{3}2^{2}3^{2}45, 1^{4}2^{2}3^{2}45, 1^{4}2^{2}3^{2}4^{2}5$

Nr. 4 with 41 positive roots:

$1, 2, 3, 4, 5, 12, 13, 24, 45, 1^{2}2, 123, 124, 245, 1^{2}23, 1^{2}24, 1234, 1245, 1^{2}2^{2}4, 1^{2}234, 1^{2}245, 12345, 1^{2}2^{2}34, 1^{2}2^{2}45, 1^{2}2345, 1^{3}2^{2}34, 1^{2}2^{2}345, 1^{2}2^{2}4^{2}5, 1^{3}2^{2}3^{2}4, 1^{3}2^{2}345, 1^{2}2^{2}34^{2}5, 1^{3}2^{2}3^{2}45, 1^{3}2^{2}34^{2}5, 1^{3}2^{3}34^{2}5, 1^{3}2^{2}3^{2}4^{2}5, 1^{4}2^{3}34^{2}5, 1^{3}2^{3}3^{2}4^{2}5, 1^{4}2^{3}3^{2}4^{2}5, 1^{5}2^{3}3^{2}4^{2}5, 1^{5}2^{4}3^{2}4^{2}5, 1^{5}2^{4}3^{2}4^{3}5, 1^{5}2^{4}3^{2}4^{3}5^{2}$

Nr. 5 with 46 positive roots:

$1, 2, 3, 4, 5, 12, 13, 14, 23, 45, 1^{2}2, 123, 124, 134, 145, 1^{2}23, 1^{2}24, 1234, 1245, 1345, 1^{2}2^{2}3, 1^{2}234, 1^{2}245, 12345, 1^{3}234, 1^{2}2^{2}34, 1^{2}2345, 1^{2}24^{2}5, 1^{3}2^{2}34, 1^{3}2345, 1^{2}2^{2}345, 1^{2}234^{2}5, 1^{3}2^{2}3^{2}4, 1^{3}2^{2}345, 1^{3}234^{2}5, 1^{2}2^{2}34^{2}5, 1^{3}2^{2}3^{2}45, 1^{3}2^{2}34^{2}5, 1^{4}2^{2}34^{2}5, 1^{3}2^{2}3^{2}4^{2}5, 1^{4}2^{3}34^{2}5, 1^{4}2^{2}3^{2}4^{2}5, 1^{4}2^{3}3^{2}4^{2}5, 1^{5}2^{3}3^{2}4^{2}5, 1^{5}2^{3}3^{2}4^{3}5, 1^{5}2^{3}3^{2}4^{3}5^{2}$

Nr. 6 with 49 positive roots:

$1, 2, 3, 4, 5, 12, 13, 24, 45, 1^{2}2, 1^{2}3, 123, 124, 245, 1^{2}23, 1^{2}24, 1234, 1245, 1^{3}23, 1^{2}2^{2}4, 1^{2}234, 1^{2}245, 12345, 1^{3}234, 1^{2}2^{2}34, 1^{2}2^{2}45, 1^{2}2345, 1^{3}2^{2}34, 1^{3}2345, 1^{2}2^{2}345, 1^{2}2^{2}4^{2}5, 1^{4}2^{2}34, 1^{3}2^{2}345, 1^{2}2^{2}34^{2}5, 1^{4}2^{2}3^{2}4, 1^{4}2^{2}345, 1^{3}2^{2}34^{2}5, 1^{4}2^{2}3^{2}45, 1^{4}2^{2}34^{2}5, 1^{3}2^{3}34^{2}5, 1^{4}2^{3}34^{2}5, 1^{4}2^{2}3^{2}4^{2}5, 1^{5}2^{3}34^{2}5, 1^{4}2^{3}3^{2}4^{2}5, 1^{5}2^{3}3^{2}4^{2}5, 1^{6}2^{3}3^{2}4^{2}5, 1^{6}2^{4}3^{2}4^{2}5, 1^{6}2^{4}3^{2}4^{3}5, 1^{6}2^{4}3^{2}4^{3}5^{2}$

2.4. Rank 6.

Nr. 1 with 36 positive roots (type E_6):

$1, 2, 3, 4, 5, 6, 12, 13, 14, 25, 36, 123, 124, 125, 134, 136, 1234, 1235, 1236, 1245, 1346, 1^{2}234, 12345, 12346, 12356, 1^{2}2345, 1^{2}2346, 123456, 1^{2}2^{2}345, 1^{2}23^{2}46, 1^{2}23456, 1^{2}2^{2}3456, 1^{2}23^{2}456, 1^{2}2^{2}3^{2}456, 1^{3}2^{2}3^{2}456, 1^{3}2^{2}3^{2}4^{2}56$

Nr. 2 with 46 positive roots:

$1, 2, 3, 4, 5, 6, 12, 13, 14, 23, 25, 46, 123, 124, 125, 134, 146, 235, 1234, 1235, 1245, 1246, 1346, 1^{2}234, 12^{2}35, 12345, 12346, 12456, 1^{2}2345, 1^{2}2346, 12^{2}345, 123456, 1^{2}2^{2}345, 1^{2}234^{2}6, 1^{2}23456, 12^{2}3456, 1^{2}2^{2}3^{2}45, 1^{2}2^{2}3456, 1^{2}234^{2}56, 1^{2}2^{2}3^{2}456, 1^{2}2^{2}34^{2}56, 1^{3}2^{2}34^{2}56, 1^{2}2^{2}3^{2}4^{2}56, 1^{3}2^{2}3^{2}4^{2}56, 1^{3}2^{3}3^{2}4^{2}56, 1^{3}2^{3}3^{2}4^{2}5^{2}6$

Nr. 3 with 63 positive roots:
$1, 2, 3, 4, 5, 6, 12, 13, 14, 23, 35, 56, 123, 124, 134, 135, 235, 356, 1^{2}24, 1234, 1235, 1345, 1356, 2356, 1^{2}234,$
$123^{2}5, 12345, 12356, 13456, 1^{2}2^{2}34, 1^{2}2345, 123^{2}45, 123^{2}56, 123456, 1^{2}2^{2}345, 1^{2}23^{2}45, 1^{2}23456, 123^{2}456, 123^{2}5^{2}6,$
$1^{2}2^{2}3^{2}45, 1^{2}2^{2}3456, 1^{2}23^{2}456, 123^{2}45^{2}6, 1^{3}2^{2}3^{2}45, 1^{2}2^{2}3^{2}456, 1^{2}23^{2}45^{2}6, 1^{3}2^{2}3^{2}4^{2}5, 1^{3}2^{2}3^{2}456, 1^{2}2^{2}3^{2}45^{2}6,$
$1^{2}23^{3}45^{2}6, 1^{3}2^{2}3^{2}4^{2}56, 1^{3}2^{2}3^{2}45^{2}6, 1^{2}2^{2}3^{3}45^{2}6, 1^{3}2^{2}3^{3}45^{2}6, 1^{3}2^{2}3^{2}4^{2}5^{2}6, 1^{3}2^{3}3^{3}45^{2}6, 1^{3}2^{2}3^{3}4^{2}5^{2}6, 1^{4}2^{2}3^{3}4^{2}5^{2}6,$
$1^{3}2^{3}3^{3}4^{2}5^{2}6, 1^{4}2^{3}3^{3}4^{2}5^{2}6, 1^{4}2^{3}3^{4}4^{2}5^{2}6, 1^{4}2^{3}3^{4}4^{2}5^{3}6, 1^{4}2^{3}3^{4}4^{2}5^{3}6^{2}$

Nr. 4 with 68 positive roots:
$1, 2, 3, 4, 5, 6, 12, 13, 14, 35, 56, 1^{2}2, 123, 124, 134, 135, 356, 1^{2}23, 1^{2}24, 1234, 1235, 1345, 1356, 1^{2}234, 1^{2}235,$
$12345, 12356, 13456, 1^{3}234, 1^{2}23^{2}5, 1^{2}2345, 1^{2}2356, 123456, 1^{3}2^{2}34, 1^{3}2345, 1^{2}23^{2}45, 1^{2}23^{2}56, 1^{2}23456, 1^{3}2^{2}345,$
$1^{3}23^{2}45, 1^{3}23456, 1^{2}23^{2}456, 1^{2}23^{2}5^{2}6, 1^{3}2^{2}3^{2}45, 1^{3}2^{2}3456, 1^{3}23^{2}456, 1^{2}23^{2}45^{2}6, 1^{4}2^{2}3^{2}45, 1^{3}2^{2}3^{2}456, 1^{3}23^{2}45^{2}6,$
$1^{4}2^{2}3^{2}4^{2}5, 1^{4}2^{2}3^{2}456, 1^{3}2^{2}3^{2}45^{2}6, 1^{3}23^{3}45^{2}6, 1^{4}2^{2}3^{2}4^{2}56, 1^{4}2^{2}3^{2}45^{2}6, 1^{3}2^{2}3^{3}45^{2}6, 1^{4}2^{2}3^{3}45^{2}6, 1^{4}2^{2}3^{2}4^{2}5^{2}6,$
$1^{5}2^{2}3^{3}45^{2}6, 1^{4}2^{2}3^{3}4^{2}5^{2}6, 1^{5}2^{3}3^{3}45^{2}6, 1^{5}2^{2}3^{3}4^{2}5^{2}6, 1^{5}2^{3}3^{3}4^{2}5^{2}6, 1^{6}2^{3}3^{3}4^{2}5^{2}6, 1^{6}2^{3}3^{4}4^{2}5^{2}6, 1^{6}2^{3}3^{4}4^{2}5^{3}6, 1^{6}2^{3}3^{4}4^{2}5^{3}6^{2}$

2.5. Rank 7.

Nr. 1 with 63 positive roots (type E_7):
$1, 2, 3, 4, 5, 6, 7, 12, 13, 14, 25, 36, 57, 123, 124, 125, 134, 136, 257, 1234, 1235, 1236, 1245, 1257, 1346, 1^{2}234, 12345,$
$12346, 12356, 12357, 12457, 1^{2}2345, 1^{2}2346, 123456, 123457, 123567, 1^{2}2^{2}345, 1^{2}23^{2}46, 1^{2}23456, 1^{2}23457, 1234567,$
$1^{2}2^{2}3456, 1^{2}2^{2}3457, 1^{2}23^{2}456, 1^{2}234567, 1^{2}2^{2}3^{2}456, 1^{2}2^{2}345^{2}7, 1^{2}2^{2}34567, 1^{2}23^{2}4567, 1^{3}2^{2}3^{2}456, 1^{2}2^{2}3^{2}4567,$
$1^{2}2^{2}345^{2}67, 1^{3}2^{2}3^{2}4^{2}56, 1^{3}2^{2}3^{2}4567, 1^{2}2^{2}3^{2}45^{2}67, 1^{3}2^{2}3^{2}4^{2}567, 1^{3}2^{2}3^{2}45^{2}67, 1^{3}2^{3}3^{2}45^{2}67, 1^{3}2^{2}3^{2}4^{2}5^{2}67, 1^{3}2^{3}3^{2}4^{2}5^{2}67,$
$1^{4}2^{3}3^{2}4^{2}5^{2}67, 1^{4}2^{3}3^{3}4^{2}5^{2}67, 1^{4}2^{3}3^{3}4^{2}5^{2}6^{2}7$

Nr. 2 with 91 positive roots:
$1, 2, 3, 4, 5, 6, 7, 12, 13, 14, 23, 25, 46, 67, 123, 124, 125, 134, 146, 235, 467, 1234, 1235, 1245, 1246, 1346, 1467, 1^{2}234,$
$12^{2}35, 12345, 12346, 12456, 12467, 13467, 1^{2}2345, 1^{2}2346, 12^{2}345, 123456, 123467, 124567, 1^{2}2^{2}345, 1^{2}234^{2}6,$
$1^{2}23456, 1^{2}23467, 12^{2}3456, 1234567, 1^{2}2^{2}3^{2}45, 1^{2}2^{2}3456, 1^{2}234^{2}56, 1^{2}234^{2}67, 1^{2}234567, 12^{2}34567, 1^{2}2^{2}3^{2}456,$
$1^{2}2^{2}34^{2}56, 1^{2}2^{2}34567, 1^{2}234^{2}567, 1^{2}234^{2}6^{2}7, 1^{3}2^{2}34^{2}56, 1^{2}2^{2}3^{2}4^{2}56, 1^{2}2^{2}3^{2}4567, 1^{2}2^{2}34^{2}567, 1^{2}234^{2}56^{2}7, 1^{3}2^{2}3^{2}4^{2}56,$
$1^{3}2^{2}34^{2}567, 1^{2}2^{2}3^{2}4^{2}567, 1^{2}2^{2}34^{2}56^{2}7, 1^{3}2^{3}3^{2}4^{2}56, 1^{3}2^{2}3^{2}4^{2}567, 1^{3}2^{2}34^{2}56^{2}7, 1^{2}2^{2}3^{2}4^{2}56^{2}7, 1^{3}2^{3}3^{2}4^{2}5^{2}6, 1^{3}2^{3}3^{2}4^{2}567,$
$1^{3}2^{2}3^{2}4^{2}56^{2}7, 1^{3}2^{2}34^{3}56^{2}7, 1^{3}2^{3}3^{2}4^{2}5^{2}67, 1^{3}2^{3}3^{2}4^{2}56^{2}7, 1^{3}2^{2}3^{2}4^{3}56^{2}7, 1^{4}2^{2}3^{2}4^{3}56^{2}7, 1^{3}2^{3}3^{2}4^{3}56^{2}7, 1^{3}2^{3}3^{2}4^{2}5^{2}6^{2}7,$
$1^{4}2^{3}3^{2}4^{3}56^{2}7, 1^{3}2^{3}3^{2}4^{3}5^{2}6^{2}7, 1^{4}2^{3}3^{3}4^{3}56^{2}7, 1^{4}2^{3}3^{2}4^{3}5^{2}6^{2}7, 1^{4}2^{4}3^{2}4^{3}5^{2}6^{2}7, 1^{4}2^{3}3^{3}4^{3}5^{2}6^{2}7, 1^{4}2^{4}3^{3}4^{3}5^{2}6^{2}7,$
$1^{5}2^{4}3^{3}4^{3}5^{2}6^{2}7, 1^{5}2^{4}3^{3}4^{4}5^{2}6^{2}7, 1^{5}2^{4}3^{3}4^{4}5^{2}6^{3}7, 1^{5}2^{4}3^{3}4^{4}5^{2}6^{3}7^{2}$

2.6. Rank 8.

Nr. 1 with 120 positive roots (type E_8):
$1, 2, 3, 4, 5, 6, 7, 8, 12, 13, 14, 25, 36, 57, 78, 123, 124, 125, 134, 136, 257, 578, 1234, 1235, 1236, 1245, 1257,$
$1346, 2578, 1^{2}234, 12345, 12346, 12356, 12357, 12457, 12578, 1^{2}2345, 1^{2}2346, 123456, 123457, 123567, 123578,$
$124578, 1^{2}2^{2}345, 1^{2}23^{2}46, 1^{2}23456, 1^{2}23457, 1234567, 1234578, 1235678, 1^{2}2^{2}3456, 1^{2}2^{2}3457, 1^{2}23^{2}456, 1^{2}234567,$
$1^{2}234578, 12345678, 1^{2}2^{2}3^{2}456, 1^{2}2^{2}345^{2}7, 1^{2}2^{2}34567, 1^{2}2^{2}34578, 1^{2}23^{2}4567, 1^{2}2345678, 1^{3}2^{2}3^{2}456, 1^{2}2^{2}3^{2}4567,$
$1^{2}2^{2}345^{2}67, 1^{2}2^{2}345^{2}78, 1^{2}2^{2}345678, 1^{2}23^{2}45678, 1^{3}2^{2}3^{2}4^{2}56, 1^{3}2^{2}3^{2}4567, 1^{2}2^{2}3^{2}45^{2}67, 1^{2}2^{2}3^{2}45678, 1^{2}2^{2}345^{2}678,$
$1^{2}2^{2}345^{2}7^{2}8, 1^{3}2^{2}3^{2}4^{2}567, 1^{3}2^{2}3^{2}45^{2}67, 1^{3}2^{2}3^{2}45678, 1^{2}2^{2}3^{2}45^{2}678, 1^{2}2^{2}345^{2}67^{2}8, 1^{3}2^{3}3^{2}45^{2}67, 1^{3}2^{2}3^{2}4^{2}5^{2}67,$
$1^{3}2^{2}3^{2}4^{2}5678, 1^{3}2^{2}3^{2}45^{2}678, 1^{2}2^{2}3^{2}45^{2}67^{2}8, 1^{3}2^{3}3^{2}4^{2}5^{2}67, 1^{3}2^{3}3^{2}45^{2}678, 1^{3}2^{2}3^{2}4^{2}5^{2}678, 1^{3}2^{2}3^{2}45^{2}67^{2}8, 1^{4}2^{3}3^{2}4^{2}5^{2}67,$
$1^{3}2^{3}3^{2}4^{2}5^{2}678, 1^{3}2^{3}3^{2}45^{2}67^{2}8, 1^{3}2^{2}3^{2}4^{2}5^{2}67^{2}8, 1^{4}2^{3}3^{3}4^{2}5^{2}67, 1^{4}2^{3}3^{2}4^{2}5^{2}678, 1^{3}2^{3}3^{2}4^{2}5^{2}67^{2}8, 1^{3}2^{3}3^{2}45^{3}67^{2}8,$
$1^{4}2^{3}3^{3}4^{2}5^{2}6^{2}7, 1^{4}2^{3}3^{3}4^{2}5^{2}678, 1^{4}2^{3}3^{2}4^{2}5^{2}67^{2}8, 1^{3}2^{3}3^{2}4^{2}5^{3}67^{2}8, 1^{4}2^{3}3^{3}4^{2}5^{2}6^{2}78, 1^{4}2^{3}3^{3}4^{2}5^{2}67^{2}8, 1^{4}2^{3}3^{2}4^{2}5^{3}67^{2}8,$
$1^{4}2^{4}3^{2}4^{2}5^{3}67^{2}8, 1^{4}2^{3}3^{3}4^{2}5^{3}67^{2}8, 1^{4}2^{3}3^{3}4^{2}5^{2}6^{2}7^{2}8, 1^{4}2^{4}3^{3}4^{2}5^{3}67^{2}8, 1^{4}2^{3}3^{3}4^{2}5^{3}6^{2}7^{2}8, 1^{5}2^{4}3^{3}4^{2}5^{3}67^{2}8, 1^{4}2^{4}3^{3}4^{2}5^{3}6^{2}7^{2}8,$
$1^{5}2^{4}3^{3}4^{3}5^{3}67^{2}8, 1^{5}2^{4}3^{3}4^{2}5^{3}6^{2}7^{2}8, 1^{5}2^{4}3^{4}4^{2}5^{3}6^{2}7^{2}8, 1^{5}2^{4}3^{3}4^{3}5^{3}6^{2}7^{2}8, 1^{5}2^{4}3^{4}4^{3}5^{3}6^{2}7^{2}8, 1^{6}2^{4}3^{4}4^{3}5^{3}6^{2}7^{2}8, 1^{6}2^{5}3^{4}4^{3}5^{3}6^{2}7^{2}8,$
$1^{6}2^{5}3^{4}4^{3}5^{4}6^{2}7^{2}8, 1^{6}2^{5}3^{4}4^{3}5^{4}6^{2}7^{3}8, 1^{6}2^{5}3^{4}4^{3}5^{4}6^{2}7^{3}8^{2}$

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