“On non-square order Tate–Shafarevich groups of non-simple abelian surfaces over the rationals”

For an elliptic curve (over a number field) it is known that the order of its Tate–Shafarevich group is a square, provided it is finite. In higher dimensions this no longer holds true. A non-simple abelian surface $B$ comes with an isogeny $\varphi: E_1 \times E_2 \rightarrow B$, where $E_1$ and $E_2$ are elliptic curves. We will classify all occurring non-square parts of orders of Tate–Shafarevich groups of such $B$ in case $\varphi$ is a cyclic isogeny and the ground field is the rationals. We will prove that only the cardinalities $k=1,2,3,5,6,7,10,13$ are possible, and in all cases we are able to construct an explicit example.