# Generic Marked Poset Polytopes

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## Introduction

When Stanley introduced order and chain polytopes associated to finite posets in the 1980s [7], he gave an elegant combinatorial description of the face lattices of order polytopes and stated that an analogue for chain polytopes seems messy. In 2011, Ardila, Bliem and Salazar generalized order and chain polytopes to a setting with arbitrary markings [1], now incorporating polytopes that appeared in the representation theory of $\mathfrak{gl}_n(\mathbb{C})$ [4, 3]. Based on work by Jochemko and Sanyal [5], we recently generalized the face lattice description of Stanley to the marked setting [6]. For faces of marked chain polytopes, the situation can be expected to be even more complicated.

We present a new approach to this problem, by introducing a third *generic marked poset polytope*, interpolating between marked order and marked chain polytopes. We hope to gain knowledge about marked chain polytopes by studying generic marked poset polytopes and how they degenerate to marked order and marked chain polytopes.

## Stanley's Poset Polytopes

Given a finite poset $P$ with $\hat{0}$ and $\hat{1}$, Stanley introduced two lattice polytopes in $\mathbb{R}^{\tilde{P}}$, where $\tilde{P} = P \setminus \{\hat{0}, \hat{1}\}$ [7],

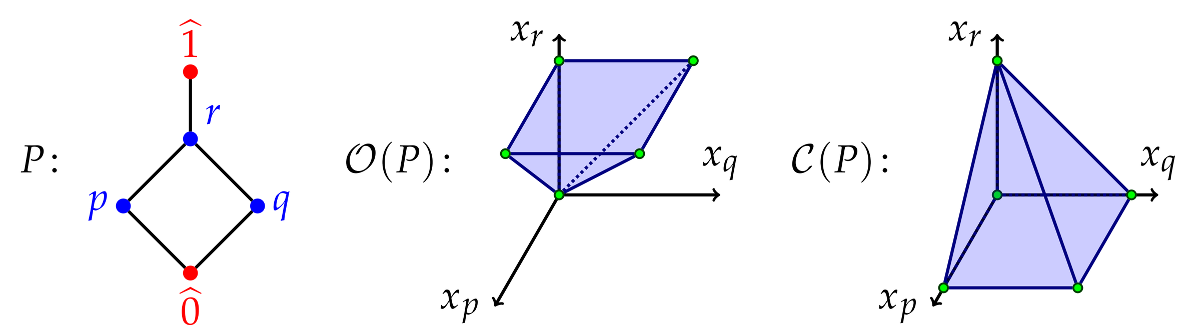
the *order polytope*

$$\mathcal{O}(P) = \left\{ x \in [0,1]^{\tilde{P}} \,\middle|\, x_p \leq x_q \text{ for } p < q \right\},$$

and the *chain polytope*

$$\mathcal{C}(P) = \left\{ x \in [0,1]^{\tilde{P}} \,\middle|\, x_{p_1} + \cdots + x_{p_k} \leq 1 \text{ for } p_1 < \cdots < p_k \right\}.$$

**Example**




The face lattice $\mathcal{F}(\mathcal{O}(P))$ has an elegant combinatorial description in terms of *face partitions* of $P$ ordered by refinement. These are partitions $\pi$ of $P$, such that each block $B \in \pi$ is connected in the Hasse diagram of $P$ and the quotient graph remains acyclic, so $P/\pi$ is a poset.

*A description of the faces of $\mathcal{C}(P)$ analogous to Theorem 1.2 seems messy and will not be pursued here.*

—R. P. Stanley, Two Poset Polytopes, 1986

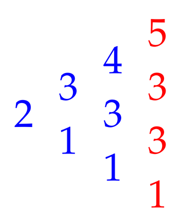
However, there is a piecewise linear bijective *transfer map*

$$\varphi\colon \mathcal{O}(P) \longrightarrow \mathcal{C}(P) \quad \text{with} \quad \varphi(x)_p = x_p - \max_{q \prec p} x_q.$$

Since $\varphi$ preserves $\frac{1}{n}\mathbb{Z}^{\tilde{P}}$, the polytopes $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same Ehrhart polynomial.

## Polytopes in Representation Theory

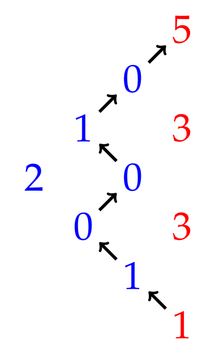
For a given tuple of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$, there is an irreducible representation $V(\lambda)$ of $\mathfrak{gl}_n(\mathbb{C})$ with heighest weight $\lambda$. It has a *Gelfand–Tsetlin basis* with elements enumerated by integral *GT-patterns* [4].




The red column contains the entries of $\lambda$. The blue entries must be integers between their upper and lower right neighbors.

This gives rise to the *Gelfand–Tsetlin polytope* GT$(\lambda)$, whose lattice points correspond to the GT basis vectors of $V(\lambda)$.

Another basis of $V(\lambda)$ called the *Feigin–Fourier–Littelmann–Vinberg basis* is enumerated by integral patterns of another kind [3]:




For each *Dyck path* between two red entries, the sum of the blue entries along the path should be at most the difference of the two red entries. In this case:

$$1 + 0 + 0 + 1 + 0 \leq 5 - 1$$

This gives rise to the *Feigin–Fourier–Littelmann–Vinberg polytope* FFLV$(\lambda)$, whose lattice points correspond to the FFLV basis vectors of $V(\lambda)$.

## Marked Poset Polytopes

To generalize $\mathcal{O}(P)$, $\mathcal{C}(P)$, GT$(\lambda)$ and FFLV$(\lambda)$, Ardila, Bliem and Salazar introduced *marked poset polytopes* [1]. To a finite poset $P$ with subset $A \subseteq P$ containing all extremal elements, and an order-preserving *marking* $\lambda\colon A \to \mathbb{Z}$, associate two lattice polytopes in $\mathbb{R}^{\tilde{P}}$, where $\tilde{P} = P \setminus A$.

The *marked order polytope*

$$\mathcal{O}(P,\lambda) = \left\{ x \in \mathbb{R}^{\tilde{P}} \,\middle|\, \begin{array}{rl} x_p \leq x_q & \text{for } p < q, \\ \lambda(a) \leq x_p & \text{for } a < p, \\ x_p \leq \lambda(a) & \text{for } p < a \end{array} \right\},$$

and the *marked chain polytope*

$$\mathcal{C}(P,\lambda) = \left\{ x \in \mathbb{R}^{\tilde{P}} \,\middle|\, \begin{array}{ll} \sum_i x_{p_i} \leq \lambda(b) - \lambda(a) & \text{for } a < p_1 < \cdots < p_k < b \\ x_p \geq 0 & \text{for all } p \in \tilde{P} \end{array} \right\}.$$

The face lattice $\mathcal{F}(\mathcal{O}(P,\lambda))$ has a combinatorial description in terms of *face partitions* of $(P,\lambda)$ ordered by refinement. These are partitions $\pi$ of $P$, such that each block $B \in \pi$ is connected in the Hasse diagram of $P$, $P/\pi$ is a poset and $\lambda$ induces a strictly order-preserving marking $\lambda/\pi$ on all blocks intersecting $A$.

The *transfer map* generalizes to the piecewise affine bijection

$$\varphi\colon \mathcal{O}(P,\lambda) \longrightarrow \mathcal{C}(P,\lambda) \quad \text{with} \quad \varphi(x)_p = x_p - \max_{q \prec p} \begin{cases} x_q & \text{if } q \in \tilde{P}, \\ \lambda(q) & \text{if } q \in A. \end{cases}$$

## Generic Marked Poset Polytopes

The idea behind generic marked poset polytopes is to add a parameter $t \in [0,1]$ to the transfer map to continuously deform $\mathcal{O}(P,\lambda)$ to $\mathcal{C}(P,\lambda)$. The images turn out to be polytopes of constant combinatorial type for $t \in (0,1)$. One can even introduce different parameters $t_p$ for all $p \in \tilde{P}$ and still obtain similar results for this family incorporating marked chain-order polytopes [2]. This is joint work in progress with Ghislain Fourier, Xin Fang and Jan-Philipp Litza, but on this poster we stick to the diagonal case.

To be precise, given $t \in [0,1]$, define

$$\varphi_t\colon \mathcal{O}(P,\lambda) \longrightarrow \mathbb{R}^{\tilde{P}} \quad \text{with} \quad \varphi_t(x)_p = x_p - t \max_{q \prec p} \begin{cases} x_q & \text{if } q \in \tilde{P}, \\ \lambda(q) & \text{if } q \in A. \end{cases}$$

**Theorem [P.]**

The image $\mathcal{O}_t(P,\lambda) = \varphi_t(\mathcal{O}(P,\lambda))$ is a polytope for all $t \in [0,1]$. The combinatorial type of $\mathcal{O}_t(P,\lambda)$ is constant for $t \in (0,1)$.

For any $t \in (0,1)$, we call $\mathcal{O}_t(P,\lambda)$ a generic marked poset polytope. All $\mathcal{O}_t(P,\lambda)$ share a common $H$-description that degenerates to those of $\mathcal{O}(P,\lambda)$ and $\mathcal{C}(P,\lambda)$ for $t = 0$ and $t = 1$, respectively. In fact, this setting produces poset surjections

$$\mathcal{F}(\mathcal{O}(P,\lambda)) \twoheadleftarrow \mathcal{F}(\mathcal{O}_{\frac{1}{2}}(P,\lambda)) \twoheadrightarrow \mathcal{F}(\mathcal{C}(P,\lambda)).$$

## The Tropical Subdivision

The marked order polytope $\mathcal{O}(P,\lambda)$ has a polyhedral subdivision into maximal regions of affine linearity with respect to the transfer map $\varphi_t$. By construction, this subdivision transfers to all $\mathcal{O}_t(P,\lambda)$. The regions are determined by the loci of non-differentiability of the affine tropical linear forms

$$\max_{q \prec p} \begin{cases} x_q & \text{if } q \in \tilde{P}, \\ \lambda(q) & \text{if } q \in A \end{cases} = \bigoplus_{q \prec p} \begin{cases} 0 \odot x_q & \text{if } q \in \tilde{P}, \\ \lambda(q) & \text{if } q \in A. \end{cases}$$

Hence, the subdivision is obtained by intersecting $\mathcal{O}(P,\lambda)$ with a tropical hyperplane arrangement. The cells are indexed by certain pairs $(\pi, c)$ where $\pi$ is a face partition of $(P,\lambda)$ and $c$ is a tropical covector. This subdivision is a coarsening of the subdivision into products of simplices introduced by Jochemko and Sanyal [5].

## Vertices of Generic Marked Poset Polytopes

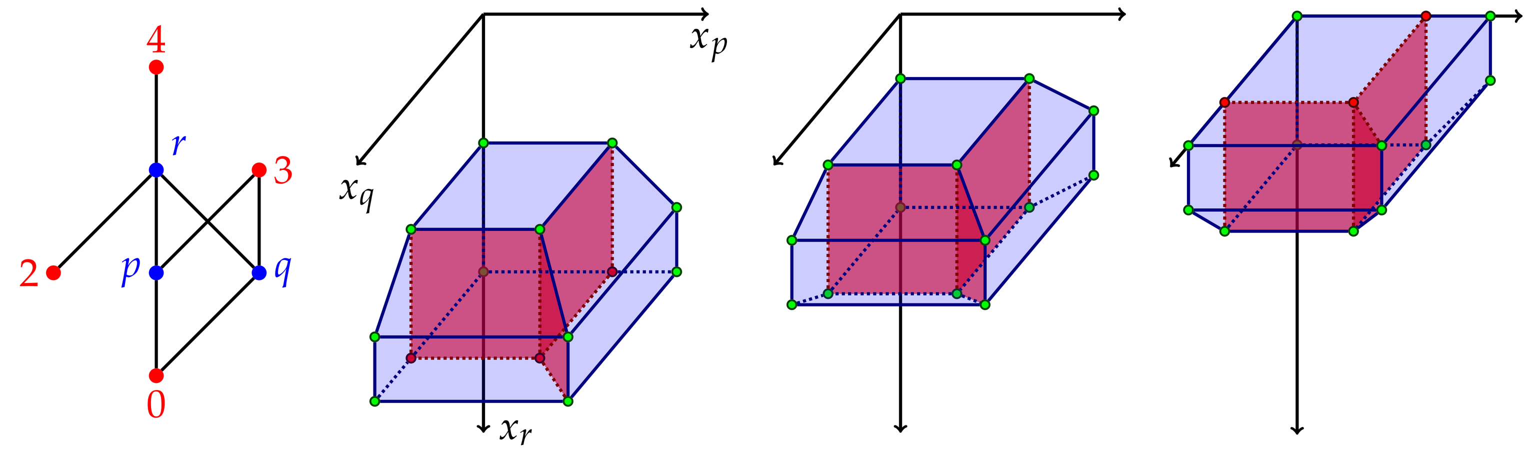
Using the tropical subdivision, we obtain the following result on the vertices of generic marked poset polytopes:

**Theorem [Litza, P.]**

When $t \in (0,1)$, the vertices in the tropical subdivision of $\mathcal{O}_t(P,\lambda)$ are exactly the vertices of the generic marked poset polytope.

We finish with an example of a marked poset $(P,\lambda)$ and the associated polytopes $\mathcal{O}(P,\lambda) = \mathcal{O}_0(P,\lambda)$, $\mathcal{O}_{1/2}(P,\lambda)$ and $\mathcal{O}_1(P,\lambda) = \mathcal{C}(P,\lambda)$. In the example, vertices of the polytopes show in green, vertices that only appear in the subdivision show in red. The theorem now states that, in the generic case, all vertices are green.

**Example**




$(P,\lambda)$ $\mathcal{O}(P,\lambda)$ $\mathcal{O}_{1/2}(P,\lambda)$ $\mathcal{C}(P,\lambda)$

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