A LECTURE ON FINITE WEYL GROUPOIDS, OBERWOLFACH 2012

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Abstract. This is a short introduction to the theory of finite Weyl groupoids, crystallographic arrangements, and their classification.

Introduction

Reflections appear in many areas of mathematics. For instance, certain groups generated by involutions may be investigated by representing them as reflection groups. In particular, the Weyl groups belong to this class. They appear naturally inside semisimple algebraic groups and are fundamental for their classification. In their reflection representation, the Weyl groups are in fact subgroups of $\mathrm{GL}(\mathbb{Z}^r)$ for some r. This integrality is a very strong and important restriction; reflection groups with this property are also called *crystallographic*.

Closely related to the algebraic group is another important structure, the Lie algebra. Lie algebras arise in nature as vector spaces of linear transformations, for example differential operators. It turns out that finite dimensional semisimple complex Lie algebras decompose into a direct sum labeled by *roots* and a Cartan subalgebra. These roots are (up to signs) the normal vectors defining the reflection hyperplanes of a Weyl group. Again, we have an integrality property for the roots: Let $\mathcal{A}$ be the real hyperplane arrangement given by the orthogonal complements of the roots. Then this is a simplicial arrangement and for each chamber K, the roots labeling the walls of K form a simple system Δ, and in particular all other roots are integer linear combinations of the roots in Δ.

So apparently the combinatorics of root systems and Weyl groups play an important role in mathematics and moreover, a certain integrality is an essential feature of these structures. In the last decades, the concept of a Lie algebra has been generalized in many directions. For example, deformations of Lie algebras called quantum groups have proved to be useful in physics. More generally the theory of Hopf algebras seems to be a further natural direction. Recent results on pointed Hopf algebras have led to yet another symmetry structure, the *Weyl* *groupoid*. Again one has vectors called "roots", but this time the object acting on the roots is in general a groupoid and not a group anymore. A remarkable fact is that even in this much more general setting, the above integrality still plays a crucial role.

The Weyl groupoid historically appeared as an invariant needed to classify Nichols algebras. However, the situation has changed during the last year: Recent observations have considerably increased the importance of the Weyl groupoid.

If the *real roots* of a Weyl groupoid form a finite root system (as defined in Section 1.3), then we will say that the Weyl groupoid is finite. As for root systems from Coxeter groups, a finite Weyl groupoid $\mathcal{W}$ defines a hyperplane arrangement: Fix an object a and its set of positive roots R^a_+. The arrangement associated to $\mathcal{W}$ and a is the set of orthogonal complements of the elements of R^a_+ in $\mathbb{R}^r$. It turns out that these arrangements are simplicial, i.e. the complement of the union of these hyperplanes decomposes into open simplicial cones (see [HW11]). Moreover, it turns out that in terms of simplicial arrangements the axioms of a finite Weyl groupoid reduce to one single integrality property (I) (see [Cun11] or Section 1.2 for details). We call simplicial arrangements satisfying this axiom (I) *crystallographic arrangements*.

Among other things, this explains why the class of arrangements obtained from Weyl groupoids is so large. In fact this class is so large that for example in rank three, 53 of the 67 known sporadic simplicial arrangements (see [Grü09]) over $\mathbb{Q}$ are crystallographic. Since the classification of simplicial arrangements in the real projective plane is still an open problem, it could be very valuable to have a complete classification of a large subclass. But also in higher rank generalizations of crystallographic arrangements could have a great impact. Like reflection arrangements, all crystallographic arrangements are free [BC12]. They could provide examples or counterexamples in geometry or topology, especially since they may also be viewed as compact smooth toric varieties (see [CRT12] or Section 1.6);

1. Weyl groupoids and crystallographic arrangements

1.1. **Simplicial arrangements.**

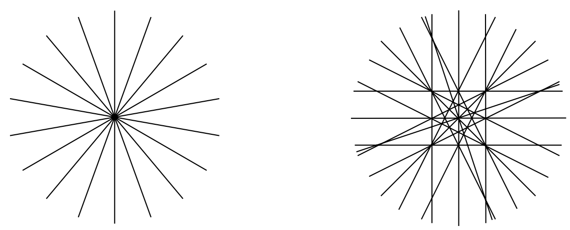
Let $r \in \mathbb{N}$, $V := \mathbb{R}^r$. For $\alpha \in V^*$ we write $\alpha^\perp = \ker(\alpha)$. We first recall the definition of a simplicial arrangement (compare [OT92, 1.2, 5.1]).

Definition 1.1. An *arrangement of hyperplanes* $\mathcal{A}$ is a finite set of hyperplanes in V. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components (*chambers*) of $V \setminus \bigcup_{H\in\mathcal{A}} H$. If every chamber K is an *open simplicial cone*, i.e. there exist $\alpha_1^\vee, \dots, \alpha_r^\vee \in V$ such that

$$K = \Big\{ \sum_{i=1}^r a_i \alpha_i^\vee \mid a_i > 0 \quad \text{for all} \quad i = 1, \dots, r \Big\} =: \langle \alpha_1^\vee, \dots, \alpha_r^\vee \rangle_{>0},$$

then $\mathcal{A}$ is called a *simplicial arrangement*.

Example 1.2. (1) $r = 2$ and $r = 3$ (projective plane)



(2) Let W be a real reflection group, R the set of roots of W. Then $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ is a simplicial arrangement.

1.2. **Crystallographic arrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$, $|\mathcal{A}| = n$ be simplicial. For each H_i, $i = 1, \dots, n$ we choose an element $x_i \in V^*$ such that $H_i = x_i^\perp$ and let $R := \{\pm x_1, \dots, \pm x_n\} \subseteq V^*$.

For each chamber $K \in \mathcal{K}(\mathcal{A})$ set

$$B^K = \{ \text{ normal vectors in } R \text{ of the walls of } K \text{ pointing to the inside } \}.$$

If $\alpha_1^\vee, \dots, \alpha_r^\vee$ is the dual basis to $B^K = \{\alpha_1, \dots, \alpha_r\}$, then $K = \langle \alpha_1^\vee, \dots, \alpha_r^\vee \rangle_{>0}$ since $\mathcal{A}$ is simplicial.

We are now ready for the main definition.

Definition 1.3. Let $\mathcal{A}$ be a simplicial arrangement and $R \subseteq V^*$ a finite set such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. We call $(\mathcal{A}, R)$ a *crystallographic arrangement* if for all $K \in \mathcal{K}(\mathcal{A})$:

$$R \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha.$$

Two crystallographic arrangements $(\mathcal{A}, R)$, $(\mathcal{A}', R')$ in V are called *equivalent* if there exists $\psi \in \mathrm{Aut}(V^*)$ with $\psi(R) = R'$. We then write $(\mathcal{A}, R) \cong (\mathcal{A}', R')$.

Example 1.4. (1) Let R be the set of roots of the root system of a crystallographic Coxeter group. Then $(\{\alpha^\perp \mid \alpha \in R\}, R)$ is a crystallographic arrangement.

(2) If $R_+ := \{(1,0), (3,1), (2,1), (5,3), (3,2), (1,1), (0,1)\}$, then $\{\alpha^\perp \mid \alpha \in R_+\}$ is a crystallographic arrangement.

Question 1.5. What are the symmetries of crystallographic arrangements?

1.3. **Cartan schemes and Weyl groupoids.** We now define the notion of a Weyl groupoid which was introduced by Heckenberger and Yamane [HY08] and reformulated in [CH09b].

Example 1.6. Let $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$, $R^a_+ := \{\alpha_1, \alpha_2, \alpha_3, (0,1,1), (0,1,2), (1,0,1), (1,1,1), (1,1,2)\}$.

$$c_{i,j} := -\max\{k \mid k\alpha_i + \alpha_j \in R^a_+\}, \qquad c_{i,i} := 2 \qquad \text{for} \quad i \neq j,$$

$$(c_{i,j})_{i,j} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}.$$

This is a *generalized Cartan matrix*. It defines *reflections* via

$$\sigma_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i \qquad \text{for } j = 1,2,3.$$

For instance,

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\sigma_1(R^a_+) = \{-\alpha_1, \alpha_2, (1,0,1), (1,1,1), (2,1,2), \alpha_3, (0,1,1), (1,1,2)\}.$$

The elements of $\sigma_1(R^a_+)$ are *positive* or *negative*. Let $R^a = R^a_+ \cup -R^a_+$ and $R^b = \sigma_1(R^a) =: R^b_+ \cup -R^b_+$. Again, one can construct a Cartan matrix from R^b_+ and it gives new reflections. In this example, we obtain the diagram:

$$\begin{array}{ccccc}
\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} & \overset{\sigma_3}{\text{———}} & \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} & \overset{\sigma_2}{\text{———}} & \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \\
\sigma_1 \Big| & & & & \Big| \sigma_1 \\
\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix} & & & & \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}
\end{array}$$

Definition 1.7. Let $I := \{1, \dots, r\}$ and $\{\alpha_i \mid i \in I\}$ the standard basis of $\mathbb{Z}^I$. A *generalized Cartan matrix* $C = (c_{ij})_{i,j \in I}$ is a matrix in $\mathbb{Z}^{I \times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \leq 0$ for all $i, j, k \in I$ with $j \neq k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

Definition 1.8. Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

Definition 1.9. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{1.1}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are compositions of maps σ^a_i with $i \in I$ and $a \in A$, where σ^a_i is considered as an element in $\mathrm{Hom}(a, \rho_i(a))$. The cardinality of I is the *rank of* $\mathcal{W}(\mathcal{C})$.

Definition 1.10. A Cartan scheme is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$. The Cartan scheme is called *simply connected*, if $\mathrm{Hom}(a, a) = \{\mathrm{id}^a\}$ for all $a \in A$. There is a straight forward notion of *equivalence* of Cartan schemes which we skip here.

Let $\mathcal{C}$ be a Cartan scheme. For all $a \in A$ let

$$(R^{\mathrm{re}})^a = \{\mathrm{id}^a\sigma_{i_1}\cdots\sigma_{i_k}(\alpha_j) \,|\, k \in \mathbb{N}_0,\, i_1, \dots, i_k, j \in I\} \subseteq \mathbb{Z}^I.$$

The elements of the set $(R^{\mathrm{re}})^a$ are called *real roots* (at a). The pair $(\mathcal{C}, ((R^{\mathrm{re}})^a)_{a\in A})$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. A real root $\alpha \in (R^{\mathrm{re}})^a$, where $a \in A$, is called positive (resp. negative) if $\alpha \in \mathbb{N}_0^I$ (resp. $\alpha \in -\mathbb{N}_0^I$).

Definition 1.11. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subseteq \mathbb{Z}^I$, and define $m^a_{i,j} = |R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma^a_i(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m^a_{i,j}$ is finite, then $(\rho_i\rho_j)^{m^a_{i,j}}(a) = a$.

The root system $\mathcal{R}$ is called *finite* if for all $a \in A$ the set R^a is finite. By [CH09b, Prop. 2.12], if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R} = \mathcal{R}^{\mathrm{re}}$, and hence $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$ in that case.

Remark 1.12. If $\mathcal{C}$ is a Cartan scheme and there exists a root system of type $\mathcal{C}$, then $\mathcal{C}$ satisfies

(C3) If $a, b \in A$ and $\mathrm{id} \in \mathrm{Hom}(a, b)$, then $a = b$.

1.4. **From simplicial arrangements to groupoids.**

Lemma 1.13. *Let $(\mathcal{A}, R)$ be a crystallographic arrangement, $K_0, K \in \mathcal{K}(\mathcal{A})$ be adjacent chambers and let $B^{K_0} = \{\alpha_1, \dots, \alpha_r\}$. Write $B^K = \{-\alpha_1, \beta_2, \dots, \beta_r\}$ for some $\beta_2, \dots, \beta_r \in R$. Then there exists a permutation $\tau \in S_r$ with $\tau(1) = 1$ and such that*

$$\beta_i = c_{\tau(i)}\alpha_1 + \alpha_{\tau(i)}, \qquad i = 2, \dots, r$$

for certain $c_2, \dots, c_r \in \mathbb{N}_0$.

Proof. Let σ be the linear map

$$\sigma : V \to V, \quad \alpha_1 \mapsto -\alpha_1, \quad \alpha_i \mapsto \beta_i \quad \text{for} \quad i = 2, \dots, r.$$

With respect to the basis B^{K_0}, σ is a matrix of the form

$$\begin{pmatrix} -1 & c_2 & \cdots & c_r \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

for some $c_2, \dots, c_r \in \mathbb{N}_0$ and $A \in \mathbb{N}_0^{(r-1)\times(r-1)}$.
By the same argument with K_0 and K interchanged, with respect to $\{-\alpha_1, \beta_2, \dots, \beta_r\}$, the $\alpha_2, \dots, \alpha_r$ have coefficients in $\mathbb{N}_0$, thus $A^{-1} \in \mathbb{N}_0^{(r-1)\times(r-1)}$, and we conclude that A is a permutation matrix. □

Definition 1.14. Let K, K' be adjacent chambers and $B^K \cap -B^{K'} = \{\alpha\}$. Then by Lemma 1.13 there exist unique $c_\beta \in \mathbb{N}_0$, $\beta \in B^K \setminus \{\alpha\}$ such that

$$\varphi_{K,K'} : B^K \to B^{K'}, \quad \beta \mapsto c_\beta \alpha + \beta, \quad \alpha \mapsto -\alpha$$

is a bijection.

Now fix a chamber K_0 and an ordering $B^{K_0} = \{\alpha_1, \dots, \alpha_r\}$. Then for any sequence $\mu_1, \dots, \mu_m \in \{1, \dots, r\}$ we get a unique chain of chambers $K_0, \dots, K_m$ such that K_{i-1} and K_i are adjacent and

$$-B^{K_{i-1}} \cap B^{K_i} = \{\varphi_{K_{i-1},K_i} \cdots \varphi_{K_0,K_1}(\alpha_{\mu_i})\}$$

for $i = 1, \dots, m$.

For $i = 0, \dots, m-1$, denote by $\sigma^{K_i}_{\mu_{i+1}}$ the linear map given by

$$\sigma^{K_i}_{\mu_{i+1}}(\alpha) = \varphi_{K_i,K_{i+1}}(\alpha)$$

for all $\alpha \in B^{K_i}$. By Lemma 1.13

$$\sigma^{K_i}_{\mu_{i+1}} \in \mathrm{Aut}(V^*), \quad (\sigma^{K_i}_{\mu_{i+1}})^2 = \mathrm{id}.$$

For a sequence $\mu_1, \dots, \mu_m$ we may abbreviate

$$\sigma_{\mu_m} \dots \sigma_{\mu_2} \sigma^{K_0}_{\mu_1} := \sigma^{K_{m-1}}_{\mu_m} \dots \sigma^{K_1}_{\mu_2} \sigma^{K_0}_{\mu_1}$$

since $K_1, \dots, K_m$ are uniquely determined by the sequence of μ_i.

We now construct a Weyl groupoid for a given crystallographic arrangement $(\mathcal{A}, R)$ of rank r. Let $I := \{1, \dots, r\}$. Fix a chamber $K_0 \in \mathcal{K}(\mathcal{A})$ and an ordering $B^{K_0} = \{\alpha_1, \dots, \alpha_r\}$. Consider the set

$$\hat{A} := \{(\mu_1, \dots, \mu_m) \mid m \in \mathbb{N},\ \mu_1, \dots, \mu_m \in I\}$$

and write $a.\nu$ for the sequence $(\mu_1, \dots, \mu_m, \nu)$ if $a = (\mu_1, \dots, \mu_m)$, $\nu \in I$. We have a map

$$\pi : \hat{A} \to \mathrm{End}(V^*), \quad (\mu_1, \dots, \mu_m) \mapsto \sigma_{\mu_m} \dots \sigma_{\mu_2} \sigma^{K_0}_{\mu_1}$$

which yields an equivalence relation $\sim$ on $\hat{A}$ via

$$v \sim w \quad :\Longleftrightarrow \quad \pi(v) = \pi(w)$$

for $v, w \in \hat{A}$. Let

$$A := \hat{A}_{/\sim}.$$

Each sequence $a = (\mu_1, \dots, \mu_m) \in \hat{A}$ determines a unique map φ_a by

$$\varphi_a := \varphi_{K_{m-1},K_m} \cdots \varphi_{K_0,K_1}$$

where $K_0, \dots, K_m$ is the sequence of chambers corresponding to a. We write $K^a := K_m$.

Definition 1.15. For each $a \in \hat{A}$ we construct a Matrix $C^a \in \mathbb{Z}^{r \times r}$ in the following way:

Let $i, j \in I$ and let K' be the chamber adjacent to K^a with $B^{K^a} \cap -B^{K'} = \{\varphi_a(\alpha_i)\}$. By Lemma 1.13 there exist integers $c_{i,1}, \dots, c_{i,r}$ such that

$$\varphi_{a.i}(\alpha_j) = -c_{i,j} \varphi_a(\alpha_i) + \varphi_a(\alpha_j).$$

We set

$$C^a := (c_{i,j})_{1 \le i,j \le r}.$$

Notice that $C^a = C^b$ for $a, b \in \hat{A}$ with $\pi(a) = \pi(b)$, so we obtain a unique Cartan matrix for each element of A, which we will also denote C^a, $a \in A$.

We now define the maps $\rho_i : A \to A$, $i \in I$. First, set

$$\hat{\rho}_i : \hat{A} \to \hat{A}, \quad a \mapsto a.i.$$

Since $\pi(a) = \pi(b)$ implies $\varphi_a = \varphi_b$ for $a, b \in A$, this induces well-defined maps

$$\rho_i : A \to A, \quad \overline{a} \mapsto \overline{a.i},$$

where $\overline{a}$ is the equivalence class of a in A.

Proposition 1.16. *Let $(\mathcal{A}, R)$ be a crystallographic arrangement of rank r and K_0 a chamber. Then I, A, $(\rho_i)_{i\in I}$, $(C^a)_{a\in A}$ defined as above define a Cartan scheme which we denote $\mathcal{C} = \mathcal{C}(\mathcal{A}, R, K_0)$. Further, we obtain a Weyl groupoid $\mathcal{W}(\mathcal{C}) = \mathcal{W}(\mathcal{A}, R, K_0)$.*

We may now construct a root system of type $\mathcal{C}(\mathcal{A}, R, K_0)$. Let $a = (\mu_1, \dots, \mu_m) \in \hat{A}$ and let

$$\phi_a : V^* \to \mathbb{R}^r$$

be the coordinate map for elements of V^* with respect to the basis $\varphi_a(\alpha_1), \dots, \varphi_a(\alpha_r)$ (in this ordering). Set

$$R^a := \phi_a(R).$$

Again, notice that $R^a = R^b$ for $a, b \in \hat{A}$ with $\pi(a) = \pi(b)$, so we get a unique set which we also denote R^a for each element of $a \in A$.

Proposition 1.17. *The sets R^a, $a \in A$ as above are the real roots of $\mathcal{C} = \mathcal{C}(\mathcal{A}, R, K_0)$ and form a root system of type $\mathcal{C}$.*

1.5. From finite Weyl groupoids to arrangements.

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected simply connected Cartan scheme for which the real roots are a finite root system $\mathcal{R}$; we call $\mathcal{W}(\mathcal{C})$ a *finite Weyl groupoid*. Note that in this situation, A is finite by [CH09b, Lemma 2.11].

For each $a \in A$ we set

$$\mathcal{A}^a := \{\alpha^\perp \mid \alpha \in R^a\}.$$

Now fix some $a \in A$. Let B denote the standard basis of $\mathbb{R}^I$ and $B^\vee$ its dual basis. Then $B \subseteq R^a$ and thus

$$K_0 := \langle B^\vee \rangle_{>0} \in \mathcal{K}(\mathcal{A}^a)$$

because no hyperplane of $\mathcal{A}$ intersects K_0 by (R1). Using the definition of σ_i^a and (R1) at $\rho_i(a)$ we see that

$$\sigma_i^a(K_0) \in \mathcal{K}(\mathcal{A}^a), \quad i \in I,$$

as well. But σ_i^a is a linear automorphism, so the chambers of $\mathcal{A}^a$ and $\mathcal{A}^{\rho_i(a)}$ are isomorphic via σ_i^a. Now consider the set

$$U := \bigcup_{m\in\mathbb{N}} \bigcup_{\mu_1,\dots,\mu_m\in I} \overline{\sigma_{\mu_m}\dots\sigma_{\mu_1}^a(K_0)}.$$

By induction every subset $\sigma_{\mu_m}\dots\sigma_{\mu_1}^a(K_0)$ is a chamber of $\mathcal{A}^a$, and all of them are open simplicial cones. By construction, U is connected and closed, and since A is finite, $U = \mathbb{R}^I$. Thus $\mathcal{A}^a$ is a simplicial arrangement.

Alternatively, one can obtain the simpliciality as a corollary to results about the poset $(\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a), \leq_R)$ where $\leq_R$ is the weak order (see [HW11]).

Alltogether, we get:

Theorem 1.18 (see [Cun11])**.** *Let $\mathfrak{A}$ be the set of all crystallographic arrangements and $\mathfrak{C}$ be the set of all connected simply connected Cartan schemes for which the real roots are a finite root system. Then the map*

$$\mathfrak{A}/_{\cong} \to \mathfrak{C}/_{\cong}, \quad \overline{(\mathcal{A}, R)} \mapsto \overline{\mathcal{C}(\mathcal{A}, R, K)},$$

where K is any chamber of $\mathcal{A}$, is a bijection.

1.6. **Fans and toric varieties.**

Definition 1.19. Let $N := \mathbb{Z}^r$. A *fan* in N is a nonempty collection Σ of strongly convex rational polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying the following conditions:

(1) Every face of any $\sigma \in \Sigma$ is contained in Σ.
(2) For any $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $\sigma \in \Sigma$, and

$$\sigma^\vee := \{x \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle x, y\rangle \geq 0 \text{ for all } y \in \sigma\}.$$

Further, $U_\sigma := \mathrm{Spec}(\mathbb{C}[M \cap \sigma^\vee])$.

Theorem 1.20. *For a fan Σ in N, we can naturally glue $\{U_\sigma \mid \sigma \in \Sigma\}$ together to obtain a* toric variety $\bigcup_{\sigma\in\Sigma} U_\sigma$ *associated to (N, Σ).*

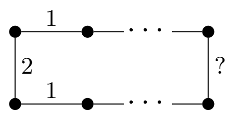
Theorem 1.21 ([CRT12])**.** *The chambers of a crystallographic arrangement are the maximal cones of a fan. There is a one-to-one correspondence between the crystallographic arrangements and the strongly symmetric smooth toric varieties.*

2. Classification of finite Weyl groupoids

Throughout this section, let $\mathcal{C}=\mathcal{C}(I,A,(\rho_i)_{i\in I},(C^a)_{a\in A})$ be a connected simply connected Cartan scheme for which the real roots are a finite root system $\mathcal{R}$. The assumption "simply connected" is not a strong rectriction: It suffices to compute automorphism groups to determine all other such Cartan schemes which are not simply connected, see [CH09a], [CH12], [CH11], and [CH10].

2.1. **Weyl groupoids: Rank two.**

In rank two, the object change diagram of $\mathcal{C}$ is a cycle,



which is locally of the form

$$\cdots \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -c_1 \\ -c_2 & 2 \end{pmatrix} \overset{\sigma_2}{\text{———}} \begin{pmatrix} 2 & -c_3 \\ -c_2 & 2 \end{pmatrix} \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -c_3 \\ -c_4 & 2 \end{pmatrix} \overset{\sigma_2}{\text{———}} \cdots$$

Hence a Cartan scheme of rank two is more or less given by a sequence $(\ldots,c_1,c_2,c_3,\ldots)$, the *characteristic sequence*. The morphism $\sigma_1^i\sigma_2^{i+1}$ is the linear map

$$\begin{aligned}\begin{pmatrix} -1 & c_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c_{i+1} & -1 \end{pmatrix} &= \begin{pmatrix} -1 & c_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c_{i+1} & -1 \end{pmatrix} \\ &= \begin{pmatrix} c_i & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_{i+1} & -1 \\ 1 & 0 \end{pmatrix} = \eta(c_i)\eta(c_{i+1})\end{aligned}$$

for

$$\eta(i)=\begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z}).$$

Remark 2.1. The columns of the matrices $\eta(c_i)\cdots\eta(c_{i+k})$ for k odd and appropriate i, are roots.

In fact, one can prove:

Proposition 2.2. *A finite sequence $(c_1,\ldots,c_n)\in\mathbb{N}^n$, $n\ge 1$ is a characteristic sequence if and only if*

- $\eta(c_1)\cdots\eta(c_n)=-\mathrm{id}$,
- *the entries in the first column of $\eta(c_1)\cdots\eta(c_i)$ are nonnegative for all $i<n$.*

Proposition 2.3. *Let $n\in\mathbb{N}$ and $b_1,\ldots,b_n\in\mathbb{Z}$. If $b_i\ge 2$ for all $i\in\{1,\ldots,n\}$, then $\eta(b_1)\cdots\eta(b_n)$ does not have finite order.*

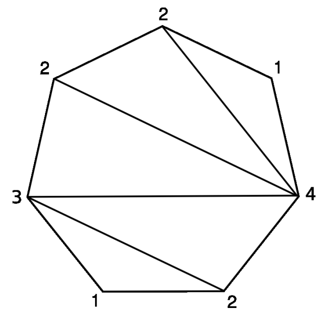


FIGURE 1. A triangulation for the characteristic sequence $(1, 4, 2, 1, 3, 2, 2)$

Since $\eta(0)^2 = -\mathrm{id}$, we always have at least one entry 1 in a characteristic sequence of length > 2. Using

$$\eta(a+1)\eta(1)\eta(b+1) = \eta(a)\eta(b)$$

for all a, b we obtain

Proposition 2.4 (see [CH09a]). *The set $\mathcal{N}$ of all charateristic sequences may be constructed recursively in the following way:*

(1) $(0, 0) \in \mathcal{N}$.
(2) *If* $(c_1, \ldots, c_n) \in \mathcal{N}$, *then* $(c_2, c_3, \ldots, c_{n-1}, c_n, c_1) \in \mathcal{N}$ *and* $(c_n, c_{n-1}, \ldots, c_2, c_1) \in \mathcal{N}$.
(3) *If* $(c_1, \ldots, c_n) \in \mathcal{N}$, *then* $(c_1 + 1, 1, c_2 + 1, c_3, \ldots, c_n) \in \mathcal{N}$.

Now to the roots. Consider on $\mathbb{N}_0^2$ the total ordering $\leq_{\mathbb{Q}}$, where $(a_1, a_2) \leq_{\mathbb{Q}} (b_1, b_2)$ if and only if $a_1 b_2 \leq a_2 b_1$. A finite sequence $(v_1, \ldots, v_n)$ of vectors in $\mathbb{N}_0^2$ is an *$\mathcal{F}$-sequence* if and only if $v_1 <_{\mathbb{Q}} v_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} v_n$ and one of the following holds.

- $n = 2$, $v_1 = (0, 1)$, and $v_2 = (1, 0)$.
- $n > 2$ and there exists $i \in \{2, 3, \ldots, n-1\}$ such that $v_i = v_{i-1} + v_{i+1}$ and $(v_1, \ldots, v_{i-1}.v_{i+1}, \ldots, v_n)$ is an $\mathcal{F}$-sequence.

Proposition 2.5 (see [CH11]). *For $a \in A$ the set R_+^a ordered by $\leq_{\mathbb{Q}}$ is an $\mathcal{F}$-sequence.*

2.2. **Some more general facts.** We now return to the case of arbitrary rank. Let $a \in A$. The last observation on root systems of rank two implies the following useful result:

Theorem 2.6. *Let $\alpha \in R_+^a$. Then either α is simple, or it is the sum of two positive roots.*

The correspondence to crystallographic arrangements yields:

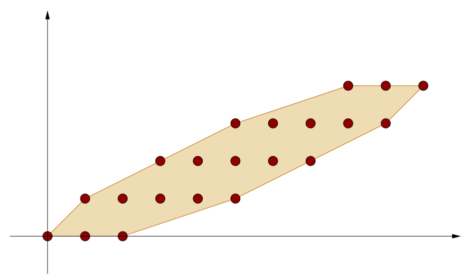
Lemma 2.7. *Let $\alpha, \beta \in R^a_+$. If $\langle \alpha, \beta \rangle_{\mathbb{Q}} \cap R^a \subseteq \pm(\mathbb{N}_0\alpha + \mathbb{N}_0\beta)$, then there exists $b \in A$ and $w \in \mathrm{Hom}(a,b)$ such that $w(\alpha), w(\beta)$ are simple.*

2.3. **Weyl groupoids: Rank three.**

2.3.1. *More facts for rank three.* View the roots as elements of $\mathbb{Z}^3$ with the lexicographic ordering $\leq$.

Lemma 2.8. *Let $\alpha, \beta \in R^a_+$ be minimal in $\langle \alpha, \beta \rangle_{\mathbb{Q}} \cap R^a_+$ with respect to $\leq$. Then $\langle \alpha, \beta \rangle_{\mathbb{Q}} \cap R^a \subseteq \pm(\mathbb{N}_0\alpha + \mathbb{N}_0\beta)$.*

Theorem 2.9 (Weak Convexity)**.** *Let $a \in A$ and $\alpha, \beta, \gamma \in R^a_+$. If $\det(\alpha,\beta,\gamma)^2 = 1$ and α, β are the two smallest elements of $\langle \alpha, \beta \rangle \cap R^a_+$, then either α, β, γ are simple, or one of $\gamma - \alpha$, $\gamma - \beta$ is in R^a.*



Roots on the plane $(1, *, *)$

Example: $\alpha = (0,1,0)$, $\beta = (0,0,1)$, $\gamma = (1,7,3)$

Theorem 2.10 (Required roots)**.** *Let $a \in A$ and $\alpha, \beta, \gamma \in R^a_+$. Assume that $\det(\alpha,\beta,\gamma)^2 = 1$ and that $\gamma - \alpha, \gamma - \beta \notin R^a$. Then $\gamma + \alpha \in R^a$ or $\gamma + \beta \in R^a$.*

Theorem 2.11 (Bound for the Cartan entries)**.** *All entries of the Cartan matrices are greater or equal to -7.*

Remark 2.12. In fact, all entries of the Cartan matrices are greater or equal to -6.

2.3.2. *An algorithm to enumerate root systems of rank three.*
Function **Enumerate**(R)

(1) If R defines a crystallographic arrangement, output R and continue.
(2) $Y := \{\alpha + \beta \mid \alpha, \beta \in R,\ \alpha \neq \beta\} \backslash R$.
(3) For all $\alpha \in Y$ with $\alpha > \max R$:
   (a) Compute all "subgroupoids of rank two" in $R \cup \{\alpha\}$.
   (b) If all Cartan entries are ≥ -7, all rank two root sets are $\mathcal{F}$-sequences, and the "Weak Convexity" is satisfied, then call **Enumerate**$(R \cup \{\alpha\})$.

Remark 2.13. We use Theorem "Required roots" as a further obstruction.

The algorithm terminates and yields the result:

Theorem 2.14 (see [CH12]). *Up to equivalences, there are* 55 *irreducible crystallographic arrangements of rank three.*

2.4. **Weyl groupoids: Rank four to eight.** With the knowledge about rank three, we can now enumerate crystallographic arrangements in higher ranks. Here is a rough sketch of the algorithm.

Function **Enumerate**(R)

(1) If $R \cup -R$ is a root set, output $R \cup -R$ and continue.
(2) For all subspaces U generated by elements of R, check that $R \cap U$ could be extended to a root set.
(3) Set $Y := \{\alpha + \beta \mid \alpha, \beta \in R,\ \alpha \neq \beta\} \backslash R$.
(4) For all $\alpha \in Y$ with $\alpha > \max R$, call **Enumerate**$(R \cup \alpha)$.

2.5. **Weyl groupoids: Higher rank.** An analysis of Dynkin diagrams leads to a complete classification in rank > 8.

Theorem 2.15 (see [CH10]). *There are exactly three families of irreducible crystallographic arrangements:*

(1) *The family of rank two parametrized by triangulations of a convex n-gons by non-intersecting diagonals.*
(2) *For each rank $r > 2$, arrangements of type A_r, B_r, C_r and D_r, and a further series of $r - 1$ arrangements.*
(3) *Further* 74 *"sporadic" arrangements of rank r, $3 \leq r \leq 8$.*

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