A Crash Course on Toric Varieties

Part 2: Toric Varieties from Polyhedral Fans

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This is the second talk I'm giving on toric varieties in the scope of my reading course. Our goal is to understand how abstract toric varieties are glued together from affine pieces given by the cones of a polyhedral fan. We start with a quick recap and then study how faces of cones correspond to open subsets of the associated affine toric variety.

Recap: From Cones to Affine Toric Varieties

- Let $N \cong \mathbb{Z}^n$ be a lattice, $N_\mathbb{R} := N \otimes \mathbb{R}$ the corresponding vector space.

  Start with a rational convex polyhedral cone

  $$\sigma = \operatorname{Cone}(\underbrace{u_1, \ldots, u_k}_{\in N}) \subseteq N_\mathbb{R}.$$

- The dual lattice $M = \operatorname{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ gives the dual vector space $M_\mathbb{R} = M \otimes \mathbb{R}$.

  By Farkas' Theorem, we have the dual cone

  $$\sigma^\vee = \left\{ v \in M_\mathbb{R} \,\middle|\, \langle u, v \rangle \geq 0 \quad \forall u \in \sigma \right\} = \operatorname{Cone}(\underbrace{v_1, \ldots, v_l}_{\in M}) \subseteq M_\mathbb{R}$$

- By Gordan's Lemma, we have a finitely generated semigroup

  $$S_\sigma = \sigma^\vee \cap M = \mathbb{N}\left\{m_1, \ldots, m_s\right\}.$$

- The semigroup determines the $\mathbb{C}$-algebra

  $$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$$

  with multiplication given by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

- This $\mathbb{C}$-algebra gives the affine toric variety

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma]).$$

Example. Take the lattice $N = \mathbb{Z}^3$ and the rational cone $\sigma = \operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq \mathbb{R}^3$ (Picture). By taking inward pointing normal vectors of the facets we see the dual cone is $\sigma^\vee = \operatorname{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3) \subseteq \mathbb{R}^3$ (Picture). The corresponding semigroup is $S_\sigma = \mathbb{N}\{e_1, e_2, e_3, e_1 + e_2 - e_3\}$ so we get the $\mathbb{C}$-algebra

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1}, \chi^{e_2}, \chi^{e_3}, \chi^{e_1+e_2-e_3}] \cong \mathbb{C}[x, y, z, xyz^{-1}] \cong \mathbb{C}[x, y, z, w]/\langle xy - zw \rangle.$$

Thus, U_σ is the affine toric variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$.

Toric ideals

From previous examples we can see that the ideals corresponding to affine toric varieties will always be generated by binomials, since we need to express relations like $w = xyz^{-1}$ without negative exponents as $xy - zw = 0$. Prime ideals generated by binomials are called *toric ideals*.

Proposition. *Let $S \subseteq M \cong \mathbb{Z}^n$ be a semigroup with generators $A = \{m_1, \dots, m_s\}$, then $V = \operatorname{Spec}(\mathbb{C}[S]) = \mathbf{V}(I) \subseteq \mathbb{C}^s$ for the toric ideal*

$$I = \left\langle x^a - x^b \,\middle|\, a, b \in \mathbb{N}^s : \sum_{i=1}^{s} (a_i - b_i) m_i = 0 \right\rangle, \quad \textit{where } x^a = x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s}.$$

Proof. We have the usual construction $\operatorname{Spec}(\mathbb{C}[S]) = \mathbf{V}(\ker \varphi) \subseteq \mathbb{C}^s$ for

$$\begin{aligned} \varphi : \mathbb{C}[x_1, \dots, x_s] &\to \mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}], \\ x_i &\mapsto \chi^{m_i}. \end{aligned}$$

Let $x^a - x^b \in I$, then $\varphi(x^a - x^b) = \chi^{a_1 m_1 + \cdots + a_s m_s} - \chi^{b_1 m_1 + \cdots + b_s m_s} = 0$, so $I \subseteq \ker \varphi$.

For $m \in S$ set $\pi(m) = \left\{ a \in \mathbb{N}^s \,\middle|\, \sum_{i=1}^{s} a_i m_i = m \right\}$.

Now let $f = \sum c_a x^a \in \ker \varphi$, so

$$\varphi(f) = \sum_{m \in S} \left(\sum_{a \in \pi(m)} c_a \right) \chi^m = 0,$$

and therefore $\sum_{a \in \pi(m)} c_a = 0$ for all $m \in S$. It suffices to show that $f_m = \sum_{a \in \pi(m)} c_a x^a$ lies in the ideal I. Let $c_{a^1}, \ldots, c_{a^k}$ be the non-zero coefficients in f_m, then

$$\begin{aligned} f_m = \sum_{i=1}^{k} c_{a^i} x^{a^i} &= c_{a^1} \left(x^{a^1} - x^{a^2} \right) + (c_{a^2} + c_{a^1}) \left(x^{a^2} - x^{a^3} \right) \\ &\quad + \cdots + \left(\sum_{i=1}^{k} c_{a^i} \right) \left(x^{a^k} - x^{a^1} \right) + \left(\sum_{i=1}^{k} c_{a^i} \right) x^{a^1}. \end{aligned}$$

The last term vanishes since $\sum_{i=1}^{k} c_{a^k} = 0$ and all other terms are elements of I. □

Background in Algebraic Geometry: Localization

Consider an affine variety $V \subseteq \mathbb{C}^n$ with coordinate ring $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]/\mathbf{I}(V)$. Assuming V is irreducible, so $\mathbb{C}[V]$ is an integral domain, for $f \in \mathbb{C}[V] \setminus \{0\}$ we can define the *localization* at f by

$$\mathbb{C}[V]_f = \left\{ \frac{g}{f^\ell} \,\middle|\, g \in \mathbb{C}[V], \ell \geq 0 \right\} = \mathbb{C}[V][1/f].$$

We have a correspondence between localizations of the coordinate ring and principal open subsets of the affine variety:

Proposition. *Let V be an irreducible affine variety, $f \in \mathbb{C}[V] \setminus \{0\}$, then*

$$\operatorname{Spec}(\mathbb{C}[V]_f) = \operatorname{Spec}(\mathbb{C}[V])_f = V_f := \left\{ p \in V \,\middle|\, f(p) \neq 0 \right\}.$$

Proof. Let $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$, then

$$V_f \cong W = \mathbf{V}(f_1, \ldots, f_s, 1 - gy) \subseteq \mathbb{C}^n \times \mathbb{C}$$

where $g \in \mathbb{C}[x_1, \ldots, x_n]$ represents $f \in \mathbb{C}[V]$ and the correspondece with V is given by the projection $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$. We obtain the coordinate ring

$$\begin{aligned} \mathbb{C}[V_f] &\cong \mathbb{C}[W] = \mathbb{C}[x_1, \ldots, x_n, y]/\langle f_1, \ldots, f_s, 1 - gy \rangle \\ &\cong \mathbb{C}[x_1, \ldots, x_n, 1/g]/\langle f_1, \ldots, f_s \rangle \\ &\cong \mathbb{C}[V][1/f] = \mathbb{C}[V]_f. \end{aligned}$$

□

Faces of Cones and Principal Open Subsets

Definition. Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational convex polyhedral cone. Given $m \in M_{\mathbb{R}}$ we define the hyperplane and half-space

$$H_m = \{ u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}},$$
$$H_m^+ = \{ u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

If $\sigma \subseteq H_m^+$ we call H_m a *supporting hyperplane*, this happens if and only if $m \in \sigma^\vee$.

Definition. A *face* of the cone σ is given as $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$, written $\tau \preceq \sigma$. Every face of a cone is a cone itself and can be expressed as $\tau = \sigma \cap H_m$ for some $m \in S_\sigma = \sigma^\vee \cap M$.

Proposition. *Let $\tau = \sigma \cap H_m \preceq \sigma$ be a face of a rational convex polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ given by some $m \in S_\sigma$, then the affine toric variety U_τ is the principal open subset $(U_\sigma)_{\chi^m} \subseteq U_\sigma$.*

Sketch of proof. From $\tau = \sigma \cap H_m$ we obtain the dual cone $\tau^\vee = \mathrm{Cone}(\sigma^\vee + \{-m\})$. Thus the corresponding semigroup is $S_\tau = S_\sigma + \mathbb{Z}(-m)$ and the coordinate ring is the localization

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma + \mathbb{Z}(-m)] = \mathbb{C}[S_\sigma]\left[\chi^{-m}\right] = \mathbb{C}[S_\sigma]_{\chi^m}.$$

From what we know about localizations of coordinate rings we have

$$U_\tau = \mathrm{Spec}(\mathbb{C}[S_\tau]) = \mathrm{Spec}(\mathbb{C}[S_\sigma]_{\chi^m}) = \mathrm{Spec}(\mathbb{C}[S_\sigma])_{\chi^m} = (U_\sigma)_{\chi^m}.$$ □

Example. In the previous example of $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq \mathbb{R}^3$ consider the facet $\tau = \mathrm{Cone}(e_1 + e_3, e_2 + e_3) = \sigma \cap H_m$ given by $m = e_1 + e_2 - e_3 \in S_\sigma$. We have

$$U_\tau = (U_\sigma)_{\chi^{e_1+e_2-e_3}}.$$

From

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1}, \chi^{e_2}, \chi^{e_3}, \chi^{e_1+e_2-e_3}] \cong \mathbb{C}[x, y, z, xyz^{-1}] \cong \mathbb{C}[x, y, z, w]/\langle xy - zw \rangle.$$

we obtained the coordinate representation $U_\sigma = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ where $\chi^{e_1+e_2-e_3} \in \mathbb{C}[S_\sigma]$ corresponds to $[w] \in \mathbb{C}[x, y, z, w]/\langle xy - zw \rangle$. Thus

$$U_\tau = (U_\sigma)_{[w]} = \left\{ (x, y, z, w) \in \mathbb{C}^4 \,\middle|\, xy - zw = 0, w \neq 0 \right\} \subseteq U_\sigma.$$

Toric Varieties from Polyhedral Fans

Definition. A *fan* in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that:

(a) Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone. (Here *strongly convex* means $\{0\}$ is a face of σ.)

(b) For all $\sigma \in \Sigma$ we have $\tau \preceq \sigma \Rightarrow \tau \in \Sigma$.

(c) For $\sigma_1, \sigma_2 \in \Sigma$ the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 (hence also in Σ).

After establishing that faces $\tau \preceq \sigma$ correspond to principal open subsets $U_\tau \subseteq U_\sigma$ we can now glue together affine toric varieties U_{σ_1} and U_{σ_2} along the principal open subset $U_{\sigma_1 \cap \sigma_2}$ of both. This gluing construction gives an abstract variety X_Σ called the *toric variety* of Σ.

Example. Consider the fan $\Sigma = \{\tau, \sigma_1, \sigma_2\}$ in $\mathbb{R}$ with lattice $\mathbb{Z}$ given by the cones $\tau = \{0\}$, $\sigma_1 = \text{Cone}(1) = \mathbb{R}_{\geq 0}$ and $\sigma_2 = \text{Cone}(-1) = \mathbb{R}_{\leq 0}$. We have the semigroup algebras

$$\begin{aligned}
\mathbb{C}[S_{\sigma_1}] &= \mathbb{C}[\chi^1] \cong \mathbb{C}[x_1], \\
\mathbb{C}[S_{\sigma_2}] &= \mathbb{C}[\chi^{-1}] \cong \mathbb{C}[x_2], \\
\mathbb{C}[S_\tau] &= \mathbb{C}[\chi^1, \chi^{-1}] \cong \mathbb{C}[x_1]_{x_1} \cong \mathbb{C}[x_2]_{x_2},
\end{aligned}$$

where U_{σ_1} and U_{σ_2} are copies of $\mathbb{C}$ (with coordinates x_1 and x_2, respectively) and U_τ is a one-dimensional torus $\mathbb{C}^*$ contained in both of them.

Our gluing rule is $x_1 \sim (x_2)^{-1}$ whenever $x_1, x_2 \neq 0$, since $\chi^1 = (\chi^{-1})^{-1}$ and we chose isomorphisms mapping $x_1 \mapsto \chi^1$ and $x_2 \mapsto \chi^{-1}$. Thus

$$X_\Sigma = \mathbb{C} \sqcup \mathbb{C} / (x \sim x^{-1})_{x \neq 0}.$$

This is exactly how the charts $(x : 1)$ and $(1 : x)$ in $\mathbb{CP}^1$ are glued, therefore $X_\Sigma \cong \mathbb{CP}^1$.

Example. Consider the fan Σ in $\mathbb{R}^2$ given by the cones

$$\begin{aligned}
\sigma_1 &= \text{Cone}(e_1, e_2), \\
\sigma_2 &= \text{Cone}(e_1, -e_1 - e_2), \\
\sigma_3 &= \text{Cone}(e_2, -e_1 - e_2)
\end{aligned}$$

and its intersections. The dual cones are

$$\begin{aligned}
\sigma_1^\vee &= \text{Cone}(e_1, e_2), \\
\sigma_2^\vee &= \text{Cone}(e_1 - e_2, -e_2), \\
\sigma_3^\vee &= \text{Cone}(e_2 - e_1, -e_1).
\end{aligned}$$

(Pictures)

The semigroup algebras of the affine patches are

$$\begin{aligned}
\mathbb{C}[S_{\sigma_1}] &= \mathbb{C}[\chi^{e_1}, \chi^{e_2}] \cong \mathbb{C}[x_1, y_1],\\
\mathbb{C}[S_{\sigma_2}] &= \mathbb{C}[\chi^{e_1-e_2}, \chi^{-e_2}] \cong \mathbb{C}[x_2, y_2],\\
\mathbb{C}[S_{\sigma_3}] &= \mathbb{C}[\chi^{e_2-e_3}, \chi^{-e_1}] \cong \mathbb{C}[x_3, y_3].
\end{aligned}$$

Thus, we have to glue 3 copies of $\mathbb{C}^2$.

Let's see how to glue U_{σ_1} and U_{σ_2}. We have

$$\begin{aligned}
\mathbb{C}[S_{\sigma_1 \cap \sigma_2}] &= \mathbb{C}[\chi^{e_1}, \chi^{e_2}, \chi^{-e_2}]\\
&= \mathbb{C}[S_{\sigma_1}]_{\chi^{e_2}} \cong \mathbb{C}[x_1, y_1]_{y_1},\\
&= \mathbb{C}[S_{\sigma_2}]_{\chi^{-e_2}} \cong \mathbb{C}[x_2, y_2]_{y_2}.
\end{aligned}$$

Thus the intersection is $\mathbb{C} \times \mathbb{C}^*$ given by $y_1 \neq 0$ for U_{σ_1} and $y_2 \neq 0$ for U_{σ_2}, where we glue according to the rules

$$\begin{aligned}
x_2 &\sim x_1 y_1^{-1},\\
y_2 &\sim y_1^{-1}.
\end{aligned}$$

For the other two intersections we obtain the gluing rules

$$\begin{aligned}
x_3 &\sim y_1 x_1^{-1}, & \qquad x_3 &\sim x_2^{-1},\\
y_3 &\sim x_1^{-1}, & \qquad y_3 &\sim y_2 x_2^{-1},
\end{aligned}$$

where $x_1, y_3 \neq 0$ and $x_2, x_3 \neq 0$ respectively.

Again, this is just how the 3 charts $(x_1 : y_1 : 1)$, $(x_2 : 1 : y_2)$ and $(1 : x_3 : y_3)$ of $\mathbb{CP}^2$ are glued, so $X_\Sigma \cong \mathbb{CP}^2$ in this case.

If time permits, we will discuss another example:

Example. The 2-dimensional fan given by

$$\begin{aligned}
\sigma_0 &= \mathrm{Cone}(e_1, e_2),\\
\sigma_1 &= \mathrm{Cone}(-e_1, -e_2)
\end{aligned}$$

gives the toric variety

$$X_\Sigma = (\mathbb{C}^2 \sqcup \mathbb{C}^2)/((x, y) \sim (x^{-1}, y^{-1}))_{x,y \neq 0},$$

which is the blow-up of $\mathbb{C}^2$ along the coordinate axes.